Revision Theory of Probability

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Introduction

Language with a type-free truth predicate (T) and type-free probability function symbol (P).

$$\lambda \leftrightarrow \neg \mathsf{T}\ulcorner\lambda\urcorner$$
$$\pi \leftrightarrow \neg \mathsf{P}\ulcorner\pi\urcorner > 1/2$$

Want to determine:

- Which sentences are true,
- What probabilities different sentences receive.

or at least some facts about these.

E.g.

- $\mathrm{T}(\mathsf{T}\ulcorner 0 = 0\urcorner) = \mathrm{t}$,
- $\mathrm{p}(\lambda) = 1/2$,
- $\mathrm{p}(\varphi) + \mathrm{p}(\neg\varphi) = 1$.

Revision theory of probability

A revision theory of probability. Gupta and Belnap (1993)

Considerations also apply to degrees of truth and any notion taking values in $\mathbb{R}$.

Fix a model M of base language $\mathcal{L}$.
Our job is to pick $\mathrm{T} : \mathsf{Sent}_{\mathsf{P},\mathsf{T}} \to \{\mathrm{t}, \mathrm{f}\}$ and $\mathrm{p} : \mathsf{Sent}_{\mathsf{P},\mathsf{T}} \to [0, 1]$.

A revision sequence is a sequence of hypotheses:

$$(\mathrm{T}_0, \mathrm{p}_0), (\mathrm{T}_1, \mathrm{p}_1), (\mathrm{T}_2, \mathrm{p}_2) \ldots$$

- To learn about these self-referential probabilities,
- To give more information about the truth revision sequence (focus on p_{On}).

Outline

Outline

What is probability?

Probability is some $\mathrm{p} : \mathsf{Sent}_L \to \mathbb{R}$ with:

- $\mathrm{p}(\varphi) \geqslant 0$ for all φ,
- $\mathrm{p}(\top) = 1$,
- $\mathrm{p}(\varphi \vee \psi) = \mathrm{p}(\varphi) + \mathrm{p}(\psi)$ for φ and ψ logically incompatible.

Many possible applications of the probability notion. E.g.

- Subjective probability, degrees of belief of an agent,
- Objective chance,
- Evidential support,
- 'Semantic probability'.

Semantic Probability

Says how true a sentence is.

This semantic probability assigns:

- 1 if φ is true
- 0 if φ is false.

E.g. $\mathrm{p}(H) = 0$ or $\mathrm{p}(H) = 1$.

But can give additional information about problematic sentences.

Add in additional probabilistic information to a usual truth construction.

- Kripkean: how many fixed points the sentence is true in.
- Revision: how often the sentence is true in the revision sequence.

Connection to Degrees of Truth

So it's very similar to a degree of truth.

Often Lukasiewicz logic used; to study vagueness.

Difference semantic probability and usual degrees of truth: Compositionality.

- $\mathrm{p}(\varphi)$ and $\mathrm{p}(\psi)$ don't fix $\mathrm{p}(\varphi \vee \psi)$ unless φ and ψ are logically incompatible.
- $\mathsf{DegTruth}(\varphi \vee \psi) = \min\{\mathsf{DegTruth}(\varphi), \mathsf{DegTruth}(\psi)\}$.

Though, e.g. Edgington (1997) for degrees of truth as probabilities.

Outline

Revision Theory of Truth

Fix M a model of $\mathcal{L}$.
Construct a model of $\mathcal{L}_\mathsf{T}$ by considering the extensions of truth.

$$\mathrm{T}_0(\dot{\lambda}) = \mathrm{f} \qquad \mathrm{T}_1(\dot{\lambda}) = \mathrm{t} \qquad \mathrm{T}_2(\dot{\lambda}) = \mathrm{f} \qquad \mathrm{T}_3(\dot{\lambda}) = \mathrm{t} \dashrightarrow$$

$$\mathcal{M}_0 \models \lambda \qquad \mathcal{M}_1 \not\models \lambda \qquad \mathcal{M}_2 \models \lambda \qquad \mathcal{M}_3 \not\models \lambda$$

$$\mathrm{T}_{n+1}(\varphi) = \mathrm{t} \iff \mathcal{M}_n \models \varphi.$$

At each finite stage some $\overbrace{\mathsf{T}\ulcorner\mathsf{T}\ulcorner\ldots\mathsf{T}\ulcorner 0 = 0\urcorner\ldots\urcorner\urcorner}^{n}$ is not satisfied.

