Modelling Exercises - Summer 2019 Edition

What are these resources? These exercises are based on the experience of the Engineering Mathematics Department at the University of Bristol. We have taught over 40 generations of students the art and practice of *ab initio* real-world problem solving, using mathematics — that is, mathematical modelling — at degree level. You can find out more about our degree programmes at the web address below.

Our core experience is that the best way to teach students to solve open-ended problems is by enabling them to investigate; allow them to think freely without fear of getting the "wrong" answer. We find it best if students are allowed to do this in pairs or small groups.

Together with the MEI and their involvement in the running the AMSP Programme to support A-level Mathematics in the UK, we have developed these set of mathematical modelling projects that are intended to be used in the school classroom.

The problems assume no more mathematics than already taught at GCSE. In particular no calculus is assumed, nor any knowledge of statistics or mechanics. Any additional formulae are included as part of the problem description. So it is also possible to use these exercises as stretching material for more able GCSE students.

Usage notes It is envisaged that the exercises will be tackled in groups of 2-5 students. Many of the exercises have multiple sheets that should be used consecutively. Copy the front sides of the sheets and then give the first sheet in a set to a group of students. We envisage that on average, the investigation of each sheet should take about 15 minutes; some much longer, especially the more open-ended ones.

Hand out the subsequent sheets only when the group has finished working on the previous one. (Sheets may contain solutions to previous ones and several sheets being handed out at the same time can overwhelm some students).

Many of the questions don't have clear cut answers, and sometimes alternative paths to solutions are possible.

The back side of each sheet shows notes on the solution. These notes are targeted at teachers and should not normally be provided to students.

The role of the teacher is to circulate around the classroom, to be ready to provide the next sheet in a sequence when appropriate. The ethos is not to encourage the students to get the "right" answers, but to encourage them to think, and to enable all group members to contribute.

New for 2019 Based on feedback so far, we have fixed some bugs and typos. We've also indicated a level of difficulty and which topics in the A-level syllabus each problem relates to; See the contents page. New problems this year are also highlighted with the symbol NEW 2019

It should be said that each problem is intended to teach the art of mathematical modelling, rather than to be associated with any particular part of the syllabus. Also, many of the 'hard' problems can easily be discussed and tackled using elementary approaches.

We are constantly working on improving the exercises and definitely welcome feedback. Please contact `Alan.champneys@bristol.ac.uk` for any suggestions for improvements, or even new problems.

The complete set of exercises can be downloaded from

bristol.ac.uk/engmaths-modelling

Contents

E=Easy, M=Medium, H=Hard

This page has been left deliberately blank[1]

[1] apart, of course, for this sentence, and its footnote

The top-notch burger joint

(by Thilo Gross)

I would like to open a top-notch burger joint. A really nice expensive restaurant, but only for burgers. If you tried my burgers, you would understand. Of course this quality cannot be mass produced, so it will have to be a small place.

I might actually need your help to figure out how small. Please estimate the number of burgers that I can prepare myself within an hour.

Given the number of burgers that I can make, what is the approximate number of customers that I can serve in an evening?

What I really want to know is, given the number of customers, how much space do I need to rent to seat all the customers on tables such that there is still enough space left that they can actually get to the tables.

Solution

This is a challenge in 3 parts with increasing difficulty. In the first part we are asked how many burgers the owner of the fictional burger joint can prepare in 1 hour. As in almost every modeling question there is no crystal clear answer. An individual burger might need as much as 15 min from start to finish, but while the patty is being cooked we can put other burgers together. Thus we can probably finish a burger every other minute, which equates to a production of 30 burgers per hour.

In the second part we are asked how many customers we can service in an evening. Lets assume that we make 30 burgers per hour and realistically we are going to have customers between 6pm and 10pm that would mean that we can feed 120 customers.

The third part asks about the seating. The first difficulty here is that we have to ask how many customers we will be in the restaurant during peak time as my 120 customers are unlikely to all hang around in the restaurant all evening. We know that we can feed 30 customers per hour, if we assume that the average customer spends two hours in the restaurant we can expect to have about 60 customers in the restaurant at the same time.

How much space do I need to seat 60 people? Let's try to work out how much space we need at least. Its not unreasonable that a person sitting on a chair with some table space needs approximately 60cm $\times$ 60cm, which is 0.36m^2. So we could actually seat 60 people on 21.6m^2. But we haven't allowed for corridor space so far. One can now have fun exploring various geometries of the restaurant. Probably the most economic one is one long corridor with tables on either side. There are now 30 people sitting on each side of the corridor, each of whom needs 60cm space so the corridor needs to be 18m long. If it a meter wide that adds 18m^2. Hence we may be able to get away with 39.6m^2. Considering more comfortable geometries with a little bit more space would result in a larger restaurant, but probably not more than twice as large.

Make sure that students use an estimate, instead of just guessing the answers. See whether they realized that not all customers are in the restaurant at the same time.

Moving Mount Fuji

(by Thilo Gross)

How many dump trucks would you need to move mount Fuji[2]?

This is a classic job-interview question and we can find an answer by a so-called a Fermi estimate: We break the answer into little pieces, each easier than the question as a whole.

The volume of a pyramid is a a third of the volume of the box into which the pyramid would fit. Fuji is a mountain, not a pyramid, but we can use this formula to roughly estimate its volume

$$V =$$

So given this volume we can estimate the approximate mass of mount Fuji

$$M =$$

How many trucks would it need to move this mass?

$$N =$$

[2]You may asume that the mountain is about 4km tall

Solution

The purpose of this worksheet is to provide a gentle introduction to Fermi estimates. The origins of this question are not completely clear but it became popular after it has been in the standard pool for job interview questions at Microsoft for several years.

We are given that Fuji is 4km high and from the photo it's about twice as wide at the base. Using the "pyramid approximation" we obtain

$$V = \frac{8 \cdot 8 \cdot 4}{3}\ \mathrm{km}^3 \approx 85\ \mathrm{km}^3 = 85 \cdot 10^9\ \mathrm{m}^3$$

From somewhere I remember that the density of rock is about 3 kg/l which means 3 tonnes per cubic meter. Hence

$$M \approx 255 \cdot 10^9\ \mathrm{t}$$

As a check I googled for typical mountain masses and various sources say $3 \cdot 10^{14}$ kg, which is in very good agreement with our estimate. A large road-going dump truck can carry approximately 30 tonnes of material. So we need approximately

$$N \approx 10^{10}, \qquad = \text{about 10 billion}$$

trucks. Of course we used wild approximations, but we can be fairly sure that the result is within the right order of magnitude. The Fermi estimate shows that moving a mountain would be huge project. If Japan devoted a fraction of its GDP to this task it may be able to build the road infrastructure and a million trucks, each of which would have to do 10.000 trips to the mountain.

If you want to challenge your students you can ask them to Fermi estimate as well how many large trucks exist in the world. There are many ways in which progress can be made, for example by estimating how many truck drivers exist, or by considering that the world burns about 10 gigatonnes of fossil fuel per year and a least the coal included in this number has been on a truck at some point.

Fermi Challenge

(by Thilo Gross)

Here are three quick questions. Try to use Fermi estimates to find answers.

I want to build a classical redbrick house in Bristol. It will be a 2 bedroom property. How many bricks do I need?

$$A =$$

A plumber in London cleans out the drains of a laundrette. This yields enough small change to fill a 5l bucket. Estimate the value of the change (in pound), assuming a typical mixture of coins.

$$B =$$

Consider a city with about 1 million residents, e.g. greater Manchester. How many playgrounds could we build on an area that is as large as the combined area taken up by parking spaces in the city?

$$C =$$

Now multiply your answers

$$A \cdot B \cdot C =$$

Solution

This worksheet is intended as a competition between groups of students. It assumes that the students are already familiar with Fermi estimates, e.g. they may have previously done our worksheet “Moving Mount Fuji”.

The first question can be answered by first estimating the total length of wall (perhaps 80 m for a 6 m by 8 m house, including some meters of double wall on the front and back and interior walls), then the total area of walls (we could assume 3 m average height of walls), and then dividing by the area of the side of a brick (about 180 cm^2).

The second question is more tricky. Here it can be useful to just take whatever coins you have in your pocket. Count their value and estimate their volume.

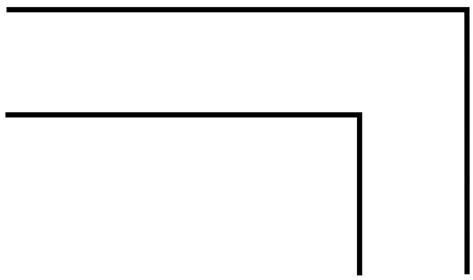
For the third question, you can estimate the number of cars (it’s more than 1 car per 2 people in the UK.) Most of the time these cars sit around parked (about 23.5h per day) so we need at least as many parking spaces as cars, perhaps a bit more. Multiply by the area of typical parking space (ca. 15 m^2) and divide by the size of a typical playground (ca. 400 m^2).

Obviously there are no definitive answers and alternative paths to solutions are possible. The multiplication of results in the last step is done to determine the winner of the competition. If you have an odd number of groups then the group which the result of $A \cdot B \cdot C$ is the median has almost certainly the best estimate. For an even number of groups the group for which the result is closest to the geometric mean of results has almost certainly the best estimates (unless one group is particularly far off the mark, in such a case ignore that group).

The Mathematical Removal Company 1

(by Alan Champneys)

The Mathematical Removal Company need to move a series of ladders from a window cleaning business through a corridor that is only 1 metre wide. The corridor consists of two long straight sections joined by a right-angled corner.



To being with, they need to calculate the longest ladder that can be manipulated around the corner, while keeping the ladder precisely horizontal. (You may ignore the thickness of the ladder.) Try drawing some pictures.

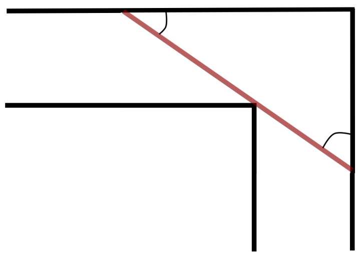
Imagine a ladder that is too long. Draw a picture of the point at which it gets stuck. What are the angles involved? Which direction can the ladder be moved?

Can you now work out what the angles would need to be if the ladder could be moved in either direction? So, what is the length of the longest ladder?

$$\ell = \qquad\qquad \text{metres}$$

Solution

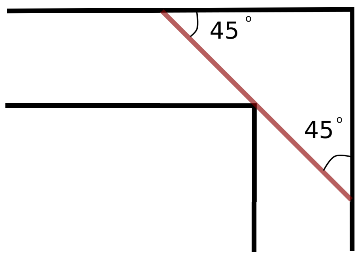
At the point a ladder gets stuck it touches at three points. Each end must be touching a wall and some point between these two ends will be touching the apex of the bend.



Drawing the angles we realise that, in general, the angle the ladder makes to one wall will be less than 45°, the other greater than 45°. Moreover, form Pythagoras' Theorem, the angles must add up to 90°.

But, in the drawing, we have not specified which directions the removers are trying to move the ladder.

Clearly, there is no problem moving the ladder away from the corner in the direction of the end making with the wall angle less than 45°. For example, dragging the ladder along that wall, the wall angle will decrease and the other end will detach from the opposite wall.



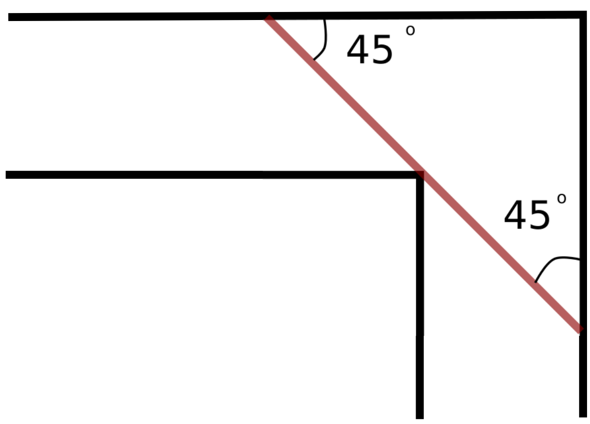
Taking this argument to its logical conclusion, we see that in the symmetric case where both angles are 45°, then the ladder can be moved in either direction. This then gives us the limiting configuration.

Simple application of Pythagoras' Theorem now gives the length of the ladder to be $2\sqrt{2}$ metres.

The Mathematical Removal Company 2

(by Alan Champneys)

You should have found something like the following diagram of how the longest possible ladder can fit around the bend.



This gives the maximum length of ladder that can be removed horizontally to

$$\ell_1 = 2\sqrt{2} = \ 2.82 \text{ metres}$$

More realistically, a professional removal worker would tilt the ladder into the third dimension in order to get it around a corner more easily.

So suppose the corridor has a uniform height of 3 metres. What is the maximum length of ladder that can be manipulated around the corner?

$$\ell_2 = \qquad\qquad \text{metres}$$

Solution

This problem is straightforward once you realise that the corner only occurs in the horizontal plane. In the vertical direction, the corridor is uniformly 3m high.

The best we can do is to tilt whatever fits in the plane so that it touches both the floor and the ceiling.

Then we can use Pythagoras' Theorem to obtain the longest ladder. So,

$$\ell_2^2 = \ell_1^2 + 3^2 = 8 + 9 = 17$$

So

$$\ell_2 = \sqrt{17} = 4.13\,\text{metres}$$

The Mathematical Removal Company 3

(by Alan Champneys)

The Mathematical Removal Company now want to decide on the optimal shape of box that can contain the most stuff to be removed, and still fits along the same corridor with the 90° bend. Assuming that the box is 3 metres tall so that it fills the entire height of the corridor, what should the cross-section of the box look like if:

the cross-section must be a square?;

the cross-section can be any rectangle?
[Hint: you can assume that the box can be rotate around the corner, and use the same idea as for the ladder.]

What do you notice about the cross-sectionaarea of A in each case?

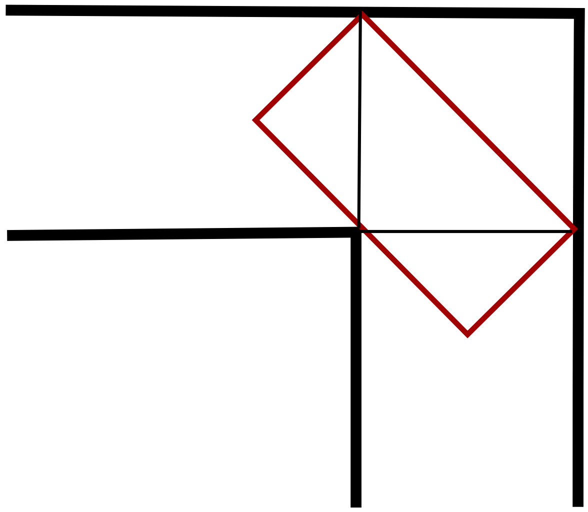
Solution

Clearly for the square box, the only thing that can be done is to slide the box up to the junction and then slide it away in the perpendicular direction, without rotation. Thus the largest square is

$$A_{\text{square}} = 1 \text{ metre } \times 1 \text{ metre } = 1\text{metre}^2.$$

Now, for a rectangle, consider the following diagram.

Appealing to the same symmetry argument as for the ladder, this is the largest rectangle that can be rotated around the corner.



Not that from Pythagoras' Theorem that the side lengths are $\sqrt{2}$ and $\sqrt{2}/2$. Therefore

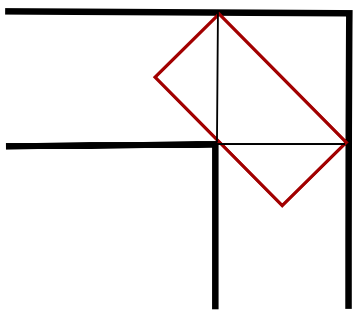
$$A_{rotatedrectangle} = \sqrt{2} \text{ metres } \times \frac{\sqrt{2}}{2} \text{ metres } = 1\text{metres}^2.$$

And so, the two areas are the same!

The Mathematical Removal Company 4

(by Alan Champneys)

You should have found, like the removers did, that the square box which is just shunted around the corner and the rectangular box that is rotated, have the same cross-sectional area of 1 metre^2. *[Note how the triangular areas of the rectangle that lie outside of the square fit perfectly into area above the rectangle to make the square.]*



One bright spark, the newest member of the Mathematical Removal Company's team, suggested they could carry more stuff if the boxes were semi-circular in cross section. Is she right?

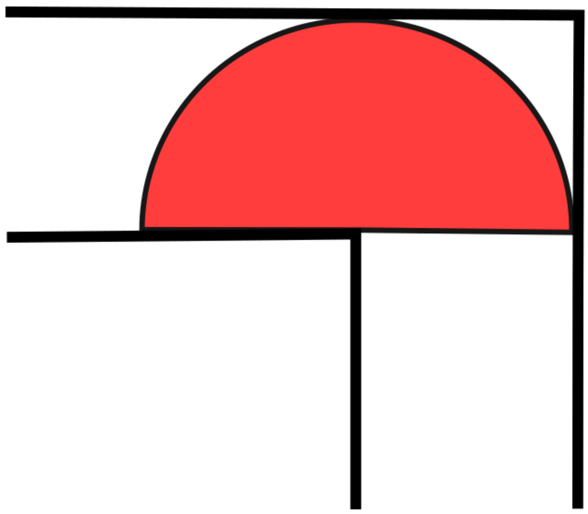
What is the largest semi-circular area that can be rotated around the corner?

Is this the largest possible area, or can you do even better? What is the largest cross-sectional area you can come up with?

[In fact this challenge is known as the Moving Sofa Problem,
see `https://en.wikipedia.org/wiki/Moving_sofa_problem`
and the solution to for the largest cross-sectional area not known]

Solution

Note how a semi-circle with its centre at the apex of the corner can be rotated around the bend in the corridor, provided the radius is less than the width of the corridor.



So the maximum radius is 1 metre. And the area is

$$\frac{1}{2}(\pi \times 1^2) = \frac{\pi}{2} \text{ metres}^2 = 1.57079 \text{ metres}^2.$$

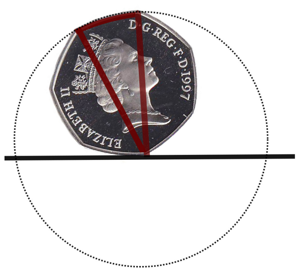
The wikipedia page `https://en.wikipedia.org/wiki/Moving_sofa_problem` Shows a so-called Hammersley sofa, which is shaped like an old fashioned telephone handset.

