Multilevel Models for Binary Responses

Preliminaries

Consider a 2-level hierarchical structure. Use ‘group’ as a general term for a level 2 unit (e.g. area, school).

Notation

- n is total number of individuals (level 1 units)
- J is number of groups (level 2 units)
- n_j is number of individuals in group j
- y_{ij} is binary response for individual i in group j
- x_{ij} is an individual-level predictor

Generalised Linear Random Intercept Model

Recall model for continuous y

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j + e_{ij}$$

$$u_j \sim N(0, \sigma_u^2) \;\text{ and }\; e_{ij} \sim N(0, \sigma_e^2)$$

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Model for binary y

For binary response $\mathrm{E}(y_{ij}) = \pi_{ij} = \Pr(y_{ij} = 1)$, and model is

$$F^{-1}(\pi_{ij}) = \beta_0 + \beta_1 x_{ij} + u_j$$

F^{-1} the link function, e.g. logit, probit clog-log

Random Intercept Logit Model: Interpretation

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- $\exp(\beta_1)$ is an odds ratio, comparing odds for individuals spaced 1-unit apart on x but in the same group

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- As for continuous y, we can obtain estimates and confidence intervals for u_j
- σ_u^2 is the level 2 (residual) variance, or the between-group variance in the log-odds that $y = 1$ after accounting for x

Response Probabilities from Logit Model

Response probability for individual i in group j calculated as

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We can also make predictions for 'ideal' or 'typical' individuals with particular values for x, but we need to decide what to substitute for u_j (discussed later).

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1. Is σ_u^2 significantly different from zero?
2. Does $\hat{\sigma}_u^2 = 0.09$ represent a large state effect?

Testing $\mathbf{H}_0 : \sigma_u^2 = 0$

- Likelihood ratio test. Only available if model estimated via maximum likelihood (not in MLwiN)

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Example

$$\text{Wald statistic} = \left(\frac{\hat{\sigma}_u^2}{\text{se}}\right)^2 = \left(\frac{0.091}{0.023}\right)^2 = 15.65$$

Compare with $\chi_1^2 \rightarrow$ reject H_0 and conclude there are state differences.

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Take p-value/2 because alternative hypothesis is one-sided ($H_A : \sigma_u^2 > 0$)

State Effects on Probability of Voting Bush

Calculate $\hat{\pi}$ for 'average' states ($u = 0$) and for states that are 2 standard deviations above and below the average ($u = \pm 2\hat{\sigma_u}$).

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u = -2\hat{\sigma_u} = -0.603 & \longrightarrow & \hat{\pi} = 0.33 \\
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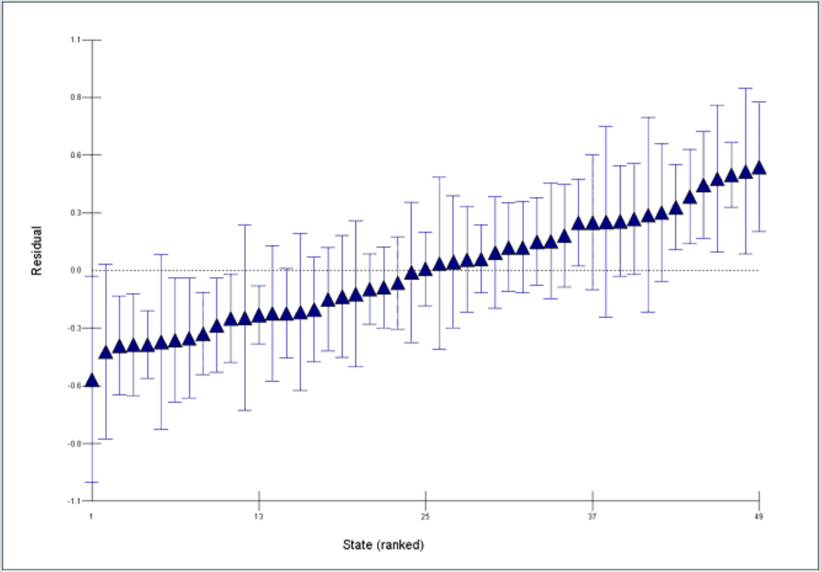
Under a normal distribution assumption, expect 95% of states to have $\hat{\pi}$ within $(0.33,\ 0.62)$.