So extend to the infinite stage and get $\mathcal{M}_\omega \models \forall n \mathsf{T}^n\ulcorner 0 = 0\urcorner$

In fact just going to ω isn't enough (E.g. $\mathsf{T}\ulcorner\forall n\mathsf{T}^n\ulcorner 0 = 0\urcorner\urcorner$) so need to go to the transfinite.

Limit stage

At a limit stage α, one "sums up" the effects of earlier revisions: if the revision process up to α has yielded a definite verdict on an element, d, ...then this verdict is reflected in the α^{th} hypothesis;
Gupta and Belnap (1993)

If a definite verdict is brought about by the revision sequence beneath μ then it should be reflected in the μ^{th} stage.

How do we define transfinite revision sequences for truth?

Characterising brought about

- $\mathrm{T}(\varphi) = \mathrm{t}$ is stable beneath $\mu \implies \mathrm{T}_\mu(\varphi) = \mathrm{t}$
- $\mathrm{T}(\varphi) = \mathrm{f}$ is stable beneath $\mu \implies \mathrm{T}_\mu(\varphi) = \mathrm{f}$

Definition

C is stable beneath μ if

$$\exists \alpha < \mu \, \forall \beta^{>\alpha}_{<\mu}, \quad (\mathrm{T}_\beta, \mathrm{p}_\beta) \in C.$$

E.g. $\mathrm{T}(0=0) = \mathrm{t}$ is stable beneath ω:

$\mathrm{T}_0(0=0) = \mathrm{f}$ $\quad$ $\mathrm{T}_1(0=0) = \mathrm{t}$ $\quad$ $\mathrm{T}_2(0=0) = \mathrm{t}$ $\quad$ $\mathrm{T}_\omega(0=0) = \mathrm{t}$

$\mathcal{M}_0 \models 0=0$ $\quad$ $\mathcal{M}_1 \models 0=0$ $\quad$ $\mathcal{M}_2 \models 0=0$

Outline

Applying revision to probability

A revision sequence is a sequence of hypotheses:

$$(\mathrm{T}_0, \mathrm{p}_0), (\mathrm{T}_1, \mathrm{p}_1), (\mathrm{T}_2, \mathrm{p}_2) \ldots (\mathrm{T}_\omega, \mathrm{p}_\omega), (\mathrm{T}_{\omega+1}, \mathrm{p}_{\omega+1}) \ldots$$

To give a revision theory we need to say:

- How to *revise* $(\mathrm{T}_\alpha, \mathrm{p}_\alpha)$ to get $(\mathrm{T}_{\alpha+1}, \mathrm{p}_{\alpha+1})$.
- How to sum up these into $(\mathrm{T}_\mu, \mathrm{p}_\mu)$ for limits μ.

Gupta and Belnap (1993) give a general revision theory.

- The revision rule gives the $(\mathrm{T}_\alpha, \mathrm{p}_\alpha) \mapsto (\mathrm{T}_{\alpha+1}, \mathrm{p}_{\alpha+1})$.
- For the limit step, one uses:

If a definite verdict is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

I.e. If $\mathrm{p}(\varphi) = r$ is stable beneath μ then $\mathrm{p}_\mu(\varphi) = r$.

Revision Rule

$\mathrm{p}_{\mu+n}(\varphi)$ is relative frequency of φ being satisfied in $\mu, \mu+1, \ldots, \mu+n-1$.

$$\mathrm{p}_{\mu+n}(\varphi) = \frac{\#\{\alpha \in \{\mu, \ldots, \mu+n-1\} \mid \mathcal{M}_\alpha \models \varphi\}}{n}.$$

$\mathcal{M}_0 \models \lambda$ $\quad$ $\mathcal{M}_1 \not\models \lambda$, $\mathrm{p}_1(\lambda) = 1$ $\quad$ $\mathcal{M}_2 \models \lambda$, $\mathrm{p}_2(\lambda) = 1/2$ $\quad$ $\mathcal{M}_3 \not\models \lambda$, $\mathrm{p}_3(\lambda) = 2/3$ $\quad$ - - - - - - - - ›