To get students to think in the direction of finding this shape, suggest they try cutting a small semicircle out of the larger semicircle so that the shape can go around the corner more easily. Then you could ask what more can be added to the two ends of the shape.

Rolling a fifty pence piece 1

(by Alan Champneys)

A fifty pence piece is a seven sided coin. But if you look at it closely, you realise that it is an unusual coin because its sides aren't exactly straight. In fact each side is an arc of a circle whose radius is the distance from that side to the opposite corner.



The fifty pence is designed this way for a specific reason. Suppose such a coin were rolled on a perfectly flat table. Can you draw a sketch of what happens to the highest point of the coin?

What do you notice? Why might this be helpful to the designer of a coin operated machine?

Solution

Let R be the radius of the arc of each of the seven sides.

Notice that as the coin is rolled, it pivots for a while on one of the corners. While this stage is happening, the highest point of the coin is on the arc of a circle that is a fixed distance R from the pivot point.

The pivoting on this corner ends when the coin starts to roll on the curved edge. During this roll, the highest point of the coin becomes the opposite corner. This is a fixed distance R from the side that it is rolling.

Now, this roll ends when the next corner point touches the table. By symmetry, this coincides with when the oppostite corner ceases to become the highest pooint. The coin now starts to pivot on the next corner and the highest point of the coin is on the opposite edge, which again is a distance R aways.

Hence, as is rolls, the top of the coin is always a fixed distance R above the table.

This is useful to the designers of coin operated machines, because no matter what orientation you put the fifty pence piece into the slot, it will always have the same width. Hence you can design the same kind of slot as you would do for a circular coin.

Rolling a fifty pence piece 2

(by Alan Champneys)

You should have found that the fifty pence piece always has a constant height no matter how you roll it! This seems remarkable, given that it is not circular.

Could you design other coins in this way, with edges that are circular arcs with radius the distance from the oppostie corner? Or is a special property of a 7-sided coin?

How about a 5-sided coin? or even a 3-sided coin? Would they also have the same height off the table no matter how you rotate them?

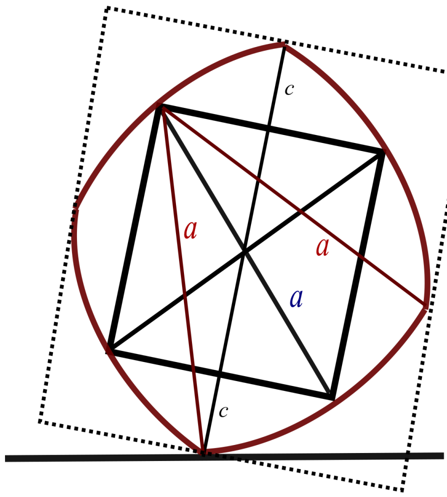
What about an even-sided coin? For example, something with four sides?

Solution

It is straightforward to see by drawing pictures, that the same trick works for any odd number of sides. Even a 3-sided object.

It is important to have an odd number of sides, because we need the corner to be opposite the mid-point of an edge.

Consider the four-sided object constructed in this way, for example.



This doesn't look like its going to have a constant height when we roll it. As we are about to show ...

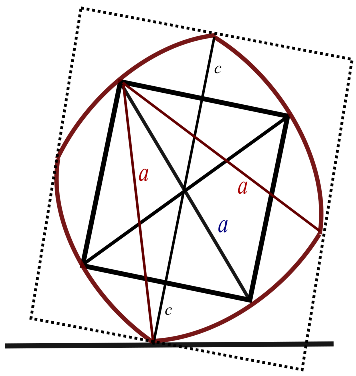
Rolling a fifty pence piece 3

(by Alan Champneys)

You should have found that any odd-sided coin can be constructed in this way so that it has a constant height when rolled.

But things go wrong for an even number of sides. Why?

Consider the four-sided 'square coin' constructed in this way.



Can you show that this does **not** have a constant height as it rolls? *[Hint: from the above diagram, assume that the smaller inner square has side length 1. Then, can you show that the width a is different from the width $1+2c$ as labelled in the diagram?]*

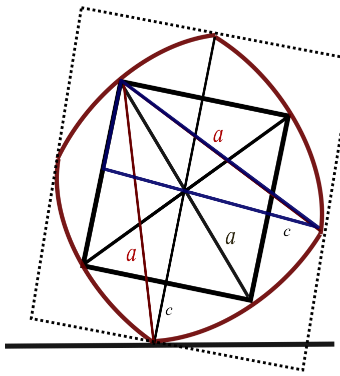
Solution

Consider the square coin as shown below. From the sketch it would appear that $a < 1 + 2c$. Let's try to show this:

If the inner square has side length 1, then from Pythagoras' Theorem we see that

$$a = \sqrt{2} = 1.4142$$

There are a number of ways to calculate the length c. For example,



the blue triangle is right-angled triangle with sides $1/2$, $1 + c$ and a. Hence

$$a^2 = (1 + c)^2 + 0.5^2$$

So

$$c = \sqrt{(a^2 - 0.5^2)} - 1 = 0.3228$$

Hence the long side of length

$$1 + 2c = 1.64575$$

And so $1 + 2c > a$ as required.

Incedentally there is a nice Numberfile video on YouTube that considers the extension to 3D

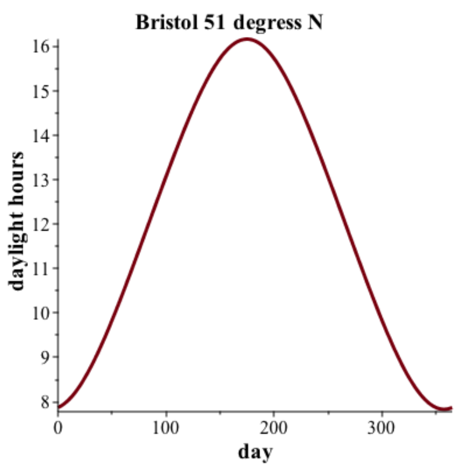
www.youtube.com/watch?v=cUCSSJwO3GU

The length of days 1

(by Alan Champneys)

Have you ever thought about what a graph of the number of hours of daylight throughout the year might look like?

Clearly it depends on lattitude. Here is a graph for Bristol which has an angle of lattitude $\phi = 51^\circ$ (north).



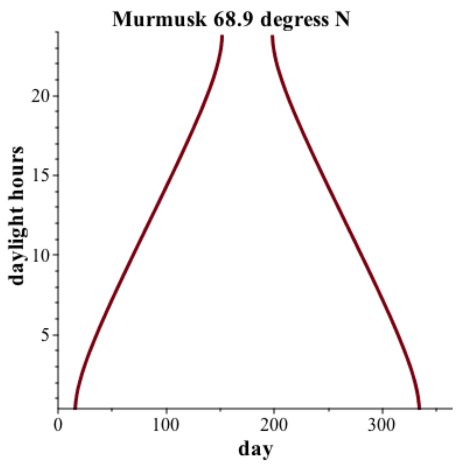
Looks like a sine wave, doesn't it? But is it exactly a sine wave? and how would we prove it? Think about what exactly causes the length of the days to change. Discuss in your group.

For the rest of this exercise we are going to try to look test a formula for the daylength anywhere on the Earth.

Solution

Daylight hours depend on the tilt of the Earth as it goes on its orbit around the sun. In the northern hemisphere Summer the earth is tilted so that the sun is over the northern tropic (of Cancer) which has an angle of latitude $\phi = 23.45°$N. In winter it is over the southern tropic (of Capricorn) at $\phi = -23.45°$S. This causes days to be longer in northern latitudes during Summer. But how much longer?

Clearly a sine wave would be a good first guess as a function for the length of the days. But there is a problem. This function is no use when we get towards the poles. For example, the northern Russian city of Murmusk is inside the arctic circle, and its graph of daylight hours is highly non-sinusoidal.



The length of days 2

(by Alan Champneys)

It turns out that there is a good approximate equation for calculating the length of a day

Sunrise equation

The **sunrise equation** defines an hour angle ω measured in terms of something called the sun's declination angle δ (the apparent angle of the sun to the vertical at noon) and the latitude ϕ.

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

where $-180° < \omega < 0$ corresponds to sunrise, and $0 < 180°$ corresponds to sunset.

The sun's declination angle δ can be defined as 23.45° times a sine wave with amplitude 1 and period of 365 days. That is the angle varies sinusoidally between +23.45o (midsummer) and -23.45° (midwinter) over the course of a year. In the northern hemisphere midsummer is 21st June which, ignoring leap years, is

$$\text{21st June } \equiv d_0 = \qquad\qquad \text{days into the year}$$

But we need to convert days into degrees. Can you therefore find an expression for can the declination angle in terms days d since New Year?

Solution

Adding the days in January, February, March, April and May to 21 days in June gives

$$d_0 = 31 + 28 + 31 + 30 + 31 + 21 = 172.$$

Converting to degrees we get

$$\delta \approx 23.45 \cos\left(\frac{360^\circ}{365}[d - 172]\right)$$

Or

$$\delta \approx 23.45 \sin\left(\frac{360^\circ}{365}[d + 10]\right)$$

The length of days 3

(by Alan Champneys)

You should get something like

$$\delta \approx 23.45 \cos\left(\frac{360^\circ}{365}[d-172]\right) \text{ or } 23.45 \sin\left(\frac{360^\circ}{365}[d+10]\right)$$

Next, given the sunrise equation

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

we need to turn the hour angle ω into a time. The angle is defined as zero at 12 noon and 180° at 12 midnight. Let $\omega > 0$ be the hour angle of sunset. Given that the Earth spins 360° in 24 hours. We can define the sunset and sunrise times in terms of ω as

$$\text{sunrise} \;=\; \qquad\qquad\qquad , \quad \text{sunset} \;=$$

Hence we can now use the sunrise equation to find an experession for the length of a day in terms of the latitude angle ϕ and day number d

$$\text{daylength} = \text{sunset} - \text{sunrise} =$$

Try feeding the numbers for Bristol at different days of the year into this formula to see if you get the right answer.

Solution

We have $360/24 = 15^\circ$ per hour. So assuming ω is measured in degrees,

$$\text{sunrise } = 12 - \omega/15, \quad \text{sunset } = 12 + \omega/15$$

And the day length is

$$\frac{2}{15}\arccos[-\tan(\phi)\tan(\delta)]$$

The length of days 4

(by Alan Champneys)

What happens to the angle ω defined by the sunrise equation, at midsummer when $\phi = 90° - 23.45 = 66.55°$?

What hour does sunrise occur for that location on that day?

Note that $\phi = 66.55°$ precisely defines the arctic circle. What happens to solutions of the sunrise equation at midsummer when $\phi > 66.55°$?

Can you explain this in terms of what you understand happens in summertime inside the arctic circle?

Solution

The smart student may be able to realise that

$$\tan(90 - x) = \cot(x) = \frac{1}{\tan(x)}$$

Hence if $\delta = 22.45$, we have

$$\cos(\omega) = -\frac{\tan(\delta)}{\tan(\delta)} = -1,$$

which gives

$$\omega = 180^\circ \quad \Rightarrow \text{sunrise} = \text{midnight}.$$

So the sun rises, at exactly the time it sets. There are 24 hours of daylight with the sun just dipping below the horizon at precisely midnight.

For $\phi > 66.55$ there are days around midsummer for which

$$\cos(\omega) < -1$$

which has no solution, and hence the sun never sets. Thus, the arctic circle is sometimes called   The land of the midnight sun.

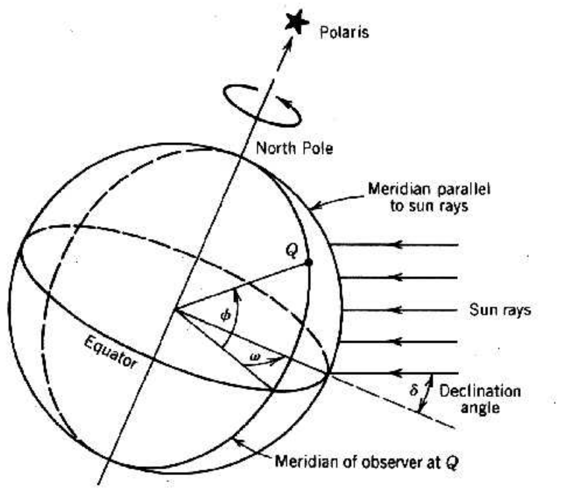
The length of days 5

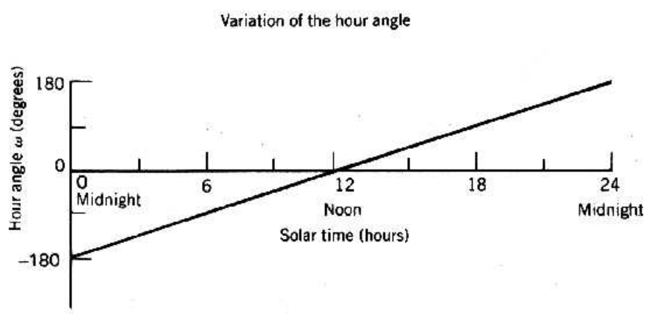
(by Alan Champneys)

Can you derive the sunrise equation?

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

The following diagram may help.





Solution

I ran out of time to try to do this for myself. But all that it should involve is elementary trigonometry. There are lots of resources you can Google, but I haven't found a simple derivation online.

Thus I think this would be good to set as a whole class challenge.

Greek geometry 1

(by Ksenia Shalonova)

Practical lessons from ancient geometers. We are going to learn of the contribution from two ancient Greeks, Pythagoras and Thales. Both visited Egypt to gain understanding.

Right-angled triangles

Pythagoras' theorem connects the side lenghts of right-angled trianges. Can you come up with some right-angled triangles with whole numbers for side lengths? This may give you a clue how they made right angles in Ancient Egypt using only a rope.

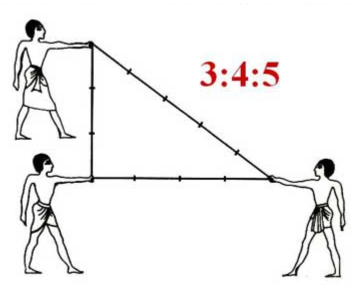
Suppose you want to make a badminton court in your garden.


- How will you make right angles using a rope? How many people would you ideally need to construct it?
- How many poles would you need to draw a long straight line (you can stick poles in the ground)?
- How will you check in the end that your quadrilateral is a rectangle?

Solution

This sesssion can be carried out in the classroom. But it can also be outside, enabling the students to make practical measurements, e.g. in the school playground. In the later tasks they can measure the height of the building or construct a rectangular shape on the terrain. Teachers may need to provide more contextual scaffolds to give them starters for working on similar triangles.

The ancient Egyptians could make right-angled triangles using a rope which was knotted to make 12 equal sections. This is called the 3-4-5 method.



It seems that ancient Egyptians were using the 3-4-5 method before Pythagoras! Pythagoras managed to prove the theorem.

How would you draw a straight line on the terrain? *(Answer: you will need three poles).*

How do you check if your quadrilateral is a rectangle? *(Answer: measure diagonals with the rope - they should be equal).*

Greek Geometry 2

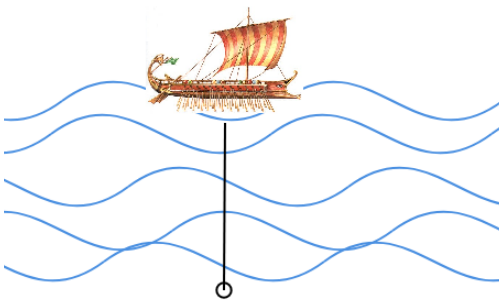
(by Ksenia Shalonova)

Measuring the distance to a ship. Thales used geometry for measuring distances. He measured distances to the ships in navigation, distances to the stars and even worked out the height of the pyramid in Egypt.

Similar triangles

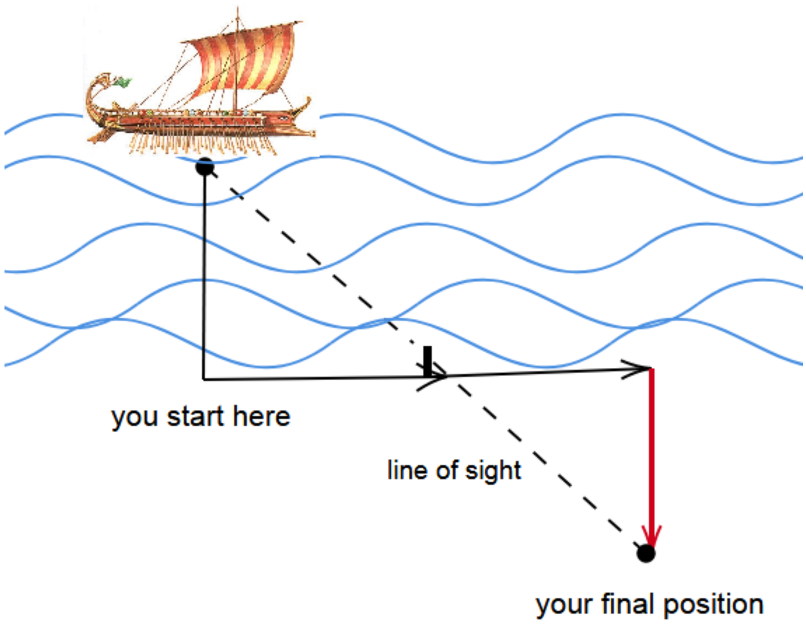
In Ancient Greece they used the properties of the similar triangles to measure distances. In similar triangles, the angles are the same and corresponding sides are proportional. Can you sketch two similar triangles? Can similar triangles be right-angled?

You are standing on the shore and want to calculate distance to the ship. For measuring the distance you are only allowed to use a pole, but you are free to move on the shore.



Solution

Firstly find the point of the shore which is opposite the ship. Then walk along the shore a particular distance (e.g. 30 steps) where you place a stick. When we have done this we continue walking along a shore and cover an equal distance. Afterwards, we walk (perpendicularly) inland up to the point where we can line up the ship and our stick in our vision. Then (due to the shaping of two equal triangles) our distance from the shore equals the desired distance of the ship from the shore.



