Ball bouncing 1

(by Alan Champneys)

What happens if you put a tennis ball on top of a basketball and then drop them both? The result is surprising. To explore this, start by finding out how to mathematically model bouncing balls:

Newton's law of bouncing*

$$[\text{ speed afterwards }] = e \cdot [\text{ speed before }]$$

e = 'coefficient of bouncing' (a property of the ball): $0 \le e \le 1$
$e = 1$ – a perfectly elastic ball; $e = 0$ – a squashy tomato;
$e = 0.8$ – a reasonable value for a well pumped ball

Suppose a basketball has $e = 0.8$ and is travelling at $v^{\mathrm{b}} = -5$ m/s as it hits the floor (here, negative velocity signifies going down, and a and b indicate before and after). How fast will it be going upwards as it lifts off?

$$v^{\mathrm{a}} = -ev^{\mathrm{b}} =$$

Can you think of any physical situation in which $e > 1$?

* actually Newton's law of restitution.

Solution

We suggest that you do the experiment. If you do, try to get the tennis ball directly over the center of mass of the basketball. Alternatively you can show this "Physics Girl" video on You Tube

`www.youtube.com/watch?v=2UHS883_P60`

(it's best to stop after about 1:30)

The answer to the question is that the ball would lift off with speed

$$v^{\mathrm{a}} = e \cdot 5 \text{ m/s} \ = 0.8 \cdot 5 \text{ m/s} \ = 4 \text{ m/s}$$

Note that the sign has changed as the ball is now travelling up.

Values $e < 1$ account for loss of energy in the bounce: An example with $e > 1$ would have to include a source of energy, e.g. a ball hitting an active bumper in a pinball machine.

Ball bouncing 2

(by Alan Champneys)

Let's check if our assumption that the ball hits the floor at 5 m/s is about right. A very useful principle is

Conservation of energy

$$\text{kinetic energy} = \tfrac{1}{2}Mv^2; \qquad \text{potential energy} = Mgh$$

$$M = \text{mass of object}; \qquad v = \text{its speed};$$
$$h = \text{height above floor}; \qquad g = 9.81 \text{ m/s}^2.$$

Make reasonable assumptions about M and the height h_b from which the ball is dropped. The kinetic energy of the ball when it hits the floor must equal the potential energy that it had in the beginning, so with how much energy does the ball hit the floor? (Hint: 1 Joule = 1000 gm^2/s^2)

$$E_\mathrm{b} =$$

What velocity does the balls kinetic energy correspond to? Was our assumption of -5 m/s approximately right?

$$v^\mathrm{b} =$$

Solution

When we release the ball it has the energy

$$E_{\mathrm{b}} = Mgh_{\mathrm{b}}$$

This must equal the kinetic energy on impact

$$E_{\mathrm{b}} = \frac{1}{2}M(v^{\mathrm{b}})^2$$

So the velocity of the ball on impact is

$$v^{\mathrm{b}} = -\sqrt{\frac{2E_{\mathrm{b}}}{M}} = -\sqrt{\frac{2Mgh_{\mathrm{b}}}{M}} = -\sqrt{2gh_{\mathrm{b}}}$$

For example for a 600g basketball released from 1m height we get

$$E_{\mathrm{b}} = 600\ \mathrm{g} \cdot 9.8\ \mathrm{m/s^2} \cdot 1\ \mathrm{m} = 5880\ \mathrm{gm^2/s^2} = 5.88\ \mathrm{J}$$

$$v^{\mathrm{b}} = -\sqrt{2 \cdot 9.8\ \mathrm{m/s^2} \cdot 1\ \mathrm{m}} = -\sqrt{19.6\ \mathrm{m/s^2}} \approx -4.427\ \mathrm{m/s}$$

If we dropped it from just a bit higher than 1 m we would reach the -5 m/s.