- -› $\mathcal{M}_\omega \models \lambda$ $\quad$ $\mathcal{M}_{\omega+1} \not\models \lambda$, $\mathrm{p}_1(\lambda) = 1$ $\quad$ $\mathcal{M}_{\omega+2} \models \lambda$, $\mathrm{p}_2(\lambda) = 1/2$ $\quad$ $\mathcal{M}_{\omega+3} \not\models \lambda$, $\mathrm{p}_3(\lambda) = 2/3$ $\quad$ - - - - - - - - ›

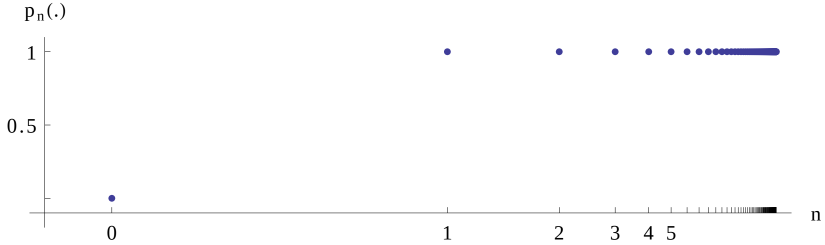
- For all limits μ, $\mathrm{p}_{\mu+1}(\varphi) = 0$ or $\mathrm{p}_{\mu+1}(\varphi) = 1$.
- Alternative: Horsten (ms).

Limit stage

If a definite verdict is brought about beneath μ then it should be reflected at μ.

If $\mathrm{p}(\varphi) = r$ is stable beneath μ then $\mathrm{p}_\mu(\varphi) = r$.

$0 = 0$.



So we get: $\mathrm{p}_\omega(0 = 0) = 1$

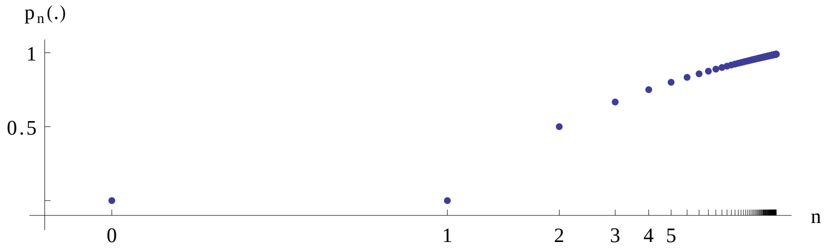
It's not powerful enough here

Converging probability

Desired:

If $\mathrm{p}_n(\varphi) \longrightarrow r$ as $n \longrightarrow \omega$ then $\mathrm{p}_\mu(\varphi) = r$.

$\mathrm{T}\ulcorner 0 = 0 \urcorner$.



Want: $\mathrm{p}_\omega(\mathrm{T}\ulcorner 0 = 0 \urcorner) = 1$

But this isn't yet a "definite verdict".

The limit stage

Original proposal:

If a definite verdict is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ

The limit stage

Reformulated:

If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ
- Properties to focus on:
  - All definite verdicts: $\mathrm{p}(\varphi) = r$, $\mathrm{T}(\varphi) = \mathrm{t}/\mathrm{f}$.

The limit stage

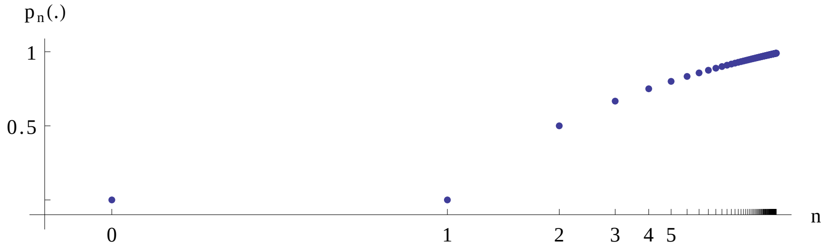
Extended to capture convergence:

If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ
- Properties to focus on:
  - All intervals: $r \leqslant \mathrm{p}(\varphi) \leqslant q$, and $\mathrm{T}(\varphi) = \mathrm{t}/\mathrm{f}$.

Considering intervals now captures converging probability

$T\ulcorner 0=0 \urcorner$.