Greek Geometry 3

(by Ksenia Shalonova)

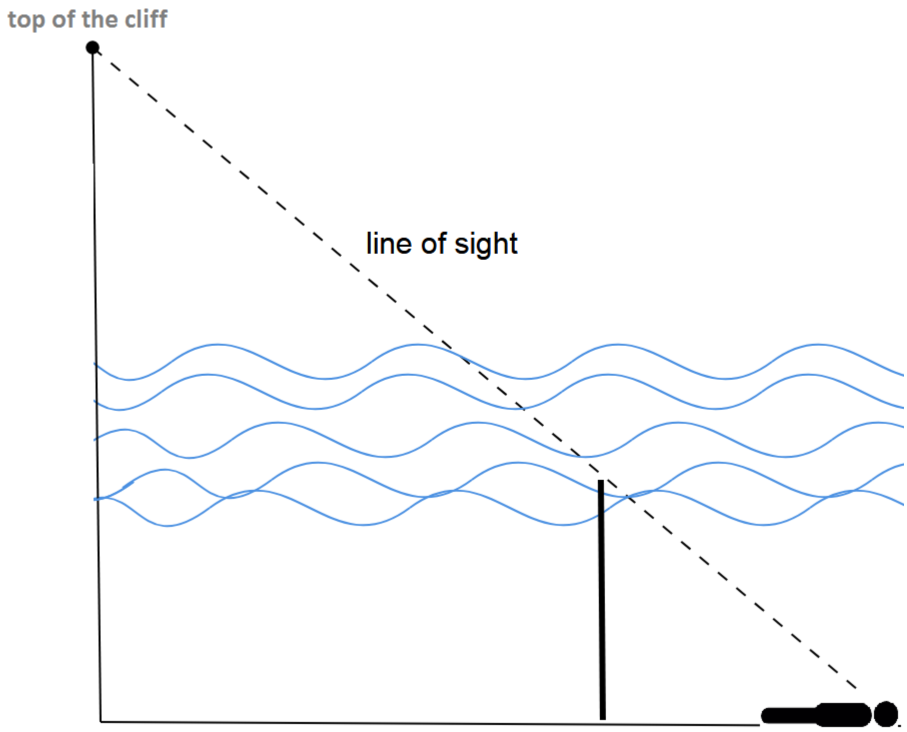
Measuring heights There are a number of methods for measuring heights,

for example, you can use the length of the shadow (on a sunny day). You can also use a mirror or a hand-drawn triangle.

In Jules Verne's "The Mysterious Island" Captain Harding wanted to find the height of a cliff and for this he used a tall poll. There is only one disadvantage of using this method - you have to lie down on the ground! Can you figure out how he did it by using properties of similar triangles and the line of sight?

Solution

We have two right-angled similar triangles with two sides in one triangle whose lengths we know (distance from a person to the pole and pole length) and one known side (the distance from a person to the cliff) in another triangle. By solving the proportion we can find the height of the cliff.



Greek Geometry 4

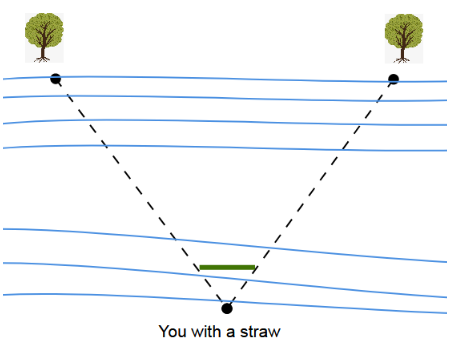
(by Ksenia Shalonova)

Measuring river width with a straw

Thales' Theorems

There are several theorems that are attributed to Thales. For example, the circle is bisected by its diameter or the angle in a semi-circle is right angle. Knowledge of the angles in a circle can help you to solve the next problem.

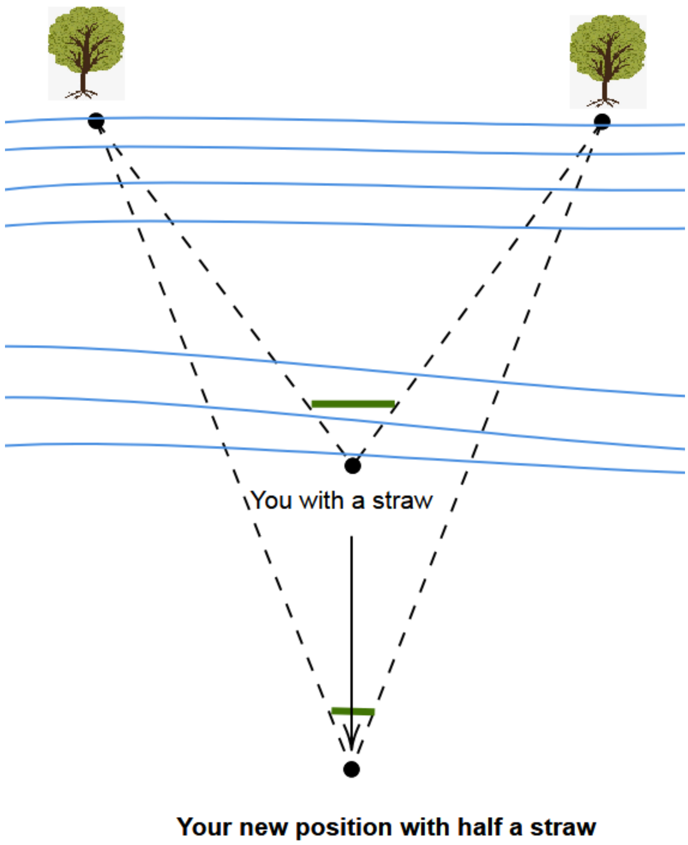
You are standing on a river shore and want to measure an approximate width of a river. You can use a straw (or a small stick). Try to solve the problem first without looking at the hint!



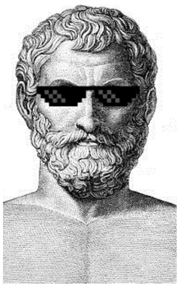
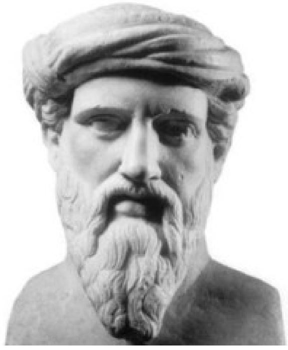
Hint. Notice any two objects on the opposite side of the river (e.g., flowers or trees). Hold a stick horizontally with an outstreched hand where the first object is directly behind the left end of the straw and another object is directly behind the right end of the straw. Reduce the size of the straw - you have to decide by how much. Start moving back until the position of one object is directly behind the left end of the straw and ...

Solution

Notice any two objects on the opposite side of the river (e.g., flowers or trees). Hold a stick horizontally with an outstreched hand where one object is directly behind the left end of the straw and another object is directly behind the right end of the straw. Now you have to use half of the straw. Start moving back until the position of the first object is directly behind the left end of the reduced straw and the position of the second object is directly behind the right end of the reduced straw. The distance you traveled back is the approximate width of the river.



Further information

These two guys from Ancient Greece - Thales of Miletus (left) and Pythagoras (right) - made Mathematics a science. They did Maths not only because it was practical, but because Maths was fun.
Pythagoras created his own school - a cult that worshiped numbers. There is still a lot of mystery around it. Thales became famous for shocking everybody by accurately predicting a solar eclipse in Ancient Greece. As both of them enjoyed the process of understanding the world, they called themselves `philosophers.`
The word geometry has its roots in the Greek language and means **earth measuring**. Both Pythagoras and Thales visited Egypt to learn more about Mathematics.

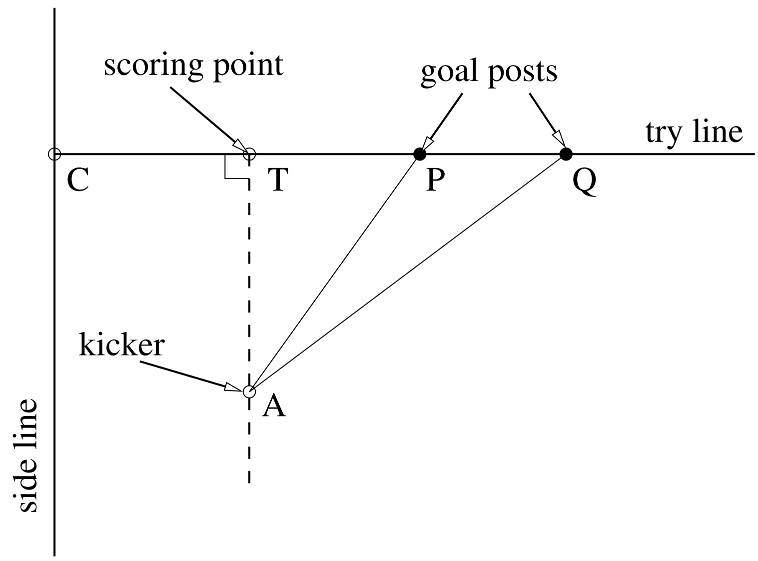
An extension to this topic can be introducing simple mechanisms such as a compass or a basic theodolite (or a protractor) for measuring angles on terrain. This will allow to solve more complicated tasks such as creating topographic maps.

In fact, the basic properties in geometry developed in Ancient Greece are widely used in a large variety of modern engineering applications: GPS positioning systems, space mission design, design of tall skyscrapers, precision manufacture of tiny parts, motion of robotic manipulation, and many more.

Rugby kicking 1

(by Alan Champneys)

In rugby, when a try is scored, a conversion kick has to be taken form a perpendicular line that intersects the try line where the try was scored (the dashed line in the diagram). The kicker is trying to get the ball through the posts. The kicker is free to choose the point A on the dashed line. But where should A be chosen to maximise the angle <PAQ between the goalposts?



What happens to the angle <PAQ if A is chosen just next to T? What happens to the angle if A is chosen to be at the far end of the pitch?

We are going to find a formula for chosing the optimal point A.

Solution

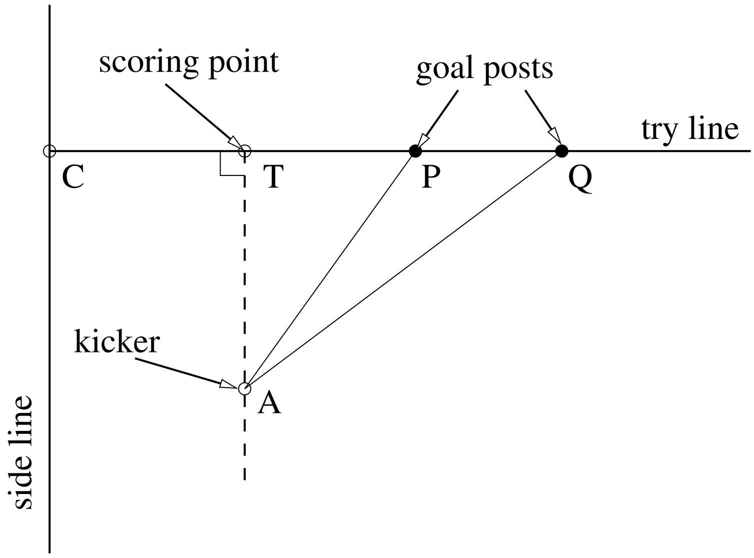
We suggest you find a video of a rugby kicker to introduce this problem.

You should also stress that (for the time being) we are doing only the 2D problem and are ignoring the height of the kick.

Rugby kicking 2

(by Alan Champneys)

Let's start with some numbers. For a typical rugby pitch, the width is 70 m and the goal width PQ is 5.6 m. Given these dimensions (assuming the goal is in the middle of the try line) calculate the distances CP.



As an example, let's suppose the try is scored at 10 m from the side line. That is, CT= 10 m. Calculate the distances TP and TQ.

Suppose the kicker chooses to kick from 10 m away from the try line. That is, TA= 10 m. Use trigonometry to calculate

$\tan(< \mathrm{TAP}) =$ and $\tan(< \mathrm{TAQ}) =$

Hence, calculate the angle <PAQ

for TA= 10 m: $< \mathrm{PAQ} =$

Solution

We have that

$$CP = \frac{\text{pitch width}}{2} - \frac{\text{post width}}{2} = 35 - 2.8 \ = 32.2 \text{ m}$$
$$CQ = \frac{\text{pitch width}}{2} + \frac{\text{post width}}{2} = 35 - 2.8 \ = 37.8 \text{ m}$$

Hence

$$\tan(< \text{TAP}) = \frac{32.2 - 10}{10} = 2.22 \ \text{ and } \ \tan(< \text{TAQ}) = \frac{37.8 - 10}{10} = 2.78$$

and so

$$< \text{PAQ} = \arctan(2.78) - \arctan(2.22) = 70.216 - 65.751 = 4.465^\circ$$

Rugby kicking 3

(by Alan Champneys)

Now suppose the kicker instead stands 30 m away from T (with again the try being scored at TC$=10$ m).

Repeating the calculation for TA$=30$ m:

$\tan(< \text{TAP}) =$ and $\tan(< \text{TAQ}) =$

Hence

for TA$=10$ m: $< \text{PAQ} =$

Which is the better position to kick from; 10 m or 30 m?

What about if they stood right at the far end of the pitch, TA$=100$ m?

Can you see that there must be an intermediate distance (between 0 and 100 m) such that the angle <PAQ is maximised?

Solution

Now for TA $= 30$ m, we have

$$\tan(< \text{TAP}) = \frac{22.2}{30} = 0.740 \quad \text{and} \quad \tan(< \text{TAQ}) = \frac{27.8}{30} = 0.927$$

and so

$$< \text{PAQ} = \arctan(0.927) - \arctan(0.74) = 42.820 - 36.501 = 6.319^{\circ}$$

This is better (a wider angle) than the TA $= 10$ m kick.

If we start on the try line we find (trivially) $< \text{PAQ} = 0$, so you might think that the angle keeps getting bigger as we increase TA.

But, if we kick from 100 meters (the far end of the pitch), we find

$$< \text{PAQ} = \arctan(0.278) - \arctan(0.222) = 15.536 - 12.517 = 3.019^{\circ}$$

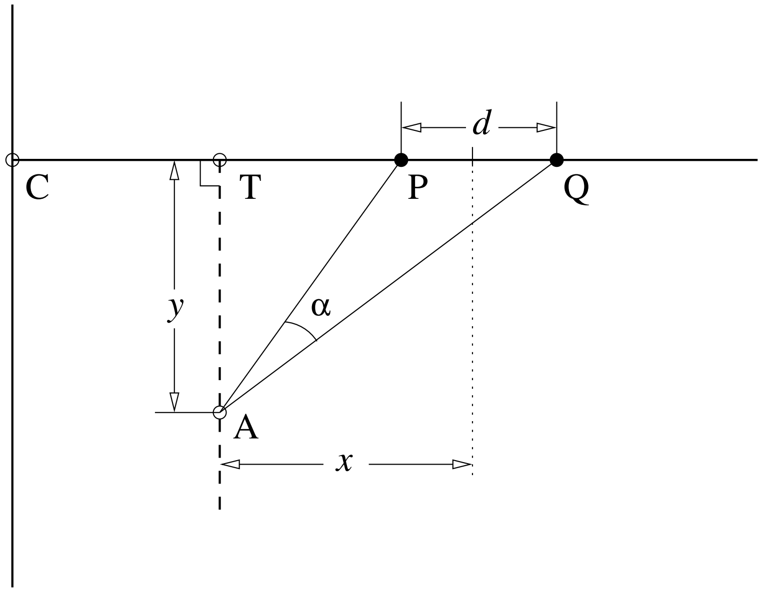
which is worse than TA$= 30$ m.

Hence there must be an optimum distance somewhere between TA$= 10$ m and 100 m

Rugby kicking 4

(by Alan Champneys)

Now we are going to try to generalise using algebra.



Suppose that the post width d and the distance x of T from the centre line of the pitch are fixed. We want to find the optimum value of y, which maximises α.

Previously we calculated the angle PAQ $= \alpha$ for some given values of d, x and y. Repeat the calculation to find a general expression for

$$\alpha = \arctan(\qquad\qquad\qquad\qquad) - \arctan(\qquad\qquad\qquad\qquad).$$

Solution

Proceeding as before ,we find

$$\tan(< \text{TAP}) = \frac{x - (d/2)}{y} \quad \text{and} \quad \tan(< \text{TAQ}) = \frac{x + (d/2)}{y}$$

and so

$$\alpha = \arctan\left(\frac{x + (d/2)}{y}\right) - \arctan\left(\frac{x - (d/2)}{y}\right)$$

Rugby kicking 5

(by Alan Champneys)

The function from the previous part can be written

$$\alpha = \arctan\left(\frac{2x+d}{2y}\right) - \arctan\left(\frac{2x-d}{2y}\right)$$

Taking the realistic value for the goal width $d = 5.6\,\mathrm{m}$, and taking $x = 25$ m (the same position of T used in parts 2 and 3) find a table of values of α for $y = 0\,\mathrm{m}$, $5\,\mathrm{m}$, $10\,\mathrm{m}$, $15\,\mathrm{m}$, etc. up to $50\,\mathrm{m}$:

y (m)	0	5	10	15	20	...
α (°)						

Plot these values on a graph of α versus y. Estimate the value of y for which α is maximised.

An A-level extension

Use calculus to find the maximum of α as a function of y

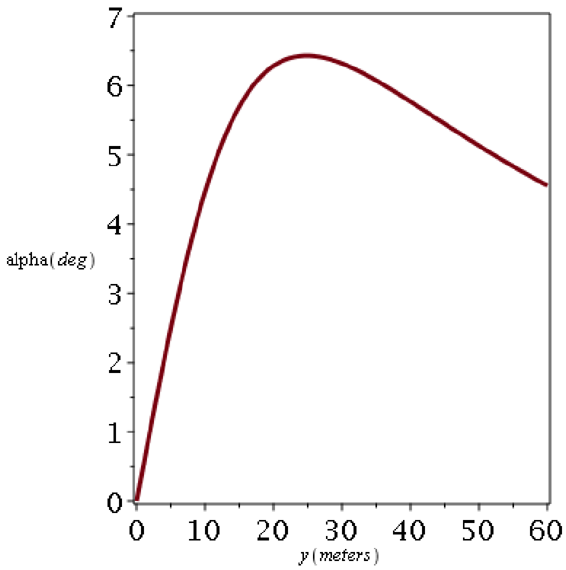
But we are going to find another way ...

Solution

Using $d = 5.6$ and $x = 25$ we find

y (m)	0	5	10	15	20	25	30	35	40	45	50
α (°)	0	2.52	4.41	5.70	6.28	6.43	6.31	6.07	5.77	5.43	

See the graph:



For A-level students we can differentiate $\alpha(y)$ to find

$$\begin{aligned}\frac{d\alpha}{dy} &= \frac{x-p}{(x-p)^2+y^2} - \frac{x+p}{(p+x)^2+y^2} \\ &= -\frac{2p(x^2-y^2-p^2}{[(p+x)^2+y^2][(x-p)^2+y^2]}\end{aligned}$$

where $p = d/2$ is half of the goal width.

