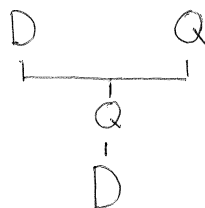


A secret of bees 1

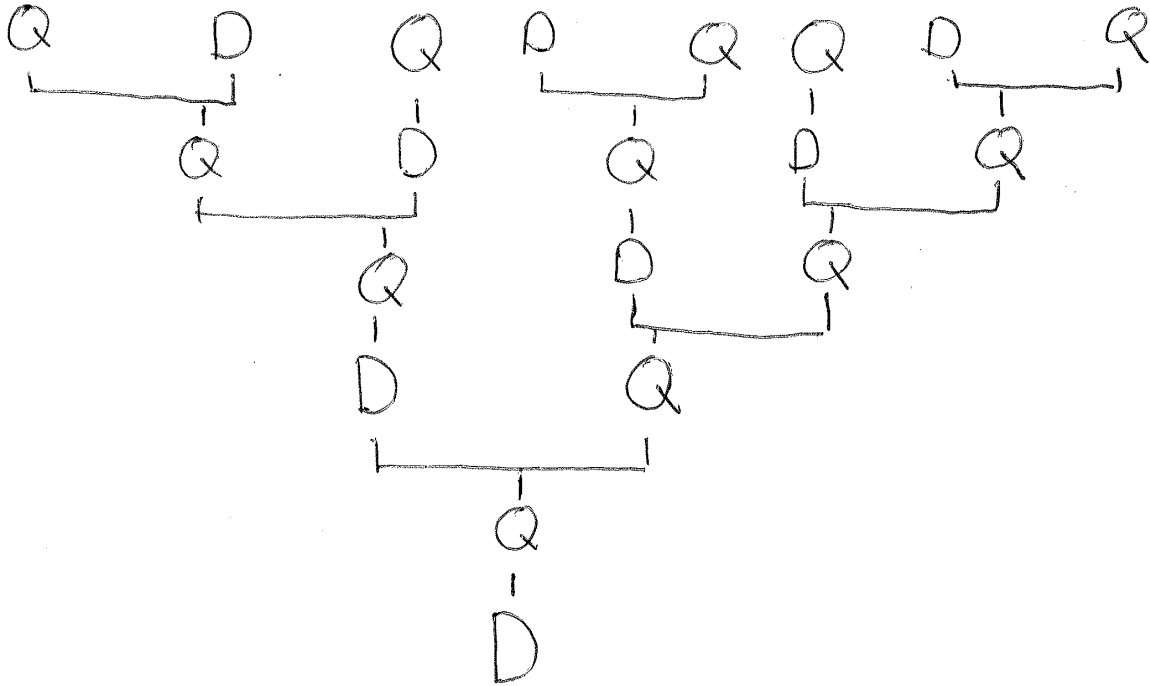
(by Thilo Gross)

Bees have interesting family trees. A male bee, a so-called drone (D) only has one parent, who is a queen (Q). A queen has two parents, a queen and a drone. So a drone has only one parent, and only 2 grandparents. Got it? Continue the family tree that I have started to draw below for at least 3 more generations.



Solution

The finished family tree should look like this.



Note that the sequence of the number of bees in each layer should be the Fibonacci sequence 1,1,2,3,5,8,13,21,...

A secret of bees 2

(by Thilo Gross)

The ancestral trees of bees hide a secret. The first step to discover it is to write down the number of queens, drones, and the total number of bees for each generation.

Use the family tree that you have drawn to fill in the next 3 lines of this table

| Gen. n | Queens Q_n | Drones D_n | Total T_n |
|----------|--------------|--------------|-------------|
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| 9 | | | |
| 10 | | | |
| 11 | | | |
| 12 | | | |

Do you see a pattern? Can you use it to fill in the remaining lines?

Solution

The pattern that can be discovered here is that each column of the table follows a Fibonacci sequence. Even if students not know it by name the underlying pattern is easy to spot.

The finished table should look like this

| Gen. n | Queens Q_n | Drones D_n | Total T_n |
|----------|--------------|--------------|-------------|
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | 2 | 1 | 3 |
| 4 | 3 | 2 | 5 |
| 5 | 5 | 3 | 8 |
| 6 | 8 | 5 | 13 |
| 7 | 13 | 8 | 21 |
| 8 | 21 | 13 | 34 |
| 9 | 34 | 21 | 55 |
| 10 | 55 | 34 | 89 |
| 11 | 89 | 55 | 144 |
| 12 | 144 | 89 | 233 |

A secret of bees 3

(by Thilo Gross)

Let's write some equations! Suppose in generation n the number of drones is D_n , and the number of queens is Q_n .

We know that every bee has a queen as a parent, and queens have an additional drone parent.

Let's compute the numbers in the generation before n that's the generation $n + 1$. According to the above, the number of queens and drones in generation $n + 1$ is

$$Q_{n+1} =$$

$$D_{n+1} =$$

Fill in the right-hand-side of these equations, using only D_n and Q_n and whatever arithmetical symbols you need.

Solution

One queen for every bee in the previous generation means

$$Q_{n+1} = D_n + Q_n$$

and one drone for every queen in the previous generation means

$$D_{n+1} = Q_n$$

A secret of bees 4

(by Thilo Gross)

When you filled the ancestry table you may have discovered the rule

$$Q_{n+1} = Q_n + Q_{n-1}$$

You can check that it holds in your table, but we really want to prove it. On the previous sheet we discovered

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

can you use these to prove the rule for queens above?

Can you prove a similar rule for drones and the total number of bees?