For each $\epsilon > 0$,

$$1-\epsilon \leqslant \mathrm{p}(T\ulcorner 0=0\urcorner) \leqslant 1$$

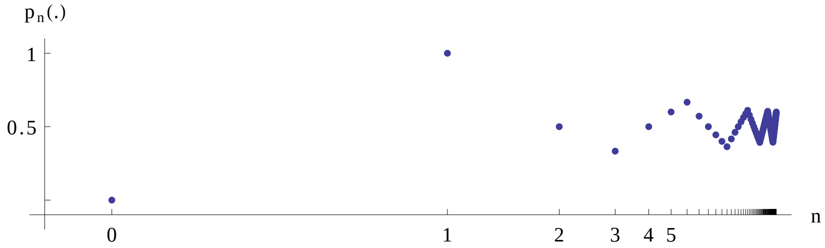
is stable beneath ω.

$$\text{So: } \mathrm{p}_\omega(T\ulcorner 0=0\urcorner)=1$$

We get more than just convergence

δ.

$$\delta \leftrightarrow (\mathrm{P}\ulcorner\delta\urcorner < 0.4 \vee (0.4 \leqslant \mathrm{P}\ulcorner\delta\urcorner \leqslant 0.6 \wedge \mathrm{T}\ulcorner\delta\urcorner)).$$



So: $0.4 \leqslant \mathrm{p}_\omega(\delta) \leqslant 0.6$

Go further?

Current proposal:

If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ
- Properties to focus on:
  - All intervals $r \leqslant \mathrm{p}(\varphi) \leqslant q$, and $\mathrm{T}(\varphi) = \mathrm{t}/\mathrm{f}$

Go further?

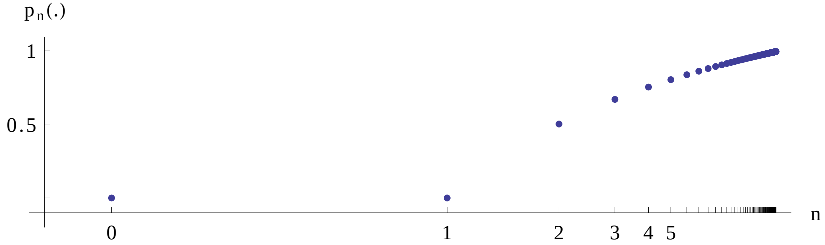
Extending further:

If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ
- Properties to focus on:
  - As many as possible

Cannot consider *any* properties

$T\ulcorner 0 = 0 \urcorner$



$p(T\ulcorner 0 = 0 \urcorner) < 1$ is stable beneath ω.
But this *should not* be reflected at ω! We want $p_\omega(T\ulcorner 0 = 0 \urcorner) = 1$.

The problem is: $p(T\ulcorner 0 = 0 \urcorner) < 1$ does not act nicely under limits.

Closed properties

Closed properties are those:

- That "act nicely" under limiting operations.
- Formally: closed in the product topology on $(\{\mathrm{t},\mathrm{f}\}\times[0,1])^{\mathsf{Sent}_{\mathrm{P},\mathrm{T}}}$

E.g.

Closed	Not closed
$r \leqslant \mathrm{p}(\varphi) \leqslant q$	$r < \mathrm{p}(\varphi) < q$
$\mathrm{T}(\varphi)=\mathrm{t}$; $\mathrm{T}(\varphi)=\mathrm{f}$	$\mathrm{p}(\varphi)\in\mathbb{Q}$
$\mathrm{p}(\varphi)+\mathrm{p}(\psi)=\mathrm{p}(\varphi\vee\psi)$	$\mathrm{p}(\varphi)+\mathrm{p}(\psi)\neq\mathrm{p}(\varphi\vee\psi)$
p is finitely additive probability	p is $\mathbb{N}$-additive.
T is maximally consistent	T is ω-consistent.
$\mathrm{p}(\varphi)\in\{0,1\}$	

Going further

Extending further:

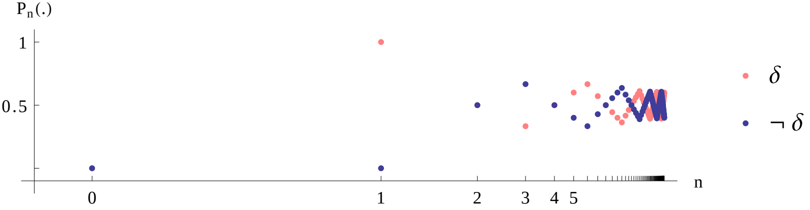
If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Stable beneath μ
- Properties to focus on:
  - As many as possible
    - All closed properties of hypotheses