To find an extremum we set this derivative to zero. Note the denominator is positive definite, so this is equivalent to setting the numerator to zero. Thus to get the extremal value we set

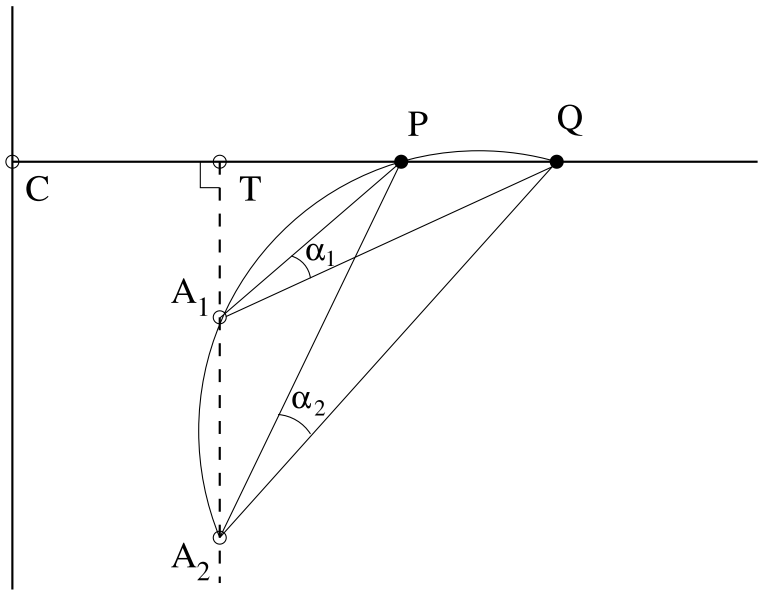
$$x^2 - y^2 - p^2 = 0 \qquad \text{hence} \quad y = \sqrt{x^2 - p^2}$$

Is it obvious that this extremum has to be a maximum?

Rugby kicking 6

(by Alan Champneys)

We are now going to use geometry. Consider the two possible kicking positions A_1 and A_2, depicted in the diagram below.



Note that A_1 and A_2 lie on the same circle through the posts P and Q. Hence what can you say about the angles α_1 and α_2? Why?

What happens to the angles α_1 and α_2 if you make the circle larger or smaller?

Can you draw the circle on which the optimum kicking point on the dashed line must lie?

Solution

One of the Circle Theorems states

"All angles inscribed in a circle and subtended by the same chord are equal."

Hence $\alpha_1 = \alpha_2$

Making the circle larger clearly makes α smaller. *Suggest students see this by drawing different circles using a compass and measuring the required angles.*

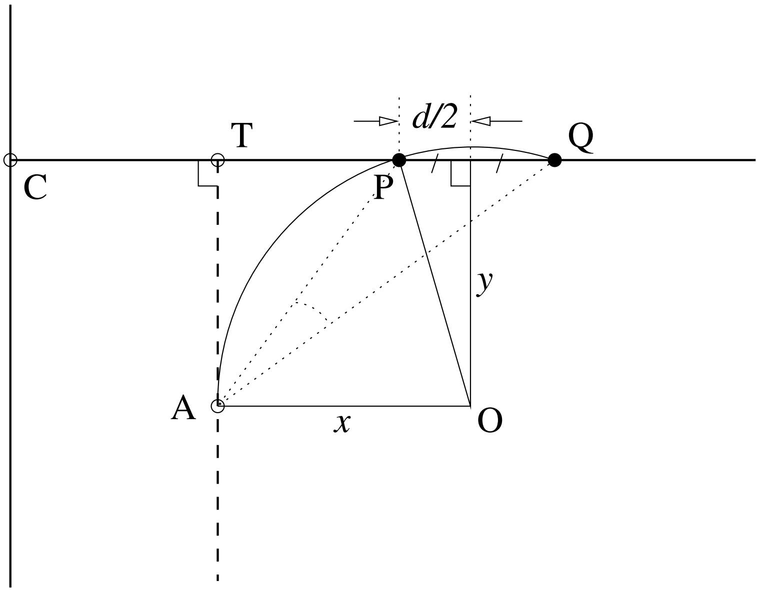
Hence the optimal distance y which makes α the largest possible, is to make the circle as small as possible.

That is, we **choose the circle that just touches the line** on which the kicker must kick.

Rugby kicking 7

(by Alan Champneys)

Now we are going to use this touching circle to calculate the optimal distance y. Consider this diagram



What is the distance OP?

Hence use Pythagoras' Theorem to find an expression for y.

$$y =$$

Congratulations you have found the optimal kicking distance!

What does curve of y against x look like? Can you sketch it?
What does it mean?

Solution

OP is a radius of the circle. Hence OP$=x$.

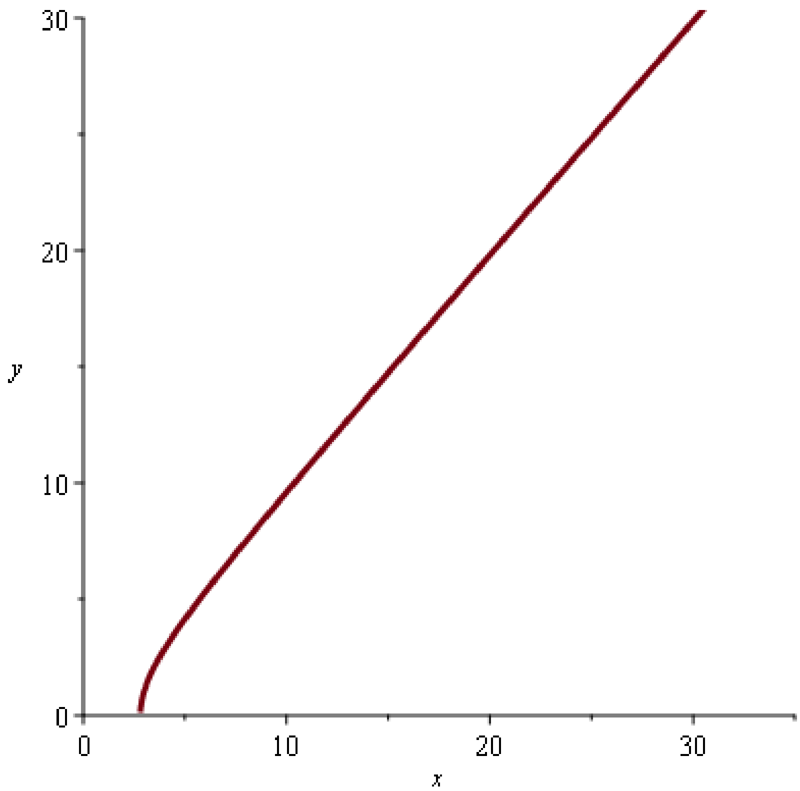
Consider now the triangle OPM where M is the midpoint between the posts. Applying Pythagoras' Theorem we get

$$x^2 = y^2 + (d/2)^2$$

Or,

$$y = \sqrt{x^2 - (d/2)^2}$$

Note this is a rectangular hyperbola. You might like to suggest the students try plotting it.



Further information

There are many possible more realistic extensions to this problem ….

What if we had to take the height of the posts into consideration? The kick must clear a bar that is $3\,m$ off the ground. Does this affect the optimal y? How would you optimise the height at which the kicker should kick?

What if $-d/2 < x < d/2$; that is, if the kick is in front of the posts? What would be optimal distance y then?

What if we take wind into account? What if we took the statistics of a particular kicker's past performance?

For more detailed information. Here are two reports that can be found online that provide further mathematical insight:

`https://epubs.siam.org/doi/pdf/10.1137/130913225`
`https://www.qedcat.com/articles/rugby.pdf`

In fact, these days, the science of sport is a lucrative business.

The payday loan 1

(by Alan Champneys)

Loans-R-Us, a so-called pay-day loan shop, is offering three different interest rates

(a) 3500% per year, (b) 35% per month, (c) 1% per day,

Interest is charged at the beginning of each period. So that if I borrow £1 for just one hour, under interest rate (a) I would pay pay £36, under (b) I would pay .35, and under (c) I pay £1.01.

Each time interest is charged, it is calculated on the whole amount owed, including any unpaid interest.

Calculate the total amount I would owe under each interest rate, if I were to borrow £1 for a whole year, under each interest rate scheme, assuming that I don't pay anything back until the end of the year.

Which interest rate should I choose?

Solution

This question is designed so that some students will jump to the wrong to the wrong conclusion, as follows:

Paying back 1% of £1 each day will result in a penny a day or £3.65 over a year. Paying back 35 pence per month will result in £4.20 over a year, whereas the high annual rate will result in £35 over a year. So clearly the daily rate is best, right?

But, of course, compound interest does not work like that. Using interest rate (a) over one year, clearly I pay back a total of

$$\text{(a)}: \;= £36 \quad = £35\,\text{interest} \;+\; £1$$

Instead, under (b) I owe $£(1+0.35)$ in the first month and $£[(1+0.35)+0.35(1+0.35)]$ in the second month. So this is a total of $£(1+0.35)(1+0.35) = 1.82$ after two months, not $1 + 0.35 + 0.35 = 1.70$. So after two months I owe $£(1 + 0.35)^2$. Thus after 12 months I owe

$$\text{(b)}: \; £(1 + 0.35)^{12} \quad = £36.64.$$

Proceeding similarly for (c) after 365 days I owe

$$\text{(c)}: \; £(1 + 0.01)^{365} \quad = £37.78.$$

So, actually, the annual interest rate gives me the least to pay back.

The payday loan 2

(by Alan Champneys)

You should have found that the amounts you pay back under each rate are:

$$\text{(a) 3500\%per year} \text{Total} = £36, \quad (1)$$
$$\text{(b) 35\%per month} \text{Total} = £(1+0.35)^{12} = £36.64, \quad (2)$$
$$\text{(c) 1\%per day} \text{Total} = £(1+0.01)^{365} = £37.78. \quad (3)$$

So you should choose option (c) even though it looks like the worst deal.

This is because of the exponential growth of compound interest, and shows how so-called "pay day loan" companies prey on the poor and vulnerable to make huge profits. It is also why, by law, all loans must specify an equivalent *Anuual Percentage Rate* (APR). This calculation shows that the APR of a loan that is advertised as ‘1% per day’ is actually 3678 % , which does not sound so appealing

But it gets worse. Suppose I choose rate (c) and borrow £1 for 5 years. How much would I owe at the end of 5 years?

What if I were to only pay back after 10 years? where could I find such money from?

Solution

If I kept things going for 5 years, I would then owe

$$£(1+0.01)^{5\times365} \quad = £77.003\times10^{7}\approx £770\,\text{million}$$

Which is quite a lot of money based on a £1 loan and a "1%" interest rate! Actually, because of leap years, its actually slightly more than this either

$$(1+0.01)^{(4\times365)+366} = 777\,\text{million} \;\text{ or }\; (1+0.01)^{(3\times365)+(2\times366)} = 785\,\text{million}\;.$$

And after 10 years (assuming two leap years):

$$£(1+0.01)^{3652}\approx 5.929\times10^{15} = 6048\,\text{billion}, \quad \text{or } £6.084\,\text{quadrillion}$$

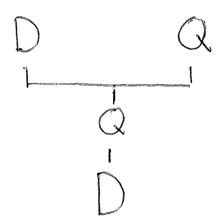
But this could never be repaid, because it is about 100 times more than the total amount of money estimated to exist in the global economy (about 100 trillion US dollars) !!

This calculation further shows the problem with "payday loan" companies.

A secret of bees 1

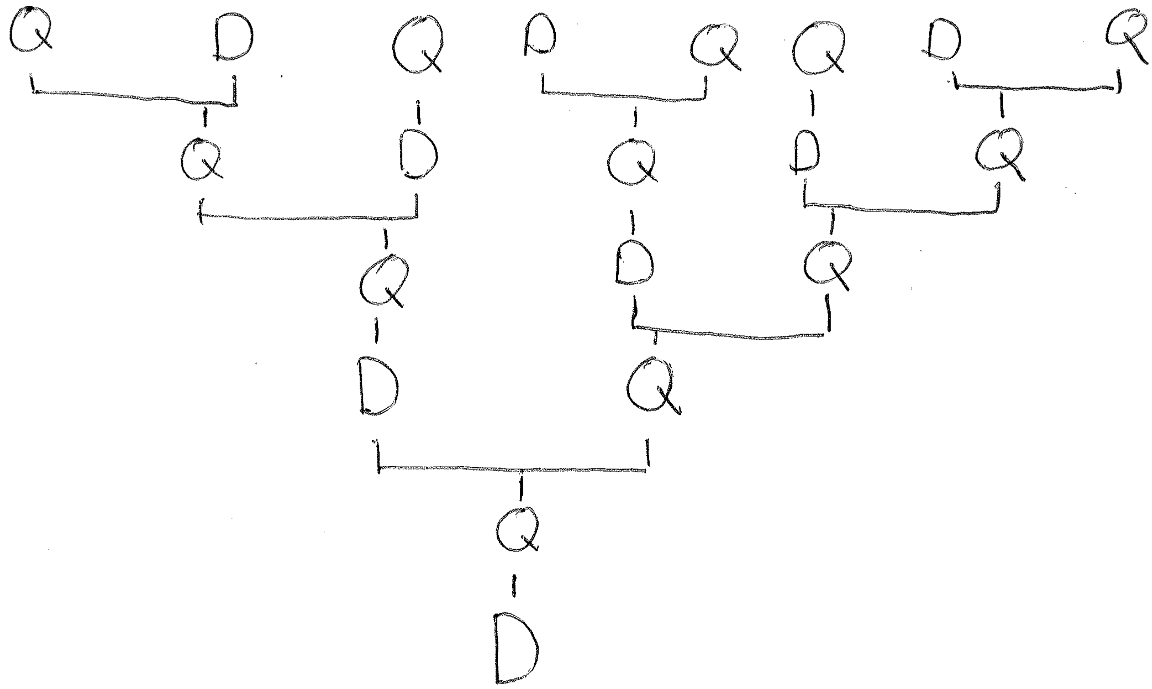
(by Thilo Gross)

Bees have interesting family trees. A male bee, a so-called drone (D) only has one parent, who is a queen (Q). A queen has two parents, a queen and a drone. So a drone has only one parent, and only 2 grandparents. Got it? Continue the family tree that I have started to draw below for at least 3 more generations.



Solution

The finished family tree should look like this.



Note that the sequence of the number of bees in each layer should be the Fibonacci sequence 1,1,2,3,5,8,13,21,...

A secret of bees 2

(by Thilo Gross)

The ancestral trees of bees hide a secret. The first step to discover it is to write down the number of queens, drones, and the total number of bees for each generation.

Use the family tree that you have drawn to fill in the next 3 lines of this table

Gen. n	Queens Q_n	Drones D_n	Total T_n
0	0	1	1
1	1	0	1
2	1	1	2
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Do you see a pattern? Can you use it to fill in the remaining lines?

Solution

The pattern that can be discovered here is that each column of the table follows a Fibonacci sequence. Even if students not know it by name the underlying pattern is easy to spot.

The finished table should look like this

Gen. n	Queens Q_n	Drones D_n	Total T_n
0	0	1	1
1	1	0	1
2	1	1	2
3	2	1	3
4	3	2	5
5	5	3	8
6	8	5	13
7	13	8	21
8	21	13	34
9	34	21	55
10	55	34	89
11	89	55	144
12	144	89	233

A secret of bees 3

(by Thilo Gross)

Let's write some equations! Suppose in generation n the number of drones is D_n, and the number of queens is Q_n.

We know that every bee has a queen as a parent, and queens have an additional drone parent.

Let's compute the numbers in the generation before n that's the generation $n+1$. According to the above, the number of queens and drones in generation $n+1$ is

$$\begin{aligned} Q_{n+1} &= \\ D_{n+1} &= \end{aligned}$$

Fill in the right-hand-side of these equations, using only D_n and Q_n and whatever arithmetical symbols you need.

Solution

One queen for every bee in the previous generation means

$$Q_{n+1} = D_n + Q_n$$

and one drone for every queen in the previous generation means

$$D_{n+1} = Q_n$$

A secret of bees 4

(by Thilo Gross)

When you filled the ancestry table you may have discovered the rule

$$Q_{n+1} = Q_n + Q_{n-1}$$

You can check that it holds in your table, but we really want to prove it. On the previous sheet we discovered

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

can you use these to prove the rule for queens above?

Can you prove a similar rule for drones and the total number of bees?

Don't forget that you can shift the indices, and $T_n = Q_n + B_n$.

Solution

If we shift the second of observation from the previous sheet down we get

$$D_n = Q_{n-1}$$

So the first observation can be rewritten as

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ &= Q_n + Q_{n-1}. \end{aligned}$$

Similary,

$$\begin{aligned} D_{n+1} &= Q_n \\ &= Q_{n-1} + D_{n-1} \\ &= D_n + D_{n-1}, \end{aligned}$$

using the second observation, first observation down-shifted, second observation down-shifted, in this order.

Finally,

$$\begin{aligned} T_{n+1} &= Q_{n+1} + D_{n+1} \\ &= Q_n + Q_{n-1} + D_n + D_{n-1} \\ &= (Q_n + D_n) + (Q_{n-1} + D_{n-1}) \\ &= T_n + T_{n-1} \end{aligned}$$

A secret of bees 5

(by Thilo Gross)

Using the very convenient formula

$$T_{n+1} = T_n + T_{n-1}$$

we can compute the number of bees for some more generations while we are on it let us also compute the factor by which the number of ancestors increases in every generation. For humans that would obviously be 2 but for bees the ratio is more intriguing (find about 6 digits after the decimal point)

Gen. n	Bees T_n	Ratio T_n/T_{n-1}
$\vdots$	$\vdots$	$\vdots$
10	89	$\vdots$
11	144	1.617977
12	233	1.618056
13		
14		
15		
16		
17		
18		

Solution

The finished table is

Gen. n	Bees T_n	Ratio T_n/T_{n-1}
$\vdots$	$\vdots$	$\vdots$
10	89	$\vdots$
11	144	1.618055
12	233	1.618026
13	377	1.618026
14	610	1.618037
15	987	1.618033
16	1597	1.618034
17	2584	1.618034
18	4181	1.618034

A secret of bees 6

(by Thilo Gross)

We are getting closer to what the bees are hiding. On the last sheet we discovered the ratio by which the number of bees increases for sufficiently large n. Let's call this ratio f and also compute its inverse

$$\begin{aligned} f &= 1.618034\ldots \\ 1/f &= \end{aligned}$$

Notice anything odd? (Perhaps you would like more digits:
$f = 1.6180339887498948482045868$ what's $1/f$ now?)