Ball bouncing 3

(by Alan Champneys)

We noticed on the first sheet that some velocity is lost as the ball hits the floor. How fast is your ball on lift off? And what is its kinetic energy?

$$v^{\mathrm{a}} =$$

$$E_{\mathrm{a}} =$$

As the ball rises again its kinetic energy gets converted into potential energy. Compute the height, h_{a}, that it can still reach after the bounce.

$$h_{\mathrm{a}} =$$

Now here is a challenge: Can you find a general formula for $h_{\mathrm{a}}/h_{\mathrm{b}}$?

$$\frac{h_{\mathrm{a}}}{h_{\mathrm{b}}} =$$

Solution

In the bounce the velocity is multiplied by e. Hence, the velocity immediately after the bounce is

$$v^{\mathrm{a}} = -ev^{\mathrm{b}} = e\sqrt{2gh_{\mathrm{b}}}$$

which is less than v^{b} due to $e < 1$. At this velocity the kinetic energy is

$$E_{\mathrm{a}} = \frac{1}{2}M(v^{\mathrm{a}})^2 = \frac{1}{2}M\left(e\sqrt{2gh_{\mathrm{b}}}\right)^2 = e^2Mgh_{\mathrm{b}},$$

at the highest point h_{a} this energy has been transformed into potential energy Mgh_{a}. Equating this potential energy with E_{a}, the kinetic energy after impact yields

$$Mgh_{\mathrm{a}} = e^2Mgh_{\mathrm{b}}$$

and hence

$$\frac{h_{\mathrm{a}}}{h_{\mathrm{b}}} = e^2$$

Note that the height is not proportional to the coefficient of restitution, e, but its square.

In the example of basketball with $M = 600$ g and $e = 0.8$ the velocity after impact is

$$v^{\mathrm{a}} = ev = 0.8 \cdot 4.427\ \mathrm{m/s} = 3.542\ \mathrm{m/s}$$

This means the ball has a kinetic energy of

$$E_{\mathrm{a}} = \frac{1}{2}M(v^{\mathrm{a}})^2 = 0.5 \cdot 600\ \mathrm{g}\ \cdot (3.542\ \mathrm{m/s})^2 = 3.762\ \mathrm{J}$$

With this energy it can reach a height of

$$h_{\mathrm{a}} = \frac{E_{\mathrm{a}}}{Mg} = \frac{3762\ \mathrm{gm^2/s^2}}{600\ \mathrm{g}\ \cdot 9.8\ \mathrm{m/s^2}} \approx 0.64\ \mathrm{m}$$

The ball bounces 64 cm high. This is consistent with the general formula that we have already derived

$$0.64\ \mathrm{m} = (0.8)^2 \cdot 1\ \mathrm{m}.$$

Ball bouncing 4

(by Alan Champneys)

Now let's return to the case where a tennis ball and a basketball are dropped together. For simplicity assume that both balls reach a velocity of -5m /s before the basketball hits the floor.

Here is a trick: Assume basketball first bounces off the floor, then, rising at velocity v_1^{a}. An instant later it collides with the tennis ball, which is still moving downward at velocity v_2^{b}. What is the relative speed at which the balls now approach each other? (it's not 10 m/s)

$$v_1^{\mathrm{a}} - v_2^{\mathrm{b}} =$$

Newton's law of bouncing for 2 objects

$$[\text{ speed of separation }] = e \cdot [\text{ speed of approach }]$$

Using the law above, and $e = 0.8$ for the collision between the two balls, find the speed at which they separate.

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} =$$

Here we have used the c to indicate the velocities after the balls collide.

Solution

From sheet 1, we know that after lifting off from the floor, the basketball is going upwards at 4 m/s. The tennis ball is still falling at -5 m/s so their speed of approach is

$$v_1^{\mathrm{a}} - v_2^{\mathrm{b}} = 9 \text{ m/s}$$

Using the law and $e = 0.8$, the speed of separation after the collision is

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} = 9 \text{ m/s} \cdot 0.8 = 7.2 \text{ m/s}$$

This solution assumes that the two collisions are independent, an unrealistic assumption unless the tennis ball and the basketball are separated by about 5cm when the basketball hits the floor. In reality all bouncing collisions take a finite amount of time and if the basketball and tennis ball are less than 5 cm apart, the tennis ball hits the basketball before it has lifted off from the ground.