Relationships between probabilities

δ; where

$$\delta \leftrightarrow (\mathsf{P}\ulcorner\delta\urcorner < 0.4 \vee (0.4 \leqslant \mathsf{P}\ulcorner\delta\urcorner \leqslant 0.6 \wedge \mathsf{T}\ulcorner\delta\urcorner)).$$



$\mathrm{p}(\delta) + \mathrm{p}(\neg\delta) = 1$ is closed and stable.

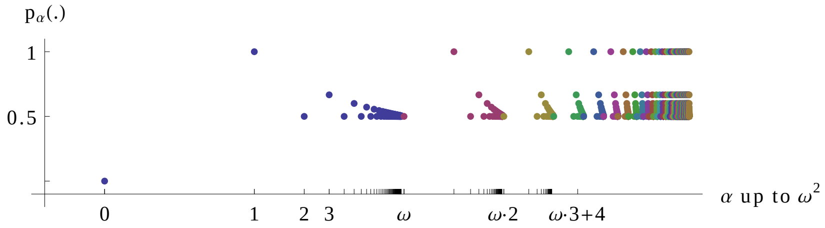
So: $\mathrm{p}_\omega(\delta) + \mathrm{p}_\omega(\neg\delta) = 1$

We similarly get:

If each p_α is a finitely additive probability ($\alpha < \mu$), then so is p_μ

Transfinite

λ.



Want: $p_{\omega\cdot\omega}(\lambda) = 1/2$

But $1/4 \leqslant p(\lambda) \leqslant 3/4$ isn't stable beneath $\omega \cdot \omega$.
Instead it's nearly stable.

Getting the transfinite right

If a property of the hypotheses is brought about by the sequence beneath μ then it should be reflected in the μ^{th} stage.

- Brought about:
  - Nearly stable beneath μ
- Properties to focus on:
  - All closed properties of hypotheses

Definition

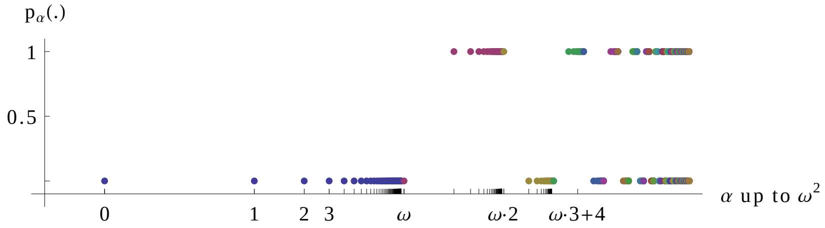
C is *nearly stable* beneath μ if

$$\exists \beta < \mu \, \forall \alpha_{>\beta}^{<\mu} \, \exists N < \omega \, \forall n_{>N}^{<\omega} \quad (\mathrm{T}_{\alpha+n}, \mathrm{p}_{\alpha+n}) \in C$$

Non-convex features

$$\delta \leftrightarrow \left(\begin{array}{cc} & (\text{limstage} \wedge \neg \mathsf{T}\ulcorner\delta\urcorner) \\ \vee & (\neg\text{limstage} \wedge \mathsf{T}\ulcorner\delta\urcorner) \end{array} \right)$$

Where limstage is $\neg\gamma$ for $\gamma : \neg\forall n \mathsf{T}^n\ulcorner\gamma\urcorner$.



$p(\delta) \in \{0,1\}$ is closed and stable beneath ω.

So: $p_{\omega\cdot\omega}(\delta) \in \{0,1\}$.

Is this desirable behaviour?
Perhaps restrict to closed *convex* sets?

Outline

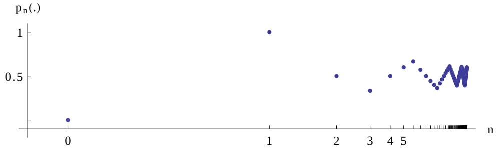
Revision sequences exist

Theorem

The limit rule given is always satisfiable

Using the topological compactness of $(\{\mathrm{t},\mathrm{f}\}\times[0,1])^{\mathsf{Sent}_{\mathsf{P},\mathsf{T}}}$.