If you noted some curious property of f, let's write it as an equation

$$1/f =$$

this time we want a mathematical expression on the right hand side, not digits.

Can you use the equation to compute the exact (!) value of f?

Solution

We notice

$$\begin{aligned} f &= 1.618034\ldots \\ 1/f &= 0.618034\ldots \end{aligned}$$

or with more digits

$$\begin{aligned} f &= 1.6180339887498948482045868\ldots \\ 1/f &= 0.6180339887498948482045868\ldots \end{aligned}$$

This is of course the golden ratio. We observe that all digits of f and $1/f$ are identical except the first. So we can write

$$1/f = f - 1$$

We can now compute f exactly

$$\begin{aligned} 1/f &= f - 1 \\ 1 &= f^2 - f \\ 1 &= (f - 1/2)^2 - 1/4 \\ 5/4 &= (f - 1/2)^2 f = 1/2 \pm \sqrt{5/4} \end{aligned}$$

The larger solution is actually f and the smaller one $-1/f$, so

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}$$

A secret of bees 7

(by Thilo Gross)

Congratulations, you have made it through this set of sheets. We have discovered the secret of bees. The number of ancestors of a bee follows the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

this is the famous Fibonacci sequence. In the long run the number of bees increases by a factor

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}.$$

in each generation. This is the famous golden ratio. A magical number that that appears all over mathematics. It has other surprising properties, for instance, by some measure, it is the most irrational of all numbers.

Here is a little bonus. On the previous sheet you may have noticed that $1/f = 1 - f$ has a second solution which is actually $-1/f$, this is the golden ratio's little brother g, note

$$\begin{aligned} f &= 1/2 + \sqrt{5/4} \\ g &= 1/2 - \sqrt{5/4} \end{aligned}$$

These two numbers work great as a team for instance they can give you the number of bees in generation n

$$T_n = \frac{f^n + g^n}{\sqrt{5}}$$

you can check that this works exactly, but why it works is a different story.

Further information

There is not really any open question, but it is good to encourage the students to actually check the formula. Seeing the integer Fibonacci numbers emerge from a sum of square roots is impressive.

The way to derive the formula is actually to write the the system

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

in matrix form and solve it by an eigendecomposition. Alternatively, the recursion $T_{n+1} = T_n + T_{n-1}$ can be solved using generating functions.

The golden ratio is the most irrational number in the sense that its continued fractions expansion is

$$f = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

Because all the coefficients are 1 this number is maximally far from any rational number.

Also consider mentioning some other places where the golden ratio appears. Plenty of material on this can be found on the internet.

When discussing this exercise you may want to emphasize how progress was made by exploration. In the first 3 sheets we drew the tree, and filled the table to try and spot a pattern. Finding this pattern beforehand meant that we already knew the result we were trying to derive on sheet 4. Likewise we were able to find the exact value of the golden mean by spotting a pattern on sheet 6. This illustrates that very pedestrian exploration (“What happens when I compute the inverse?”, “Lets write out a the numbers for a few simple cases”) often puts us on the right track to the solution.

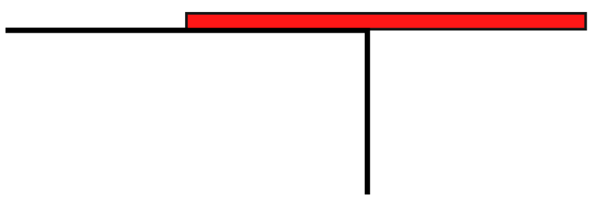
The one-sided arch 1

(by Alan Champneys)

Your task is to construct the longest possible overhanging arch to reach out over the sea, from a rigid clifftop, using only a very large collection of identical planks made of uniform material and length 2 metres. No nails, no glue, just planks, balanced one on top of the other.

We shall build up a solution bit by bit.

Using just one plank you can reach out precisely 1 metre with the plank teeting on the very edge of the cliff.



Can you think of a way of making a longer overhanging structure using two planks? What about three planks? or four? What is the longest overhang you can make?

Start drawing some pictures before doing any maths.

Solution

See the subsequent sheets for the solution that we have in mind. Of course, all kinds of other disigns with counterbalancing weights are also possible. These should be encouraged.

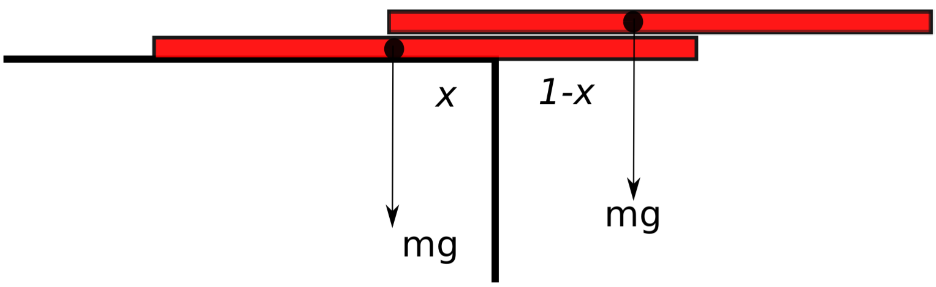
But the purpose of this exercise is to simply put one plank on top of any one plank.

The one-sided arch 2

(by Alan Champneys)

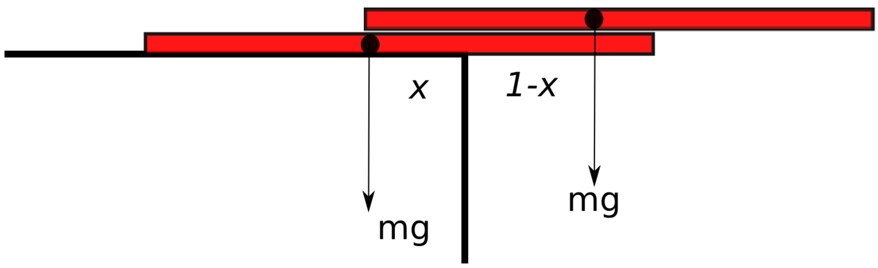
We are going to constrain things by saying that you can only place one plank on top of any other one and that all planks must point in the same direction.

Under this constraint, how far out can you place two planks, one on top of the other so that the top plank is teeting on the edge of bottom plank, and the bottom one is teeting on the edge of the cliff? *[The following diagram might help, where m is the mass of each plank]*



Solution

This can be solved either by taking moments or by trying to impose that the centre of mass should be at precisely the position of the pivot.



In either case, we find that a balance can only be obtained by setting that the mass times length on the left must equal to mass times length to the right of the pivot point. So, referring to the diagram.

$$mgx = mg(1-x), \qquad \Rightarrow 2x = 1 \quad \Rightarrow x = 1/2$$

So, the total overhang is

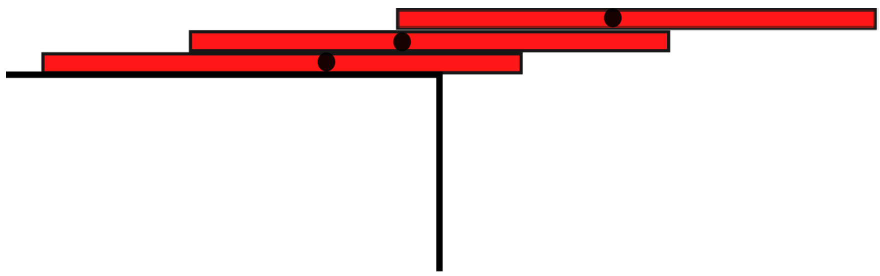
$$O = \text{overhang}_{\text{topplank}} + \text{overhang}_{\text{2ndplank}} = 1 + 1/2 = 3/2 \,\text{metres}$$

The one-sided arch 3

(by Alan Champneys)

You should have found that if $x = 0.5$ metres then the two planks just teeter on the cliff edge, with a total overhang of $1 + x = 1.5$ metres.

Now we are going to proceed using the same approach, essentially taking the solution we already have for two planks, which we know is just teeters on the edge of the cliff, and add one more plank underneath, with the upper two just teetering on it. Take a look at the diagram

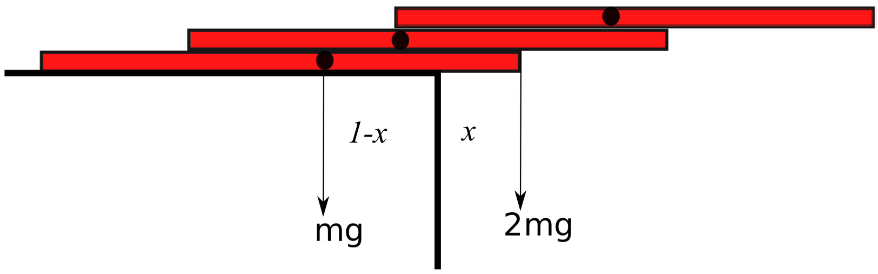


in which the centres of mass of each plank is marked by a large black blob.

Can you calculate the maximum overhang of the lower plank, so that the three planks just teeter on the edge of the cliff?

Solution

There are several ways to solve this, but the most efficient is to notice that we can treat the upper two planks as a single body with mass $2m$. Then the problem is just like the two-plank case, but with the upper plank having twice the mass.



Letting x be the additional overhang, balancing forces we get

$$2mgx = mg(1-x) \qquad \Rightarrow 3x = 1 \quad \Rightarrow x = 1/3$$

So

$$\text{Total overhang } O_3 = O_2 + 1/3 = 1 + 1/2 + 1/3$$

By now the smarter students may have begun to see where this is going ...

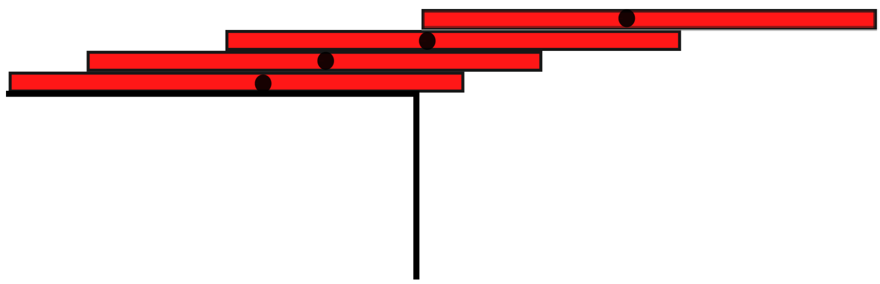
The one-sided arch 4

(by Alan Champneys)

You should have found the answer that the new piece of overhang is 1/3 metre, so that

$$\text{Total overhang} = 1 + 1/2 + 1/3 = 1.83333 \text{ metres}$$

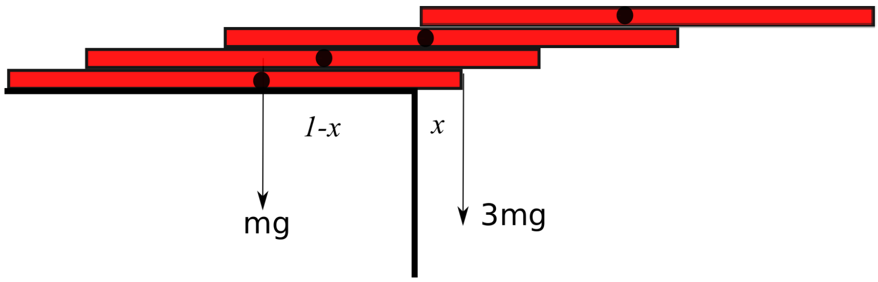
Now lets repeat this process by slotting a fourth plank underneath. Can you guess what the total overhang is now? How do you prove this?



Can you generalise this result to an arbitrary number of planks, N?

Solution

The case for four planks is similar,

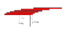


so the additional overhang x_4 is given by

$$3mgx_4 = mg(1 - x_4) \qquad \Rightarrow 4x_4 = 1 \quad \Rightarrow x_4 = 1/4$$

and

$$O_4 = O_3 + x_4 = 1 + 1/2 + 1/3 + 1/4.$$



Generalising to $n + 1$ planks, we get

$$x_{n+1} = 1/(n + 1), \qquad O_{n+1} = O_n + 1/(n + 1)$$

So, by induction

$$O_N = \sum_{n=1}^{N} \frac{1}{n}$$

The one-side arch 5

(by Alan Champneys)

So you should have found that with a total of N planks the overhang is equal to

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/N \text{ metres}$$

Note how the additional overhang gets smaller and smaller each time.

But what is the maximum length of overhang I can produce via this method if I keep taking more and more planks?

Mathematically, what we want to know is what is the sum as N tends to infinity of

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$$

This is known as the Harmonic Series. Try summing this on a calculator.

What do you think the limit is a $N \to \infty$?

How would you prove this?

Solution

This is a standard series. It tends to infinity. But very slowly. There are many ways to prove this and lots of possible extension ideas. See for example:

`https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)`

Interstellar flight 1

(by Thilo Gross)

I want to establish a base in another solar system and I need your help planning it. We will do this in 7 steps. First we need to figure out how long the trip to the destination will take, then we can think about what to bring and whom to take along.

Our target will be the closest star (well, second closest actually), Proxima Centauri. Proxima is only 4.243 light years away, which means light from the sun needs 4.243 years to get to Proxima.

But light is pretty fast, $300,000$ km/s, so how far is Proxima actually away in m?

$$d =$$

Solution

We have 4.243 lightyears, about 365 days in a year, 24 hours in a day, 3,600 s in an hour and 300,000 km/s and 1,000 m in a km. We multiply

$$4.243 \cdot 365 \cdot 24 \cdot 3,600 \cdot 300,000 \cdot 1,000 = 4.0142174 \cdot 10^{16}.$$

Thats is approximately 40 quadrillion meters. Because quadrillion is not uniquely defined it is better to say 40 petameters. (peta is the prefix that comes after tera, as in terabytes).

Interstellar flight 2

(by Thilo Gross)

I want the spaceship to constantly accelerate for half the way, and constantly decelerate for the second half of the way.

The spaceship's engines can accelerate it at

$$a = 10\text{m/s}^2$$

So if we accelerate for $t = 100\text{s}$ what will be our velocity?

$$v(100\text{s}) =$$

Can you write an equation for the velocity after accelerating for time t?

$$v(t) =$$

Solution

For constant acceleration velocity is acceleration times time. Therefore

$$v(100\text{s}) = 100\text{s} \cdot 5\text{m}/\text{s}^2 = 500\text{m}/\text{s}$$

the general formula (for constant acceleration) is

$$v = at$$

Note that we have chosen an acceleration that is similar to the gravitational acceleration on the surface of the earth. In other words, if we set our spaceship to this value of a the acceleration will feel like the earth gravity pushing us toward the rear of the ship, so we can walk around normally (for us 'rear' will feel like 'down'). You may want to point this out to the students, it's nice physics and also shows that we are not assuming 'crazy' acceleration.

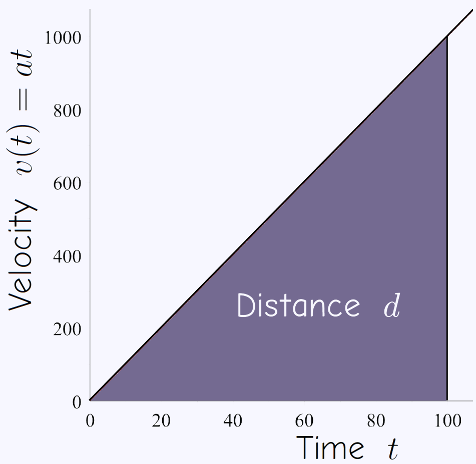
Interstellar flight 3

(by Thilo Gross)

After half the way we need to turn the ship around and fire our engines in the direction of Proxima to start decelerating. How long will it actually take to reach the halfway point? To work this out, we use the following trick:

Distance covered

If you plot your speed over time, the area under the curve is the distance covered up to a certain time. For constant acceleration this becomes a triangle, so we can use our knowledge of the area of triangles to find the distance covered.



Can you write an equation for d, the distance covered, as a function of acceleration a and time t?

$$d =$$

Solution

The equation that we seek is of course

$$d = \frac{1}{2}at^2$$

Interstellar flight 4

(by Thilo Gross)

Use the equation from the previous sheet to derive a formula for the time t it takes to go a distance, d, at a constant acceleration a

$$t =$$

How, long in years do we need to get to the half way point?

$$t_{\text{halfway}} =$$

So how long, in years, does the whole journey to Proxima take? (its shorter than you would think!)

$$t_{\text{Proxima}} =$$

Solution

Solving the equation from the previous sheet for the time yields

$$t = \sqrt{\frac{2d}{a}}$$

The distance to the halfway point is $d = 2 \cdot 10^{16}$ m and our acceleration was $a = 10$ m/s^2.

So the time we need is approximately

$$t_{\text{halfway}} = \sqrt{4 \cdot 10^{15}\text{s}^2} \approx 63 \cdot 10^6\text{s}$$

63 million seconds sounds like a lot, but it actually isn't. The second half of the journey takes as long as the first (good students may actually check this, less good students seem to assume it intuitively, both is ok). So the total journey only takes

$$t_{\text{Proxima}} = \sqrt{4 \cdot 10^{15}\text{s}^2} \approx 126 \cdot 10^6\text{s} \approx 4 \text{ yrs}$$

So that's not too much.

A note on relativity
Although we seem to travel faster than the light we can actually do the calculation in this way (with very good accuracy). In our co-moving frame of reference the journey takes just 4 years. From the perspective of an observer on earth the trip takes longer (about 6 years), so we cannot overtake a the light that left earth before us. This effect is known as relativistic time dilation. Compared to earth, time runs slower on our space ship, but we won't even notice that on board the space ship.

The assumption of constant acceleration is also not a problem. In the co-moving frame of reference typical drive technologies will provide constant acceleration. However, a rocket providing constant acceleration for 2 years would have to be huge. More realistic are ion drives, which could run that long. But current ion drives don't provide nearly as much thrust as would be necessary for the 10m/s^2 that we assumed. Presently, $10 \cdot 10^{-6}$m/s^2 is more realistic, which would increase travel time by a factor 1000.

Interstellar flight 5

(by Thilo Gross)

We could go on and compute the amount of fuel we need. But let's focus on something more important: food. Water and air can be recycled readily, but humans need to consume biomass to survive. Suppose we want to bring 1,000 people to Proxima, how much food do we need to bring for the journey and to survive the first year there? (Hint: You can start by thinking about how much food you eat per day)

An alternative to bringing food is growing biomass on board the ship. Adding vitamins as needed is relatively easy and because we will have to put up with nuclear power anyway, we can generate enough light to keep things growing.