$\hat{u}_j$ with 95% Confidence Intervals for u_j



Adding Income as a Predictor

x_{ij} is household annual income (grouped into 9 categories), centred at sample mean of 5.23

Parameter	Estimate	Standard error
β_0 (constant)	-0.099	0.056
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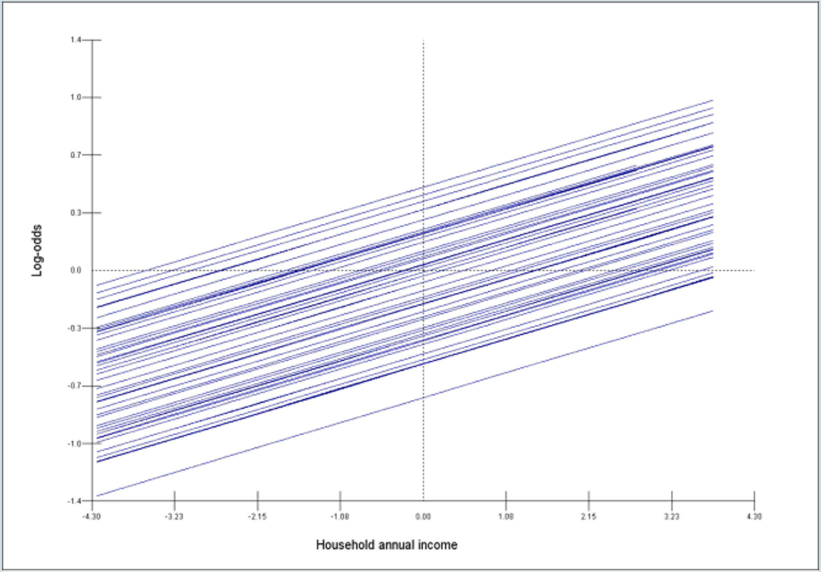
- -0.099 is the log-odds of voting Bush for household of mean income living in an ‘average’ state
- 0.140 is the effect on the log-odds of a 1-category increase in income
- expect odds of voting Bush to be $\exp(8 \times 0.14) = 3.1$ times higher for an individual in the highest income band than for an individual in the same state but in the lowest income band

Prediction Lines by State: Random Intercepts



Latent Variable Representation

As in the single-level case, consider a latent continuous variable y^* that underlines observed binary y such that:

$$y_{ij} = \begin{cases} 1 & \text{if } y_{ij}^* \geq 0 \\ 0 & \text{if } y_{ij}^* < 0 \end{cases}$$

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So the level 1 residual variance, $\text{var}(e^*_{ij})$, is fixed.

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Adding random effects has increased the residual variance
$\rightarrow$ scale of y^* stretched out
$\rightarrow$ β_0 and β_1 increase in absolute value.

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NOTE: Adding random effects to a continuous response model does not 'scale up' coefficients because the level 1 variance is not fixed and so: $\text{var}(e_i) \simeq \text{var}(u_j) + \text{var}(e_{ij})$

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Variable	β^{SL}	β^{RI}	β^{RI}/β^{SL}
Constant	0.221	0.257	1.163
x_1	0.430	0.519	1.207
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In practice, RI:SL ratio for a given x may be quite different from that expected if distribution of x differs across level 2 units.

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- $\rightarrow$ increase in absolute value of coefficients of other variables

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In voting intentions example, $\hat{\sigma}_u^2$=0.125, so VPC=0.037. Adjusting for income, 4% of the remaining variance in the propensity to vote Bush is attributable to between-state variation.