But there might be many hypotheses satisfying the limit criterion. E.g.



one where $\mathrm{p}_\omega(\delta) = 0.5$, and
one where $\mathrm{p}_\omega(\delta) = 0.6$.

Probability at limits

Theorem

- p_μ *is a finitely additive probability function*
- T_μ *is maximally consistent*

This contrasts to usual revision theory.

Limit stages are legitimate models of the language.

Countable additivity

Things don't work so well with quantifiers:

Theorem

There is a formula $\varphi(x)$ with:

- *for each n,* $\mathrm{T}_\mu(\varphi(\overline{n})) = \mathrm{t}$*;* $\mathrm{T}_\mu(\forall n\, \varphi(n)) = \mathrm{f}$
- *for each n,* $\mathrm{p}_\mu(\varphi(\overline{n})) = 1$*;* $\mathrm{p}_\mu(\forall n\, \varphi(n)) = 0$

I.e.

- T_μ *is ω-inconsistent*
- p_μ *is not $\mathbb{N}$-additive*

Proof.

McGee (1985). Consider $\gamma \leftrightarrow \neg\forall n\, \mathsf{T}^{n+1}\ulcorner\gamma\urcorner$.
$\varphi(n) := \mathsf{T}^{n+1}\ulcorner\gamma\urcorner$ □

ω-consistency and $\mathbb{N}$-additivity are not closed properties.

Probabilistic Convention T

Theorem

p_μ *satisfies Probabilistic Convention* T*:*

$$\mathrm{p}_\mu(\varphi) = \mathrm{p}_\omega(\mathsf{T}\ulcorner\varphi\urcorner).$$

Proof.

$|\mathrm{p}_{\beta+n}(\varphi) - \mathrm{p}_{\beta+n}(\mathsf{T}\ulcorner\varphi\urcorner)| \longrightarrow 0$ as $n \longrightarrow \omega$. □

Note: depends on the revision rule.

Outline

Horsten (ms)

Define:

- For X a finite set of ordinals $< \beta$:

$$\mathrm{p}_X(\varphi) := \frac{\#\{\alpha \in X \mid \mathcal{M}_\alpha \models \varphi\}}{\#X}.$$

- If for all $\epsilon > 0$, enough X have $|\mathrm{p}_X(\varphi) - r| < \epsilon$ then put $\mathrm{p}_\beta(\varphi) = r$.

What does ‘enough’ mean?
Use a non-principal ultrafilter $\mathcal{U}_\beta$ on $(\beta)^{<\omega}$.

Theorem

$\mathrm{p}_\beta(\varphi)$ *receives a value for all* φ.
And is a (finitely additive) probability function.

Connection to closed properties definition

Theorem

If $C \subseteq [0,1]^{\mathsf{Sent}_{\mathrm{P},\mathrm{T}}}$ is closed and $\{X \in \beta^{<\omega} \mid \mathrm{p}_X \in C\} \in \mathcal{U}_\beta$ then $\mathrm{p}_\beta \in C$.

For choices of ultrafilters at limits that cohere with the near stability characterisation, it'll then satisfy my conditions.

Horsten's can also apply at the successor stage.

- E.g. previous one had $\mathrm{p}_{\mu+1}(\varphi) \in \{0,1\}$.
- This can instead 'look through' the limit stages by taking samples.

Limiting behaviour

Horsten also says: treat 'arbitrarily close' different to 'equal'.

Only put $\mathrm{p}_\beta(\varphi) = r$ if $\mathrm{p}_X(\varphi) = r$ often enough (instead of close enough).

If $|\mathrm{p}_X(\varphi) - r| < \epsilon$ often enough for each ϵ then $\mathrm{p}_\beta(\varphi) \approx r$.
Then $\mathrm{p}_\beta(\varphi)$ is a non-standard number.

Outline

Summary

We've constructed a revision theory for probability.
Where probability is a semantic notion like degree of truth.
Which measures how often φ was satisfied in the revision sequence.

I focused on the limit stages and focused on closed properties being 'taken over' instead of definite verdicts.

And this should apply more generally to revision sequences with other value spaces.

Horsten's proposal is related and is a generalisation of mine for certain choices of ultrafilters.
His allows nicer successor stages.

Thanks!

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