The highest biomass production is achieved with algal growth tanks, which produce about 4g per day per litre of tank volume. Suppose that for our spaceship we can use tank units which weigh 300kg and each produce 1kg of dry biomass per day.

How long does a journey have to be so that the tank units are more efficient than just bringing the food? And, how many units do you want to install in our spaceship?

Solution

Now we are in the realm of Fermi estimates. One 1kg of food per day (dry mass) for an average human sounds about right to me (e.g. 1 kg of dry pasta is a lot of pasta). Hence, for 1000 people we need a tonne of food (literally!) per day.

Given that we want to prepare for 4 yrs of voyage and an additional year we need in total 1,825t of food. That's a lot.

The algal alternative is interesting. The tank units produce 1kg biomass per day so enough for one person. Since their weight is 300kg they are more efficient for every journey lasting more than 300 days.

We will need to install 1000 growth tank units to feed all passengers, but the mass of the 1000 tanks is just 300t. Also the tanks have the advantage that we can keep them running as long as we need them after arrival.

Interstellar flight 6

(by Thilo Gross)

We should also think about who we should bring. For instance our algal growth tanks need maintenance every 100 days and it takes a biosphere technician 6 hours to carry out the maintenance procedures on one tank. So how many biosphere technicians do we need?

What about physicians? Could one doctor look after all us?

If it is fun, go one, who else would you want to bring to keep the ship going and get a colony on a distant planet started?

Solution

The first part of this question is relatively clear cut. Since we have 1000 tanks and they need maintenance every 100 days, that means our technicians have to service 10 a day, that's 60 hours of work every day, so we will need at least 8 technicians if they don't mind some extra hours. If they work strictly 40 hour weeks then we will need 11 technicians.

The second part of the question is harder. Given a healthy crew, it seems plausible that they create less than 1h of work for the physician per person per year. So in total one physician would only work 1000h per year (he or she gets a much better deal than the biosphere technicians, it seems). We might want a second one anyway, in case of accidents etc.

Interstellar flight 7

(by Thilo Gross)

Congratulations you have made it through this set of exercises. Good work!

Of course if you enjoyed it you can always go on. There are plenty of questions related to the planned colony that benefit from mathematical considerations. For example how much energy do we need to produce to keep the spaceship going? How many people do we expect to die on the way to the destination? How many will be born? What else would we want to bring. Say, we want to build concrete domes as living space, how much concrete would be required? Or would it be more efficient to manufacture the concrete at the destination? In that case how long would it take till everybody would have a place to live on the surface of the planet?

Further information

In principle this challenge can be taken much further. By combining web search and mathematical estimates one can have a lot of fun. The concrete domes question that is mentioned leads up to the volume of spherical shells. The question regarding energy consumption is a tricky one, but it turns out that the two big consumers of energy are our algal tanks, and the propulsion. Without knowing the details, we can say that our drive system will use electrical energy to create momentum. Since we already know the acceleration $a = 10\text{m/s}^2$ we only need to estimate the mass of the spaceship to get an idea of how much power we will need.

The weightless girl 1

(by Alan Champneys)

Some people say gravity is caused by the Earth's rotation. But a little thought shows that centrifugal force acts outwards not towards the ground. (Think of a wet dog spinning its body to throw off excess water).

Lauren, a girl of mass M, is standing on the equator where acceleration due Centrifugal force will make experience a weight that is slightly less than Mg. But how much less?

Some useful facts

- Centrifugal force $= mr\omega^2$, where r is distance to spin axis and ω is rotation speed in radians/s ($2\pi\times$ revs/s)
- The radius of the earth is 6371 km. The earth spins one revolution per 24 hours. g =9.81.

First, what is the earth's rotation speed in radians per second?

$\omega =$ revs/hour = rad/s

Hence calculate Laruen's centrifugal force as a function of her mass M. What percentage of her gravitational weight is lost due to centrifugal force?

centrifugal force = $\times M =$ % of Mg

Solution

You may need to talk about the difference between centrifugal force (which is experienced only if you are in a rotating frame) and centripetal acceleration (that enables the body to rotation to occur).

Let $\Omega = 1/24$ be Earth's rotation in revolutions per hour. We want ω, the same quantity, but in radians per second. There are 3600 seconds in an hour. So

$$\omega = \frac{2\pi\Omega}{3600}$$

Hence

$$\omega = \frac{2\pi}{24 \times 3600} = 7.2722 \times 10^{-5} \text{ rad/s}$$

Suppose Lauren's weight is M kg. The she will experience

$$\begin{aligned} \text{centrifugal force} &= Mr\omega^2 \\ &= m6.371 \times 10^6 (7.72722 \times 10^{-5})^2 \\ &= 0.0337 \times M \end{aligned}$$

This we should compare with her gravity force

$$\text{gravitational force} = Mg = 9.81 \times M$$

So the fraction of gravitational force that is counterbalanced by centrifugal force is

$$\frac{Mr\omega^2}{Mg} = \frac{0.0337M}{9.81M} = \frac{0.0337}{9.81} = 0.0034$$

Hence only about 0.3% of Lauren's weight is compensated by centrifugal force. Or, put another way, the centrifugal force she feels is about

$$\frac{1}{0.0034} \approx 291 \quad \text{times weaker than gravity}$$

The weightless girl 2

(by Alan Champneys)

So, the centrifugal force experienced by Lauren, standing at the equator, is about 291 times weaker than gravity.

Now imagine that the Earth was spinning faster. Centrifugal force increases with rotation speed, but gravity is unaffected. But how much faster would it to spin in order for Lauren to feel weightless. That is, is there a critical rotation speed ω_c for which centrifugal force exactly balances gravity.

The critical angular velocity (in radians per second) is

$$\omega_c = \qquad\qquad\qquad \text{rad/s}$$

which is $\qquad\qquad\qquad$ times faster than the earth currently spins

How long would a day last on such a fast rotating Earth?

$$\text{day length} = \qquad\qquad\qquad \text{hours}$$

Solution

To find ω_c we need to set

$$Mr\omega_c^2 = Mg$$

this gives

$$\omega_c = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.81}{6.371 \times 10^6}} = 1.24 \times 10^{-3} \text{ rad/s}$$

But dividing by the current rotation speed, this gives

$$\frac{\omega_c}{\omega_{\text{Earth}}} = \frac{1.24 \times 10^{-3}}{7.2722 \times 10^{-5}} = 17.0633$$

Thus, the Earth only need spin about 17 times faster!

Actually we could have worked this out because we needed

$$\frac{\omega_c^2}{\omega_{\text{Earth}}^2} = \frac{291}{1}$$

and $\sqrt{291} = 17.06$

On such an Earth, a day would last 24/17 hours = 1 hour and 24 minutes.

Further information

There are many possible more realistic extensions to this problem ....

What if Lauren were standing in Bristol, namely at 51° North? What about at the North Pole?

What if Lauren was standing on the moon who's gravity is about 80 times less than Earth but who's radius is about a quarter of Earth's?

At what height (in radial distance from the centre of the Earth) can a satellite be geostationary? That is, a distance for which the sate-lite can orbit at precisely the Earth's rotation speed, with centrifugal force precisely balancing gravity, so that the satellite appears from Earth to be in the same place. (To answer this you will need to use Newton's law of gravitational force between two bodies who's centres of mass are a given distance r apart).

How would the required distance from the Earth change if the satellite were over the equator or if it were above Bristol?

Geostationary satellites are extremely important in telecommunications and in GPS systems. Can you think why?

Ball bouncing 1

(by Alan Champneys)

What happens if you put a tennis ball on top of a basketball and then drop them both? The result is surprising. To explore this, start by finding out how to mathematically model bouncing balls:

Newton's law of bouncing*

$$[\text{ speed afterwards }] = e \cdot [\text{ speed before }]$$

e = 'coefficient of bouncing' (a property of the ball): $0 \leq e \leq 1$
$e = 1$ – a perfectly elastic ball; $e = 0$ – a squashy tomato;
$e = 0.8$ – a reasonable value for a well pumped ball

Suppose a basketball has $e = 0.8$ and is travelling at $v^{\mathrm{b}} = -5$ m/s as it hits the floor (here, negative velocity signifies going down, and a and b indicate before and after). How fast will it be going upwards as it lifts off?

$$v^{\mathrm{a}} = -ev^{\mathrm{b}} =$$

Can you think of any physical situation in which $e > 1$?

* actually Newton's law of restitution.

Solution

We suggest that you do the experiment. If you do, try to get the tennis ball directly over the center of mass of the basketball. Alternatively you can show this “Physics Girl” video on You Tube

www.youtube.com/watch?v=2UHS883_P60

(it’s best to stop after about 1:30)

The answer to the question is that the ball would lift off with speed

$$v^{\mathrm{a}} = e \cdot 5 \text{ m/s} \ = 0.8 \cdot 5 \text{ m/s} \ = 4 \text{ m/s}$$

Note that the sign has changed as the ball is now travelling up.

Values $e < 1$ account for loss of energy in the bounce: An example with $e > 1$ would have to include a source of energy, e.g. a ball hitting an active bumper in a pinball machine.

Ball bouncing 2

(by Alan Champneys)

Let's check if our assumption that the ball hits the floor at 5 m/s is about right. A very useful principle is

Conservation of energy

$$\text{kinetic energy} = \tfrac{1}{2}Mv^2; \qquad \text{potential energy} = Mgh$$

$$M = \text{mass of object}; \qquad v = \text{its speed};$$
$$h = \text{height above floor}; \qquad g = 9.81 \text{ m/s}^2.$$

Make reasonable assumptions about M and the height h_b from which the ball is dropped. The kinetic energy of the ball when it hits the floor must equal the potential energy that it had in the beginning, so with how much energy does the ball hit the floor? (Hint: 1 Joule = 1000 gm^2/s^2)

$$E_\mathrm{b} =$$

What velocity does the balls kinetic energy correspond to? Was our assumption of -5 m/s approximately right?

$$v^\mathrm{b} =$$

Solution

When we release the ball it has the energy

$$E_{\mathrm{b}} = Mgh_{\mathrm{b}}$$

This must equal the kinetic energy on impact

$$E_{\mathrm{b}} = \frac{1}{2}M(v^{\mathrm{b}})^2$$

So the velocity of the ball on impact is

$$v^{\mathrm{b}} = -\sqrt{\frac{2E_{\mathrm{b}}}{M}} = -\sqrt{\frac{2Mgh_{\mathrm{b}}}{M}} = -\sqrt{2gh_{\mathrm{b}}}$$

For example for a 600g basketball released from 1m height we get

$$E_{\mathrm{b}} = 600\ \mathrm{g} \cdot 9.8\ \mathrm{m/s^2} \cdot 1\ \mathrm{m} = 5880\ \mathrm{gm^2/s^2} = 5.88\ \mathrm{J}$$

$$v^{\mathrm{b}} = -\sqrt{2 \cdot 9.8\ \mathrm{m/s^2} \cdot 1\ \mathrm{m}} = -\sqrt{19.6\ \mathrm{m/s^2}} \approx -4.427\ \mathrm{m/s}$$

If we dropped it from just a bit higher than 1 m we would reach the -5 m/s.

Ball bouncing 3

(by Alan Champneys)

We noticed on the first sheet that some velocity is lost as the ball hits the floor. How fast is your ball on lift off? And what is its kinetic energy?

$$v^{\mathrm{a}} =$$

$$E_{\mathrm{a}} =$$

As the ball rises again its kinetic energy gets converted into potential energy. Compute the height, h_{a}, that it can still reach after the bounce.

$$h_{\mathrm{a}} =$$

Now here is a challenge: Can you find a general formula for $h_{\mathrm{a}}/h_{\mathrm{b}}$?

$$\frac{h_{\mathrm{a}}}{h_{\mathrm{b}}} =$$

Solution

In the bounce the velocity is multiplied by e. Hence, the velocity immediately after the bounce is

$$v^{\mathrm{a}} = -ev^{\mathrm{b}} = e\sqrt{2gh_{\mathrm{b}}}$$

which is less than v^{b} due to $e < 1$. At this velocity the kinetic energy is

$$E_{\mathrm{a}} = \frac{1}{2}M(v^{\mathrm{a}})^2 = \frac{1}{2}M\left(e\sqrt{2gh_{\mathrm{b}}}\right)^2 = e^2Mgh_{\mathrm{b}},$$

at the highest point h_{a} this energy has been transformed into potential energy Mgh_{a}. Equating this potential energy with E_{a}, the kinetic energy after impact yields

$$Mgh_{\mathrm{a}} = e^2Mgh_{\mathrm{b}}$$

and hence

$$\frac{h_{\mathrm{a}}}{h_{\mathrm{b}}} = e^2$$

Note that the height is not proportional to the coefficient of restitution, e, but its square.

In the example of basketball with $M = 600$ g and $e = 0.8$ the velocity after impact is

$$v^{\mathrm{a}} = ev = 0.8 \cdot 4.427 \text{ m/s} = 3.542 \text{ m/s}$$

This means the ball has a kinetic energy of

$$E_{\mathrm{a}} = \frac{1}{2}M(v^{\mathrm{a}})^2 = 0.5 \cdot 600 \text{ g } \cdot (3.542 \text{ m/s})^2 = 3.762 \text{ J}$$

With this energy it can reach a height of

$$h_{\mathrm{a}} = \frac{E_{\mathrm{a}}}{Mg} = \frac{3762 \text{ gm}^2/\text{s}^2}{600 \text{ g } \cdot 9.8 \text{ m/s}^2} \approx 0.64 \text{ m}$$

The ball bounces 64 cm high. This is consistent with the general formula that we have already derived

$$0.64 \text{ m} = (0.8)^2 \cdot 1 \text{ m}.$$

Ball bouncing 4

(by Alan Champneys)

Now let's return to the case where a tennis ball and a basketball are dropped together. For simplicity assume that both balls reach a velocity of -5m /s before the basketball hits the floor.

Here is a trick: Assume basketball first bounces off the floor, then, rising at velocity v_1^{a}. An instant later it collides with the tennis ball, which is still moving downward at velocity v_2^{b}. What is the relative speed at which the balls now approach each other? (it's not 10 m/s)

$$v_1^{\mathrm{a}} - v_2^{\mathrm{b}} =$$

Newton's law of bouncing for 2 objects

$$[\text{ speed of separation }] = e \cdot [\text{ speed of approach }]$$

Using the law above, and $e = 0.8$ for the collision between the two balls, find the speed at which they separate.

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} =$$

Here we have used the c to indicate the velocities after the balls collide.

Solution

From sheet 1, we know that after lifting off from the floor, the basketball is going upwards at 4 m/s. The tennis ball is still falling at -5 m/s so their speed of approach is

$$v_1^{\mathrm{a}} - v_2^{\mathrm{b}} = 9 \text{ m/s}$$

Using the law and $e = 0.8$, the speed of separation after the collision is

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} = 9 \text{ m/s} \cdot 0.8 = 7.2 \text{ m/s}$$

This solution assumes that the two collisions are independent, an unrealistic assumption unless the tennis ball and the basketball are separated by about 5cm when the basketball hits the floor. In reality all bouncing collisions take a finite amount of time and if the basketball and tennis ball are less than 5 cm apart, the tennis ball hits the basketball before it has lifted off from the ground.

Yet, it has been discovered that as long as the two balls are at least a few millimeters apart, the assumption gives accurate predictions. It is still an open question of scientific research why this is so. See for example

`dx.doi.org/10.1098/rspa.2015.0286`

Ball bouncing 5

(by Alan Champneys)

To solve for the speed v_1^{c} of the basketball and v_2^{c} of the tennis ball after the collision we have to use another principle:

Conservation of momentum

$$\text{Total momentum} = \sum[\text{mass} \cdot \text{velocity of balls}]$$

where $\sum$ means 'sum over all balls' and
Total momentum before collision = Total momentum after collision

Make a reasonable assumption about the masses of the basketball, M_1, and the tennis ball, M_2, and find the total momentum just before the basketball and tennis ball collide (after the basketball lifts off from the floor)

$$P = M_1 v_1^{\mathrm{a}} + M_2 v_2^{\mathrm{b}} =$$

Since the momentum is conserved in the collision

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}}$$

is also true.

Solution

A basketball is about 10-12 times heavier than a tennis ball, make sure that the students use a sufficiently high ratio of masses or the results will be less impressive.

For a 600g basketball colliding with a 50g tennis ball. The total momentum is

$$P = 4\ \mathrm{m/s}\ \cdot 600\ \mathrm{g}\ - 5\ \mathrm{m/s}\ \cdot 60\ \mathrm{g}\ = 2100\ \mathrm{gm/s}\ = M_1 v_1^c + M_2 v_2^c$$

Ball bouncing 6

(by Alan Champneys)

We just discovered

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}},$$

where we know P, M_1, and M_2 already. So there are only two unknowns, the velocities after the collision. We also know from sheet 4 that

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} = 7.2 \text{ m/s}$$

We have two equations for two unknowns! Substitute all the values and compute the velocity of the tennis ball after the collision

$$v_2^{\mathrm{c}} =$$

This velocity should take the tennis ball to a height h_{c}, higher than the height from which it was released. Use your insights from sheet 3 to determine the factor $h_{\mathrm{c}}/h_{\mathrm{b}}$. Furthermore, by what factor, $h_{\mathrm{c}}/h_{\mathrm{a}}$ is this bounce higher than the height that would be reached if we dropped the tennis ball without the basketball?

Solution

For our example the two equations read

$$\begin{aligned} v_2^c - v_1^c &= 7.2 \text{ m/s} \\ 600 \text{ g } \cdot v_1^c + 50 \text{ g } \cdot v_2^c &= 2100 \text{ gm/s} \end{aligned}$$

Multiplying the first equation with 600 g gives us

$$\begin{aligned} -600 \text{ g } \cdot v_1^c + 600 \text{ g } \cdot v_2^c &= 4320 \text{ gm/s} \\ 600 \text{ g } \cdot v_1^c + 50 \text{ g } \cdot v_2^c &= 2100 \text{ gm/s} \end{aligned}$$

adding the the two equations yields

$$650 \text{ g } \cdot v_2^c = 6420 \text{ gm/s}$$

where the minus signifies a change in direction. We are only interested in the absolute value. Dividing by 650 g, we find

$$v_2^c = 9.88 \text{ m/s}$$

this is almost twice our initial velocity of 5 m/s.

From sheet 3 we know that the height to which the ball can climb is proportional to the velocity squared. So this means our tennis ball can reach about 4 times the height from which we released it. Without the basketball it would only have reached $e^2 = 0.64$ times the release height. So we can say the bounce from the basketball increases the height by a factor $4/0.64 = 6.25$.