Yet, it has been discovered that as long as the two balls are at least a few millimeters apart, the assumption gives accurate predictions. It is still an open question of scientific research why this is so. See for example

`dx.doi.org/10.1098/rspa.2015.0286`

Ball bouncing 5

(by Alan Champneys)

To solve for the speed v_1^{c} of the basketball and v_2^{c} of the tennis ball after the collision we have to use another principle:

Conservation of momentum

$$\text{Total momentum} = \sum[\text{mass} \cdot \text{velocity of balls}]$$

where $\sum$ means 'sum over all balls' and
Total momentum before collision = Total momentum after collision

Make a reasonable assumption about the masses of the basketball, M_1, and the tennis ball, M_2, and find the total momentum just before the basketball and tennis ball collide (after the basketball lifts off from the floor)

$$P = M_1 v_1^{\mathrm{a}} + M_2 v_2^{\mathrm{b}} =$$

Since the momentum is conserved in the collision

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}}$$

is also true.

Solution

A basketball is about 10-12 times heavier than a tennis ball, make sure that the students use a sufficiently high ratio of masses or the results will be less impressive.

For a 600g basketball colliding with a 50g tennis ball. The total momentum is

$$P = 4\ \mathrm{m/s}\ \cdot 600\ \mathrm{g}\ - 5\ \mathrm{m/s}\ \cdot 60\ \mathrm{g}\ = 2100\ \mathrm{gm/s}\ = M_1 v_1^c + M_2 v_2^c$$

Ball bouncing 6

(by Alan Champneys)

We just discovered

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}},$$

where we know P, M_1, and M_2 already. So there are only two unknowns, the velocities after the collision. We also know from sheet 4 that

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} = 7.2 \text{ m/s}$$

We have two equations for two unknowns! Substitute all the values and compute the velocity of the tennis ball after the collision

$$v_2^{\mathrm{c}} =$$

This velocity should take the tennis ball to a height h_{c}, higher than the height from which it was released. Use your insights from sheet 3 to determine the factor $h_{\mathrm{c}}/h_{\mathrm{b}}$. Furthermore, by what factor, $h_{\mathrm{c}}/h_{\mathrm{a}}$ is this bounce higher than the height that would be reached if we dropped the tennis ball without the basketball?

Solution

For our example the two equations read

$$\begin{aligned} v_2^c - v_1^c &= 7.2 \text{ m/s} \\ 600 \text{ g } \cdot v_1^c + 50 \text{ g } \cdot v_2^c &= 2100 \text{ gm/s} \end{aligned}$$

Multiplying the first equation with 600 g gives us

$$\begin{aligned} -600 \text{ g } \cdot v_1^c + 600 \text{ g } \cdot v_2^c &= 4320 \text{ gm/s} \\ 600 \text{ g } \cdot v_1^c + 50 \text{ g } \cdot v_2^c &= 2100 \text{ gm/s} \end{aligned}$$

adding the the two equations yields

$$650 \text{ g } \cdot v_2^c = 6420 \text{ gm/s}$$

where the minus signifies a change in direction. We are only interested in the absolute value. Dividing by 650 g, we find

$$v_2^c = 9.88 \text{ m/s}$$

this is almost twice our initial velocity of 5 m/s.

From sheet 3 we know that the height to which the ball can climb is proportional to the velocity squared. So this means our tennis ball can reach about 4 times the height from which we released it. Without the basketball it would only have reached $e^2 = 0.64$ times the release height. So we can say the bounce from the basketball increases the height by a factor $4/0.64 = 6.25$.

(Make sure the students notice the quadratic increase in the height)

Ball bouncing 7

(by Alan Champneys)

Congratulations you have explained what happens in the two-ball bounce.

Of course there are some extensions to think about. What would happen if we used two tennis balls instead of a tennis ball and a basketball? What about if the tennis ball and the basketball were reversed? What about three balls on top of each other (of successively decreasing mass)?

Very similar principles apply also in horizontal collisions, this is important for instance in nuclear physics where particles become very fast due to collisions with heavier particles, or in traffic accidents, where heavy vehicles can transfer a lot of energy to lighter ones.

Further information

Actually there are many hidden assumptions in this model. More information can be found at

"*The two-ball bounce problem.*" *Proc. Roy. Soc. Lond. A.* 2015
dx.doi.org/10.1098/rspa.2015.0286

The principles used in this exercise are of broad importance for many applications. In engineering, the dynamics of bouncing and rattling is a fundamental area of active research. For example, car manufacturers devote whole teams of people to precisely model the whole vehicle to eliminate what is known in the industry as 'buzz, squeak and rattle'. Such phenomena usually come about through parts of the car's body or interior panelling impacting with other parts at certain frequency. This is not usually dangerous, but leads to commercially important customer satisfaction issues.