Complex numbers

C1 Write the complex number $z_1 = \sqrt{2}e^{i\pi/4}$ in the form $a + ib$ for real a and b.

C2 Write the complex number $z_2 = 1 + \sqrt{3}i$ in polar form $re^{i\theta}$.

C3 Let $\bar{z}$ denote the complex conjugate of the complex number z. Write $\overline{z_2}$ in polar form.

C4 For any complex numbers w and z, show that $z\bar{z} = |z|^2$, $\overline{zw} = \bar{z}\,\overline{w}$ and $|z+w|^2 = |z|^2 + |w|^2 + \bar{z}w + z\overline{w}$.

C5 Show that $|1 + e^{i\theta}|^2 = 4\cos^2(\theta/2)$.

C6 Let $\omega = e^{2i\pi/3}$. Compute $\overline{\omega}$ and $|\omega|$. Show that $\overline{\omega} = \omega^2$, $\overline{\omega^2} = \omega$ and $\omega^3 = 1$. Show that $1 + \omega + \omega^2 = 0$.

Matrices

Let us define the matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

M1 Show that $\sigma_x\sigma_y = -\sigma_y\sigma_x$, $\sigma_x^2 = \sigma_y^2 = I_2$, $\sigma_x^\dagger = \sigma_x$ and $\sigma_y^\dagger = \sigma_y$. [$M^\dagger$ denotes the Hermitian conjugate of the matrix M – i.e. the matrix derived from M by taking its transpose and the complex conjugate of each coefficient.]

M2 Show that the eigenvalues of σ_x are ± 1, and find normalised eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Show that these eigenvectors are orthogonal [i.e. $\overline{\mathbf{e}_1}^T \cdot \mathbf{e}_2 = 0$]

M3 Show that the eigenvalues of σ_y are ± 1, and find normalised eigenvectors $\mathbf{f}_1$ and $\mathbf{f}_2$. Show that these eigenvectors are orthogonal [i.e. $\overline{\mathbf{f}_1}^T \cdot \mathbf{f}_2 = 0$].

M4 Show how to write an arbitrary (complex) 2-component vector $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in terms of the basis $\mathbf{f}_1$ and $\mathbf{f}_2$. i.e. write $\mathbf{v} = a\mathbf{f}_1 + b\mathbf{f}_2$, where a and b are constants you should determine in terms of α and β.

Probability

Consider that we have two fair six-sided dice. The value on the first die is a and on the second, b.

P1 Compute Prob(a is odd) and Prob(a is prime). [By "Prob(a is x)" where "x" is some property we mean the total probability that a randomly chosen a has property x i.e. $\sum_{a \text{ has property x}} \text{Prob}(a)$.]

P2 Compute (i) Prob(a is odd$|a$ is prime) (ii) Prob(a is prime$|a$ is odd) (iii) Prob(a is odd and a is prime). [The | denotes "given that".]

P3 Show that
Prob(a is odd and a is prime) = Prob(a is prime$|a$ is odd) Prob(a is odd) and that
Prob(a is odd and a is prime) = Prob(a is odd$|a$ is prime) Prob(a is prime).

P4 Let $E(a)$ denote the expected (or expectation) value of a, $E(a) := \sum_a a\,\text{Prob}(a)$. Compute $E(a)$ and $E(a^2)$.

P5 What are the possible values of $a + b$ and their probabilities? Compute $E(a + b)$ and verify that $E(a + b) = E(a) + E(b)$.

P6 What are the possible values of ab and their probabilities? Compute $E(ab)$ and verify that $E(ab) = E(a)E(b)$.