(Make sure the students notice the quadratic increase in the height)

Ball bouncing 7

(by Alan Champneys)

Congratulations you have explained what happens in the two-ball bounce.

Of course there are some extensions to think about. What would happen if we used two tennis balls instead of a tennis ball and a basketball? What about if the tennis ball and the basketball were reversed? What about three balls on top of each other (of successively decreasing mass)?

Very similar principles apply also in horizontal collisions, this is important for instance in nuclear physics where particles become very fast due to collisions with heavier particles, or in traffic accidents, where heavy vehicles can transfer a lot of energy to lighter ones.

Further information

Actually there are many hidden assumptions in this model. More information can be found at

"*The two-ball bounce problem.*" *Proc. Roy. Soc. Lond. A.* 2015
`dx.doi.org/10.1098/rspa.2015.0286`

The principles used in this exercise are of broad importance for many applications. In engineering, the dynamics of bouncing and rattling is a fundamental area of active research. For example, car manufacturers devote whole teams of people to precisely model the whole vehicle to eliminate what is known in the industry as 'buzz, squeak and rattle'. Such phenomena usually come about through parts of the car's body or interior panelling impacting with other parts at certain frequency. This is not usually dangerous, but leads to commercially important customer satisfaction issues.

Sports betting 1

(by Filippo Simini)

In sporting bets, the return of a winning bet is calculated multiplying the stake by the 'odds multiplier'.

The following table shows the odds multipliers of two bookmakers for the same game, which has two possible outcomes: victory of the home team (Win) or defeat of the home team (Lose).

	Bookmaker 1	Bookmaker 2
Win	$w_1 = 3$	$w_2 = 2.5$
Lose	$l_1 = 1.5$	$l_2 = 2$

For example, a bet of £10 with Bookmaker 1 on the victory of the home team would return £$10w_1$ = £30 if the home team wins (and zero if the home team lose). Note that odd ratios are always larger than 1: $l_1, l_2, w_1, w_2 > 1$.

Suppose you have £1 to bet and you think the home team will lose the game, which Bookmaker should you pick to maximise your return?
How much will you gain if you win, and what will be your loss if you lose?

(Note that here we are ignoring any betting fee).

Solution

The Bookmaker offering the highest odds ratio for the selected outcome should be chosen. So, if $l_1 > l_2$ Bookmaker 1 should be chosen, otherwise if $l_1 < l_2$ Bookmaker 2 should be chosen.

In this case $l_2 = 2 > 1.5 = l_1$. If the home team loses, you win the bet and get $l_2 =$ £2 back. So, since you paid £1 for the bet, your gain is $l_2 - 1 =$£1. If the home team wins, you lose the bet and get nothing back, so you lose the £1 of the stake.

Given that bookmakers compete with each other to attract more bets and that betters would prefer to place their bet with the bookmaker that offers the highest odds ratio (and hence the highest potential return), bookmakers tend to increase their odds ratios above the values of their competitors.

Sports betting 2

(by Filippo Simini)

Consider the odds ratios of part 1:

	Bookmaker 1	Bookmaker 2
Win	w_1	w_2
Lose	l_1	l_2

where $w_1 > w_2$ and $l_2 > l_1$.

Suppose you have £1 to bet, but you are not sure of the outcome of the game, so you decide to bet on both outcomes.
Assume you bet £x on the victory of the home team and £$y = (1 - x)$ on the defeat of the home team.

What would be your total gain (or loss) if the home team wins? And if it loses?

Solution

If the home team wins you obtain $w_1 x$ from the winning bet with Bookmaker 1 and you lose 1, the total stake:

$$R_w = w_1 x - 1$$

If the home team loses you obtain $l_2(1 - x)$ from the winning bet with Bookmaker 2 and you lose 1, the total stake:

$$R_l = l_2(1 - x) - 1$$

Sports betting 3

(by Filippo Simini)

In gambling, a "Dutch book" is a set of odds and bets which guarantees a profit, regardless of the outcome of the gamble.

Assuming you bet £x on the victory of the home team and £$(1-x)$ on the defeat of the home team, your have

$$R_w = w_1 x - 1$$

if the home team wins, and

$$R_l = l_2(1-x) - 1$$

if the home team loses.

In this situation, is it possible to create a Dutch book and gain some money irrespective of the outcome of the game?

Under which conditions on the odds ratios w_1 and l_1 would it be possible to find a value of x which always guarantees a profit?

It is important to realise that bookmakers have lots of tricks that they always make a profit. While in the short term, some gamblers can win money. All gamblers lose money in the long term. Gambling is addictive and can lead to misery!

Solution

It is possible to create a Dutch book if both returns are positive: $R_w = w_1 x - 1 > 0$ and $R_l = l_2(1-x) - 1 > 0$.

The figure below shows that this is possible if the two lines

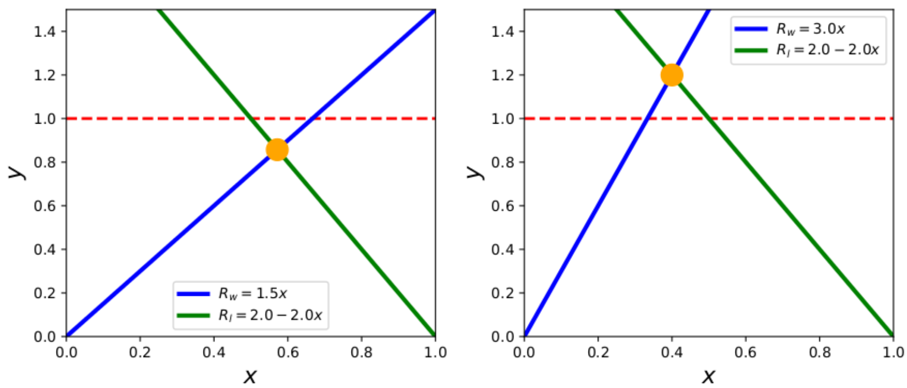
$$y = w_1 x$$

and

$$y = l_2 - l_2 x$$

intersects in a point (x^*, y^*) such that $y^* > 1$.

In the case on the left, there is no x for which both lines are above $y = 1$ (i.e. $R_w > 1$ and $R_l > 1$ simultaneously). In the case on the right, there is a range of x values for which both lines are above 1, so creating a Dutch book is possible.



In formulae, the point x^* where the two lines intersect is the solution of the equation: $w_1 x^* = l_2 - l_2 x^*$, which yields

$$x^* = \frac{l_2}{w_1 + l_2}$$

Betting £x^* ensures that you will get the same return irrespective of the game's outcome. This return will be positive if $R_w = w_1 x^* - 1 = w_1 \frac{l_2}{w_1 + l_2} - 1 > 0$, which yields

$$1 > \frac{1}{w_1} + \frac{1}{l_2}.$$

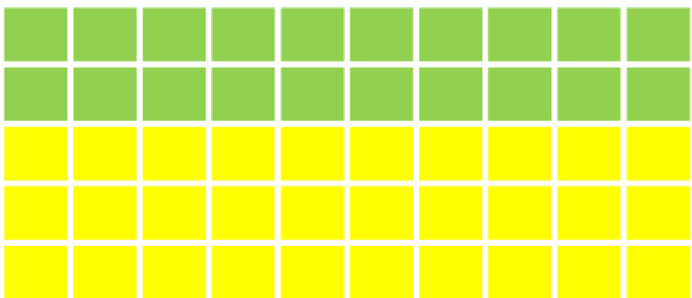
With odds w_1 and l_2 satisfying this condition it is possible to create the Dutch book.

Simpson's paradox 1

(by Filippo Simini)

"Gerrymandering" consists of the manipulation of the boundaries of constituencies in order to alter the electoral results in a non-proportional system.

For example, consider the following region where each square represents a precinct: green squares vote for party A and yellow squares vote for party B. Out of the total 50 precincts, 20 (40%) vote for party A, and 30 (60%) for party B.

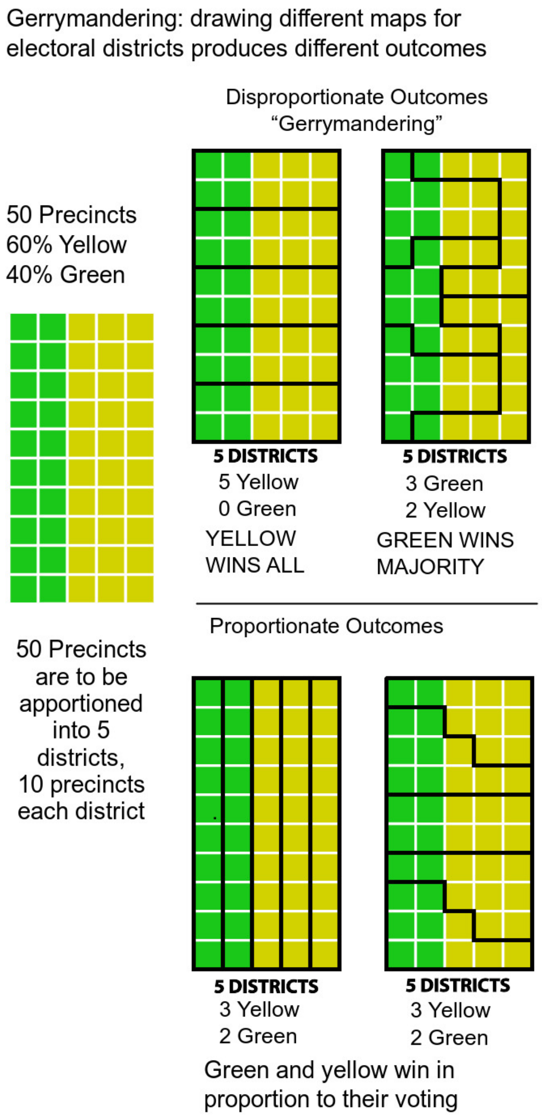


Can you draw 5 constituencies of equal size (10 neighboring precincts each) so that party A and B win in proportion to their overall voting?

Can you draw other 5 constituencies of equal size (10 neighboring precincts each) so that party A, the minority party, wins in the majority of constituencies?

Solution

From https://en.wikipedia.org/wiki/Gerrymandering:



Simpson's paradox 2

(by Filippo Simini)

Gerrymandering is a real life example of what's know as Simpson's paradox, where a trend appearing in different groups disappears when the groups are combined. Another example is the following:

A school has two classes. Class 1 has 25 students, 11 males and 14 females, and class 2 has 23 students, 13 males and 10 females.

Both classes take the same Maths test.
Overall, female students did better than male students: 12 out of 24 females passed the test (success rate 0.5), while just 11 of the 24 males passed it (success rate 0.458).

However, in each class, males had a higher success rate than females!
How is this possible?

Can you find a set of results such that male students have a higher success rate than female students in each class, but a lower success rate overall?

	Class 1	Class 2	Total
Males	**? / 11**	**? / 13**	11 / 24
Females	? / 14	? / 10	**12 / 24**

Solution

A possible solution is

	Class 1	Class 2	Total
Males	**9 / 11**	**2 / 13**	11 / 24
Females	11 / 14	1 / 10	**12 / 24**

Let m_i (f_i) be the number of male (female) students passing the test in class i.
In general, a set of scores for which males have higher success rates in each class must satisfy these conditions:

$$m_1 + m_2 = 11 \quad \Rightarrow \quad m_2 = 11 - m_1 \tag{4}$$
$$f_1 + f_2 = 12 \quad \Rightarrow \quad f_2 = 12 - f_1 \tag{5}$$

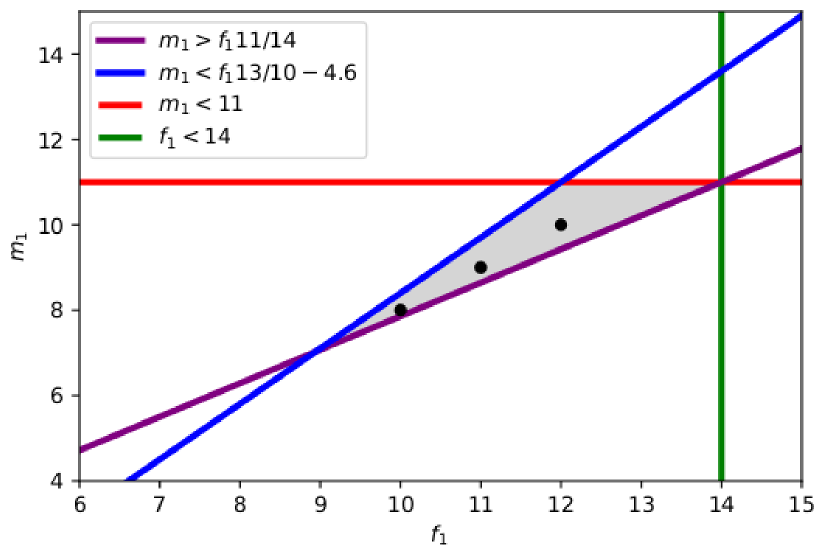
and

$$m_1/11 > f_1/14 \quad \Rightarrow \quad m_1 > f_1 11/14 \tag{6}$$
$$m_2/13 > f_2/10 \quad \Rightarrow \quad m_2 > f_2 13/10 \quad \Rightarrow \quad m_1 < f_1 13/10 - 4.6 \tag{7}$$

plus the constrains on the total number of males and females in each class

$$m_1 < 11 \tag{8}$$
$$f_1 < 14 \tag{9}$$

The solutions of the system lay in the grey region showed below:



Simpson’s paradox 3

(by Filippo Simini)

Simpson’s paradox has relevant implications on our ability to understand the results of scientific experiments, for example in medical studies.

Consider the previous example, with the same outcomes, but instead of class tests assume that the results describe two independent investigations (test 1 and 2) on the effectiveness of two drugs, A and B. The numerators now correspond to the number of patients successfully treated using the drug.

	Test 1	Test 2	Total
Drug A	**9 / 11**	**2 / 13**	11 / 24
Drug B	11 / 14	1 / 10	**12 / 24**

According to both independent experimentations (tests), drug A is more effective than B. However, when the results of the tests are combined, we reach the opposite conclusion: drug B works better than A.

What should be trusted, the unanimous conclusions of the independent tests, or the reverse indication of the aggregate data?

Solution

Ask the students how they would interpret the results of the experiments and which drug they would give to a patient.

To stimulate the discussion, propose the following variations of the experiment:

1. Suppose that the number of success cases of drug A in Test 2 is 1 instead of 2. Hence, the success rate becomes $1/13 < 1/10$, so drug B becomes more effective in Test 2. Should you prefer drug B over drug A, in this case?

2. Suppose you are told that the participants of Test 1 are all over 40 years old, while the participants of Test 2 are all below 40.
   If you know that a patient is below 40, then using the results of Test 2 obtained testing patients in this age class, you would recommend drug A. Similarly, if you know that the patient is above 40, using the results of Test 1 for old patients, you would also recommend drug A. But if you do not know the patient's age, using the combined data of all tests you would recommend drug B.
   So, if you don't know the patient's age you would recommend drug B, while if you know their age you would always recommend drug A, irrespective of the age!

Simpson's paradox 4

(by Filippo Simini)

Suppose we test the drugs on groups that are ten times bigger than the previous ones. For example, in Test 1 drug A is now tested on 110 individuals instead of 11.
We also assume that the success rates do not depend on the group sizes.

	Test 1	Test 2	Total
Drug A	? / 110	? / 130	? / 240
Drug B	? / 140	? / 100	? / 240

Does increasing the number of participants in all groups resolve the paradox?

Solution

Assuming that the success rates do not depend on the group sizes, the number of successful cases in each group must also become ten times bigger, so that the success rates do not change.

	Test 1	Test 2	Total
Drug A	**90 / 110**	**20 / 130**	110 / 240
Drug B	110 / 140	10 / 100	**120 / 240**

You can see that the success rates of the two experiments combined (in the "Total" column) do not change. Hence, increasing the number tests in each class in the same proportion does not resolve the paradox.

Simpson's paradox 5

(by Filippo Simini)

Suppose we test the drugs on groups of equal sizes, of 100 patients each:

	Test 1	Test 2	Total
Drug A	? / 100	? / 100	? / 200
Drug B	? / 100	? / 100	? / 200

Assume again that the success rates do not depend on the group sizes.

Does considering groups of equal sizes resolve the paradox?

Solution

Let's compute the success rates in each group:

	Test 1	Test 2	Total
Drug A	**9/11 = 0.82**	**2/3 = 0.15**	
Drug B	11/14 = 0.79	1/10 =0.1	

We see that now drug A is the most successful separately in both tests *and* when all data are combined:

	Test 1	Test 2	Total
Drug A	**82 / 100**	**15 / 100**	**97 / 200**
Drug B	79 / 100	10 / 100	89 / 200

Hence, choosing groups of equal sizes can resolve the paradox.
The first table shows that the overall success rate of drug A can take any value between 0.15 and 0.82, depending on the relative sizes of the groups treated with drug A in Tests 1 and 2. In particular, calling N_i the number of individuals participating in Test i, the overall success rate of drug A will be ~ 0.82 if $N_1 \gg N_2$ (N_1 much larger than N_2) and vice-versa, it will be ~ 0.15 if $N_1 \ll N_2$.
The same applies to drug B.
So, if we expect that groups belonging to Test 1 and 2 should have equal sizes, we should compute the overall success rates as the average rates of all tests.
When groups tested in Tests 1 and 2 do have different sizes (because, for example, there are more young patients than old patients), then we should compute the overall success rates of drugs A and B as weighted averages, where weights are N_1 and N_2 for Tests 1 and 2 respectively.

In general, to avoid ambiguous results, in each Test we should treat the same number of patients with the two drugs. Indeed, for any N_1 and N_2, drug A has a higher overall success rate than drug B:

$$\text{total success A} = N_1 r_{A,1} + N_2 r_{A,2} > N_1 r_{B,1} + N_2 r_{B,2} = \text{total success B} \tag{10}$$

$$= N_1(r_{A,1} - r_{B,1}) > N_2(r_{B,2} - r_{A,2}) \tag{11}$$

where $r_{A,1} = 0.82 > 0.79 = r_{B,1}$ and $r_{A,2} = 0.15 > 0.1 = r_{B,2}$, hence $(r_{A,1} - r_{B,1}) > 0 > (r_{B,2} - r_{A,2})$.

Comparative advantage 1

(by Filippo Simini)

The manager of a furniture company that manufactures tables hires you to increase productivity.

Four table legs and one table top are needed to produce one table.
On average, after a day worth of effort one worker produces $l = 10$ legs or $t = 5$ table tops.