Don't forget that you can shift the indices, and $T_n = Q_n + B_n$.

Solution

If we shift the second of observation from the previous sheet down we get

$$D_n = Q_{n-1}$$

So the first observation can be rewritten as

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ &= Q_n + Q_{n-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} D_{n+1} &= Q_n \\ &= Q_{n-1} + D_{n-1} \\ &= D_n + D_{n-1}, \end{aligned}$$

using the second observation, first observation down-shifted, second observation down-shifted, in this order.

Finally,

$$\begin{aligned} T_{n+1} &= Q_{n+1} + D_{n+1} \\ &= Q_n + Q_{n-1} + D_n + D_{n-1} \\ &= (Q_n + D_n) + (Q_{n-1} + D_{n-1}) \\ &= T_n + T_{n-1} \end{aligned}$$

A secret of bees 5

(by Thilo Gross)

Using the very convenient formula

$$T_{n+1} = T_n + T_{n-1}$$

we can compute the number of bees for some more generations while we are on it let us also compute the factor by which the number of ancestors increases in every generation. For humans that would obviously be 2 but for bees the ratio is more intriguing (find about 6 digits after the decimal point)

| Gen. n | Bees T_n | Ratio T_n/T_{n-1} |
|----------|------------|---------------------|
| ⋮ | ⋮ | ⋮ |
| 10 | 89 | ⋮ |
| 11 | 144 | 1.617977 |
| 12 | 233 | 1.618056 |
| 13 | | |
| 14 | | |
| 15 | | |
| 16 | | |
| 17 | | |
| 18 | | |

Solution

The finished table is

| Gen. n | Bees T_n | Ratio T_n/T_{n-1} |
|----------|------------|---------------------|
| \vdots | \vdots | \vdots |
| 10 | 89 | \vdots |
| 11 | 144 | 1.618055 |
| 12 | 233 | 1.618026 |
| 13 | 377 | 1.618026 |
| 14 | 610 | 1.618037 |
| 15 | 987 | 1.618033 |
| 16 | 1597 | 1.618034 |
| 17 | 2584 | 1.618034 |
| 18 | 4181 | 1.618034 |

A secret of bees 6

(by Thilo Gross)

We are getting closer to what the bees are hiding. On the last sheet we discovered the ratio by which the number of bees increases for sufficiently large n . Let's call this ratio f and also compute its inverse

$$f = 1.618034\dots$$

$$1/f =$$

Notice anything odd? (Perhaps you would like more digits:
 $f = 1.6180339887498948482045868$ what's $1/f$ now?)

If you noted some curious property of f , let's write it as an equation

$$1/f =$$

this time we want a mathematical expression on the right hand side, not digits.

Can you use the equation to compute the exact (!) value of f ?

Solution

We notice

$$\begin{aligned} f &= 1.618034\dots \\ 1/f &= 0.618034\dots \end{aligned}$$

or with more digits

$$\begin{aligned} f &= 1.6180339887498948482045868\dots \\ 1/f &= 0.6180339887498948482045868\dots \end{aligned}$$

This is of course the golden ratio. We observe that all digits of f and $1/f$ are identical except the first. So we can write

$$1/f = f - 1$$

We can now compute f exactly

$$\begin{aligned} 1/f &= f - 1 \\ 1 &= f^2 - f \\ 1 &= (f - 1/2)^2 - 1/4 \\ 5/4 &= (f - 1/2)^2 f = 1/2 \pm \sqrt{5/4} \end{aligned}$$

The larger solution is actually f and the smaller one $-1/f$, so

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}$$

A secret of bees 7

(by Thilo Gross)

Congratulations, you have made it through this set of sheets. We have discovered the secret of bees. The number of ancestors of a bee follows the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

this is the famous Fibonacci sequence. In the long run the number of bees increases by a factor

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}.$$

in each generation. This is the famous golden ratio. A magical number that that appears all over mathematics. It has other surprising properties, for instance, by some measure, it is the most irrational of all numbers.

Here is a little bonus. On the previous sheet you may have noticed that $1/f = 1 - f$ has a second solution which is actually $-1/f$, this is the golden ratio's little brother g , note

$$\begin{aligned} f &= 1/2 + \sqrt{5/4} \\ g &= 1/2 - \sqrt{5/4} \end{aligned}$$

These two numbers work great as a team for instance they can give you the number of bees in generation n

$$T_n = \frac{f^n + g^n}{\sqrt{5}}$$

you can check that this works exactly, but why it works is a different story.

Further information

There is not really any open question, but it is good to encourage the students to actually check the formula. Seeing the integer Fibonacci numbers emerge from a sum of square roots is impressive.

The way to derive the formula is actually to write the the system

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

in matrix form and solve it by an eigendecomposition. Alternatively, the recursion $T_{n+1} = T_n + T_{n-1}$ can be solved using generating functions.

The golden ratio is the most irrational number in the sense that its continued fractions expansion is

$$f = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$$

Because all the coefficients are 1 this number is maximally far from any rational number.

Also consider mentioning some other places where the golden ratio appears. Plenty of material on this can be found on the internet.

When discussing this exercise you may want to emphasize how progress was made by exploration. In the first 3 sheets we drew the tree, and filled the table to try and spot a pattern. Finding this pattern beforehand meant that we already knew the result we were trying to derive on sheet 4. Likewise we were able to find the exact value of the golden mean by spotting a pattern on sheet 6. This illustrates that very pedestrian exploration (“What happens when I compute the inverse?”, “Lets write out a the numbers for a few simple cases”) often puts us on the right track to the solution.