Suppose that n out of the N workers of the company are assigned to the production of legs, and the remaining $(N - n)$ to the production of tops.

How many full tables will be produced in one day, on average, as a function of n ?

Solution

Parameters:

l = number of legs produced in one day by one worker, on average.
t = number of tops produced in one day by one worker, on average.
N = total number of workers.

Variables:

n = number of workers assigned to legs.

Relationships:

$N - n$ = number of workers assigned to tops.
ln = total number of legs produced in one day, on average.
$t(N - n)$ = total number of T produced in one day, on average.
$P = \min\{nl/4, (N - n)t\}$ = total number of tables produced in one day, on average.

The last relationship is obtained considering that to produce one full table one top and four legs are needed.
Hence, the number of tables cannot exceed the number of tops and the number of groups of four legs.

Comparative advantage 2

(by Filippo Simini)

To maximise the production, workers should be assigned to the production of legs or tops in order to manufacture 4 legs in the time needed to complete 1 table top.

How many workers should produce legs and how many should produce tops, in order to maximise the production of tables ?

Solution

We have to find the value of n that maximises the total number of tables produced per day.

We use the result of part 1 for the total number of tables produced per day as a function of n:

$$P = \min\{nl/4, (N-n)t\}$$

Intuitively, we should maximise the production by producing legs and tops in the right proportions, i.e. 4 legs per 1 top.
Then the question becomes:
What is the n that solves

$$nl/4 = (N-n)t \quad ?$$

Solving for n yields:

$$n = N4t/(l+4t)$$

If $l = 10$ and $t = 5$, then $n/N = 2/3$ of the workers should produce legs, and $1/3$ tops.

The average number of tables produced in one day is equal to $N5/3$.

Comparative advantage 3

(by Filippo Simini)

We found that the average number of tables produced in one day is equal to $P = N5/3$. Is it possible to further increase the production of tables?

Previously we assumed that the productivity of each worker is exactly equal to the average productivity. This may not be true in general, as it is common to find more and less productive workers.

Suppose that you find out that in the company there are some workers that are faster than the average at producing table tops, and some other workers that are faster at producing legs.
Considering this information, you are able to form two groups:
Group 1 has 2/3 of the workers and they produce 12 legs and 4 tops per day, on average; Group 2 has 1/3 of the workers and they produce 6 legs and 7 tops per day, on average.

Average number of units produced by one worker per day			
	Group 1	Group 2	All
fraction of workers	2/3	1/3	1
legs	12	6	10
tops	4	7	5

How can you use this new information to increase the productivity ?

Solution

Group 1 is more efficient at producing Legs than the overall pool of workers.
Group 2 is more efficient at producing Tops than the overall pool of workers.
Hence, assign Group 1 to the production of legs only, and Group 2 to the production of tops only !

Let's compare the production of tables between the previous solution (Model 1) and this new solution considering the two groups (Model 2).

Model 1: $P_1 = 5/3N$

Model 2: $P_2 = \min\{(N2/3)12/4, (N1/3)7\} = \min\{6/3N, 7/3N\} = 6/3N$.

Since $P_2 = 6/3 > 5/3 = P_1$, productivity of Model 2 is higher than Model 1.

This situation is an example of what economists call *absolute advantage*:

If two countries produce two products of the same quality, but the first country is faster at producing one product and the second country is faster at producing the other product, then they should specialise and trade: each country will produce only the product they make faster and they will trade part of it for the other product.

Comparative advantage 4

(by Filippo Simini)

Suppose instead that you were not able to split workers into two groups such that each group is more efficient than the other at producing one item, tops or legs.
This can happen when there are some workers that are more efficient than others at both tasks, while all other workers are less efficient than the average at both tasks.

Dividing the workers into two groups of equal size according to their efficiency, you are able to form the following two groups:
Group 1 is formed by the most efficient workers that produce 14 legs and 6 tops per day, on average; Group 2 comprises the least efficient workers that produce 6 legs and 4 tops per day, on average.

Average number of units produced by one worker per day			
	Group 1	Group 2	All
fraction of workers	1/2	1/2	1
legs	14	6	10
tops	6	4	5

How should the work be divided among the two groups in order to have balanced production, such that one table top and four legs are produced at the same rate ?

Solution

Let's call $x \in [0,1]$ the fraction of workers of group 1 that are assigned to work on legs, and $(1-x)$ the fraction of workers of group 1 that work on tops.
Let's call $y \in [0,1]$ the fraction of workers of group 2 that are assigned to work on legs, and $(1-y)$ the fraction of workers of group 2 that work on tops.

Let's call $\hat{l}_i$ (t_i) the average number of legs (tops) produced by one worker of group i in one day. To simplify the notation of the following equations, let's call $l_i = \hat{l}_i/4$ the number of groups of 4 legs produced by a worker of group i in one day, on average. In our case, we have $l_1 = 3.5$, $l_2 = 1.5$, $t_1 = 6$, $t_2 = 4$.

The total daily production of 4 legs is $N/2[l_1 x + l_2 y]$, while the total production of tops per day is $N/2[t_1(1-x) + t_2(1-y)]$.
The overall production of tables is limited by the production of the least efficient component:

$$P = N/2 \, \min\{l_1 x + l_2 y, t_1(1-x) + t_2(1-y)\}$$

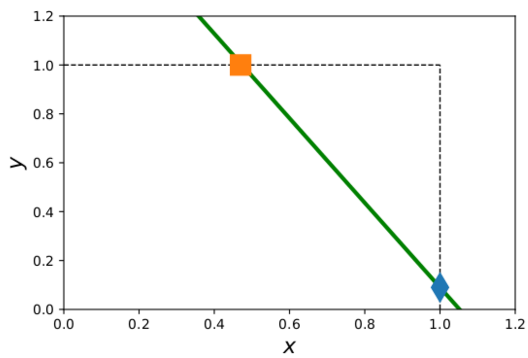
The maximum production can be achieved when 4 legs and 1 top are produced at the same rate, i.e. the production is balanced:

$$l_1 x + l_2 y = t_1(1-x) + t_2(1-y)$$

Rearranging:

$$y_{eq}(x) = \frac{t_1 + t_2}{l_2 + t_2} - x\frac{l_1 + t_1}{l_2 + t_2} \simeq 1.82 - x \cdot 1.73$$

The production is balanced for the (x, y) pairs on the green line:



The square and diamond correspond to the cases where group 2 only produces legs and group 1 only produces tops, respectively: $x = 1; y = 0.09$ and $y = 1; x = 0.47$.

Comparative advantage 5

(by Filippo Simini)

We found that the production is balanced (i.e. tops and 4-legs are produced at the same rate) if x workers of group 1 are assigned to produce legs, and $y_{eq}(x)$ workers of group 2 are assigned to produce legs, where

$$y_{eq}(x) = \frac{t_1 + t_2}{l_2 + t_2} - x\frac{l_1 + t_1}{l_2 + t_2}$$

In this case the productivity per worker is

$$P_3(x) = 1/2\,[l_1 x + l_2 y_{eq}(x)]$$

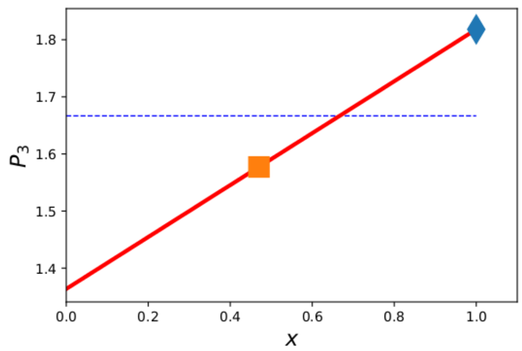
Is it still possible to divide the production among the two groups in order to increase the productivity with respect to the solution found in part 2 ?

Solution

When the production is balanced (along the green line obtained in the solution to part 4), the productivity per worker can be computed as

$$P_3(x) = 1/2[l_1 x + l_2 y_{eq}(x)]$$

and it is shown in the figure below:



The maximum productivity is achieved when $x = 1$, and all workers of group 1 produce legs. In this case the productivity is $P_3(x = 1) = 1.82 > 1.67 = P_1$, which is higher than Model 1 (dashed blue line).

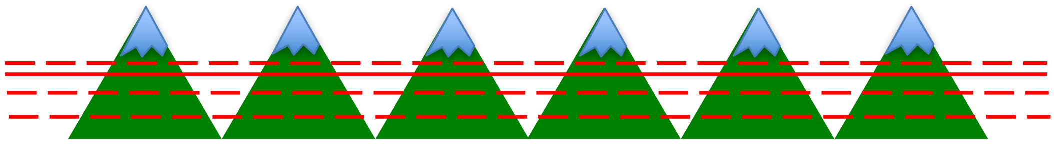
This situation is related to what economists call *comparative advantage*:

Consider two countries that produce two products of the same quality. The first country is more efficient than the other at producing both products, however the second country is relatively more efficient than the first country at producing one of the products. Then they should specialise and trade: each country should produce the product they make more efficiently and they will trade part of it for the other product.

Railway line 1

(by Filippo Simini)

Design the path of a high-speed railway line across a mountain region in order to minimise the total construction cost. The construction costs depend on the excavation of tunnels and the construction of viaducts. The high speed trains should avoid slopes, so the railway line must run horizontally. Trains should also avoid sharp corners and travel as much as possible on a straight line. For simplicity, we can consider the problem in one dimension, where the mountains are a series of triangles placed next to each other and we can assume that the railway line is a straight horizontal line. The red lines in the figure below show possible railway lines under these assumptions.

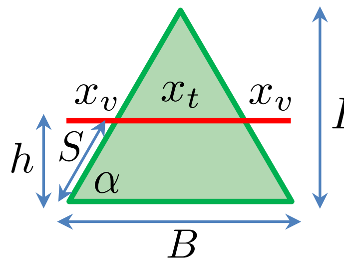


The cost of excavation of a tunnel can be assumed to be proportional to the length of the tunnel. For the cost of building a viaduct, we can consider two scenarios. In the first scenario, the cost is proportional to the viaduct length, as in the case of tunnels.

Can you find the optimal height of the railway line in this first scenario?

Solution

To find the height of the railway line, h, that minimises the total building cost we have to write the total cost, C, as a function of the height, $C(h)$. The total cost of the railway line is the sum of the costs to pass through each mountain. Since all mountains are equal, the total cost is proportional to the cost of one mountain, so it is enough to minimise the cost to build the railway line through one mountain.



The total cost is the sum of the cost of excavating tunnels and building viaducts:

$$C(h) = C_t(h) + C_v(h).$$

The cost of excavation of a tunnel is proportional to the tunnel's length, x_t: $C_t(h) = T \cdot x_t(h)$, where T is the excavation cost per unit length. In the first scenario, the cost of building a viaduct is proportional to the viaduct's length, $2x_v$, where x_v is the viaduct's length on each side of the mountain: $C_v(h) = V \cdot 2x_v(h)$, where V is the cost of the viaduct per unit length. So,

$$C(h) = T \cdot x_t(h) + V \cdot 2x_v(h)$$

Let's now find $x_t(h)$ and $x_v(h)$ as a function of h. For x_v we have $x_v(h) = S \cos\alpha$. Combining it with $h = S \sin\alpha$ we obtain

$$x_v(h) = h/\tan\alpha.$$

To find x_t we note that $B = x_t + 2x_v$. Since $B = 2L/\tan\alpha$ we obtain

$$x_t(h) = 2(L-h)/\tan\alpha$$

The total cost is

$$C(h) = \frac{2}{\tan\alpha}(T(L-h) + Vh).$$

From the formula it is clear that the height that maximises $C(h)$ is $h = 0$ if $T < V$, while it is $h \geq L$ if $T > V$. In other words, if it is cheaper to build tunnels the railway should be one long tunnel underneath all mountains, otherwise it should be one long viaduct above all mountains.

Railway line 2

(by Filippo Simini)

In the second scenario the cost to build a viaduct depends both on the length of the viaduct and on the height of its pillars, because more material is needed to build higher viaducts. So in this case we can assume that the cost is proportional to the area between the viaduct and the mountain.

Can you find the optimal height of the railway line in this second scenario?

Solution

In the second scenario, the cost of a viaduct is proportional to the area between the viaduct and the mountain, $2a_v$, where a_v is the area on each side of the mountain:

$$C_v(h) = V \cdot 2a_v = V \cdot 2\frac{h \cdot x_v(h)}{2},$$

where V is now the cost of the viaduct per unit area. Combining this with the cost of tunnels, which is the same derived in Question 1, we obtain the total cost:

$$C(h) = \frac{2}{\tan\alpha}(T(L-h) + Vh^2/2).$$

To find the maximum of this parabola we can differentiate $C(h)$ and find the h^* such that $C'(h^*) = 0$:

$$\frac{d}{dh}C(h)\mid_{h^*} = -T + Vh^* = 0 \qquad \Longleftrightarrow \qquad h^* = T/V.$$

Actually, given that the height must be smaller than L in order to minimise the cost, we have $h^* = \min(L, T/V)$. The minimum cost is then

$$C(h^* = L) = \frac{2}{\tan\alpha}(\frac{L^2V}{2})$$

if $T/V \geq L$, and

$$C(h^* = T/V) = \frac{2T}{\tan\alpha}(L - \frac{T}{2V}).$$

otherwise.

Note that if $T/V < L$ the optimum h^* is independent of α and L. This means that we can relax the assumptions of having mountains with the same slope and the same height (providing that the shortest mountain's height is larger than T/V) and the optimal railway's height will always be given by $h^* = T/V$.

Watering a sports field

(by Martin Homer)

This is an open-ended challenge, there are no right or wrong answers, it can be used as a brainstorming exercise or carried out over a prolonged period of time, like a project.

During a long, hot summer, cricket fields need regular watering. To make sure the grass is in optimal condition, your local club wants to use an automatic pop-up sprinkler system. But how should the sprinklers be laid out to ensure the best possible watering for least cost and inconvenience?

Ideally, every bit of the field would be watered equally, and the club particularly wants to avoid over- and under-watering the grass. Wasting water would be bad, too.

Most pop-up sprinklers can water either a whole circle, or a segment of a circle. You can assume that the club opts for a sprinkler with an 11m coverage radius.

Solution

First of all, what size and shape is your cricket field? Choose something simple to start with, and you can always make it more complicated later. There aren't any rules to precisely define the size of a field, but you should be able to find a good starting point online. It's probably easiest to start with a circular field.

What different layouts for the sprinklers could you choose? It's probably best to start with regular ones first. How many do you need to guarantee the whole field is watered?

To make the problem easier, it might be worth thinking about shapes that tessalate, in a way that circles don't. Squares and hexagons are an obvious starting point, since both tessalate. There are two sizes that might be worth considering:

1. the biggest square/hexagon that fits inside the sprinkler circle (*circumcircle*),
2. the smallest square/hexagon that fits outside the sprinkler circle (*incircle*).

How big do they need to be in each case? And how many do you need to cover the field?

The first guarantees that the whole cricket field will be watered (but with overlaps), while the second will leave gaps between the watered circles. Neither is ideal, though: overlaps result in over-watering, and wasted water, while gaps mean that some grass will not be watered at all. Can you measure the overlaps/gaps between the watered circles? How much area is overwatered/missed? It might help to work out the areas of the circle, inscribed, and escribed squares and hexagons.

Is there a compromise solution between the fully covered solution, and the one with gaps?

What other factors might come into play in practice?

Car Parking

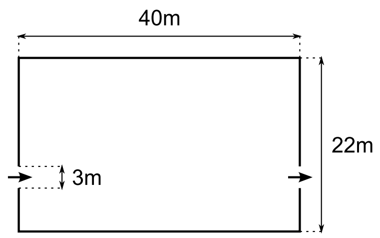
(by Martin Homer)

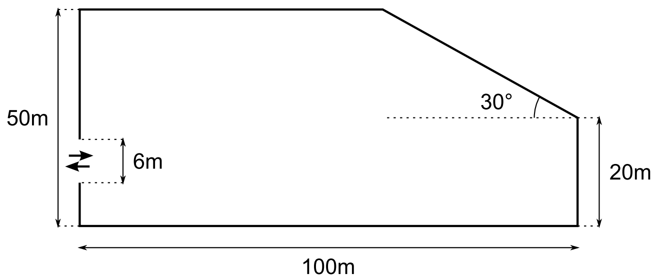
This is an open-ended challenge, there are no right or wrong answers, it can be used as a brainstorming exercise or carried out over a prolonged period of time, like a project.

Land in cities is in short supply, and so the value of parking spaces is very high; perhaps as much as £100,000 per space in central London. So when making a new car park, it is important to design them so as to fit in as many parking spaces as possible.

Your challenge is to find a way of fitting as many parking spaces as possible into a given area, whilst maintaining the overall usability of the car park.

Two areas to try are shown below. You can move the entrances and exits, provided they stay on the same side of the car park, as shown in the figure.





Solution

There are two constraints on how the spaces (known as bays) in a car park should be arranged: **One:** The bays must be big enough to accommodate the type of vehicles that will park in them. Some vehicles are bigger than others but in practice a design standard has been adopted which recommends that each bay is at least 4.8m×2.4m. Note 2.4m is a little wider than a typical car, but the surplus allows some room to open the doors etc. **Two:** There needs to be sufficient room for a car to manoeuvre into each bay and it must be possible to drive between the entrance / exit and each bay, even when every other bay is full. This concerns the manoeuvrability of vehicles and this is sometimes understood in terms of their turning circles, see e.g. `http://en.wikipedia.org/wiki/Turning_radius`.

Common sense suggests that a 90° arrangement requires a wider aisle to allow cars sufficient space to turn into the bays than angled bays, as the latter require less space for turning. To begin with, it is easiest to use a single, unform, size of parking bay; adding different sizes (e.g. disabled, parent and child) bays makes the problem significantly harder. Possible approaches to consider:

- What is the *maximum* number of bays you can fit into a given area (without aisles, as in)?
- Can you draw inspiration from the layout and dimensions of existing car parks?
- What are the typical dimensions of different makes of car? What are their turning circles?
- What is the area swept out by a car manoeuvering into a bay, and how is this related to the turning circle, and the width of an aisle?
- Does it help to experiment with setting the bays at different angles to the aisles? Thinking about an infinite car park, without boundaries, might help simplify this calculation.
- When designing the layout, should one begin by deciding where to put bays, or alternatively begin by placing the aisles?
- How do you demonstrate that your layouts are valid? — i.e., that it is possible to access all the bays from the entrance/exit?
- Are some shapes easier to pack than others (a lower total area per bay)?
- How do you know if you have achieved the best possible layout?