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1. Generalised linear model specifying relationship between π_{ij} and x_{ij}
2. Specification of structure of correlations between pairs of individuals in the same group
    - Exchangeable - equal correlation between every pair (as in random intercept model)
    - Autocorrelation - used for longitudinal data where correlation a function of time between measures
    - Unstructured - all pairwise correlations estimated

Estimated using Generalised Estimating Equations (GEE)

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- Level 2 residuals $\hat{u}_j$ interpreted as group effects
- Can allow between-group variance to depend on x

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$$\text{logit}(\pi_{ij}) = \beta_0^{PA} + \beta_1^{PA} x_{ij}$$

Plus specification of structure of within-cluster covariance matrix

Interpretation of CS and PA Effects

Cluster-specific

- β_1^{CS} is the effect of a 1-unit change in x on the log-odds that $y = 1$ for a given cluster, i.e. holding constant (or conditioning on) cluster-specific unobservables

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Population-averaged

- β_1^{PA} is the effect of a 1-unit change in x on the log-odds that $y = 1$ in the study population, i.e. averaging over cluster-specific unobservables

Example: PA vs. CS Interpretation (1)

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y_{ij} indicates whether patient j has an adverse reaction at occasion i to (time-varying) dose x_{ij}.

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For a level 2 variable, β_2^{PA} may be of more interest.

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- When there is no clustering, $\sigma_u^2 = 0$ and $\beta^{CS} = \beta^{PA}$. Coefficients move further apart as σ_u^2 increases
- Note that marginal models can also be specified for continuous y, but in that case CS and PA coefficients are equal

Predictions from a Multilevel Model

Response probability for individual i in group j calculated as

$$\pi_{ij} = \frac{\exp(\beta_0 + \beta_1 x_{ij} + u_j)}{1 + \exp(\beta_0 + \beta_1 x_{ij} + u_j)}$$

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Rather than calculating probabilities for each individual, however, we often want predictions for specific values of x. But what do we substitute for u_j?

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2. Integrate out u_j to obtain an expression for mean π that does not involve u. Leads to probabilities that have a PA interpretation, but requires some approximation.
3. Average over simulated values of u_j. Also gives PA probabilities, but easier to implement. Now available in MLwiN.

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4. Repeat 1-3 for different value of x

Predicted Probabilities for Voting Bush

	Random intercept model		
	Method 1	Method 3	Marginal model
Household income			
Low	0.374	0.378	0.377
Medium	0.444	0.446	0.445
High	0.564	0.564	0.562
Sex			
Male	0.510	0.510	0.510
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- In longitudinal applications, where $\hat{\sigma}_u^2$ can be large, there will be bigger differences between Methods 1 and 3

Random Slope Logit Model

So far we have allowed π_{ij} to vary from group to group by including a group-level random component in the intercept: $\beta_{0j} = \beta_0 + u_{0j}$.

BUT we have assumed the effect of any predictor x is the same in each group. We now consider a random slope model in which the slope of x (β_1) is replaced by $\beta_{1j} = \beta_1 + u_{1j}$.

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$$\log\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) = \beta_0 + \beta_1 x_{ij} + u_{0j} + u_{1j} x_{ij}$$

where (u_{0j}, u_{1j}) follow a bivariate normal distribution:

$$u_{0j} \sim N(0, \sigma^2_{u0}), \quad u_{1j} \sim N(0, \sigma^2_{u1}), \quad \text{cov}(u_{0j}, u_{1j}) = \sigma_{u01}$$

Example: Random Slope for Income

Extend random intercept logit model for relationship between probability of voting Bush and household income to allow income effect to vary across states.

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	Random int.		Random slope	
Parameter	Est.	se	Est.	se
β_0 (constant)	-0.099	0.056	-0.087	0.057
β_1 (Income, centred)	0.140	0.008	0.145	0.013
State-level random part				
σ^2_{u0} (intercept variance)	0.125	0.031	0.132	0.032
σ^2_{u1} (slope variance)	-	-	0.003	0.001
σ_{u01} (intercept-slope covariance)	-	-	0.018	0.006

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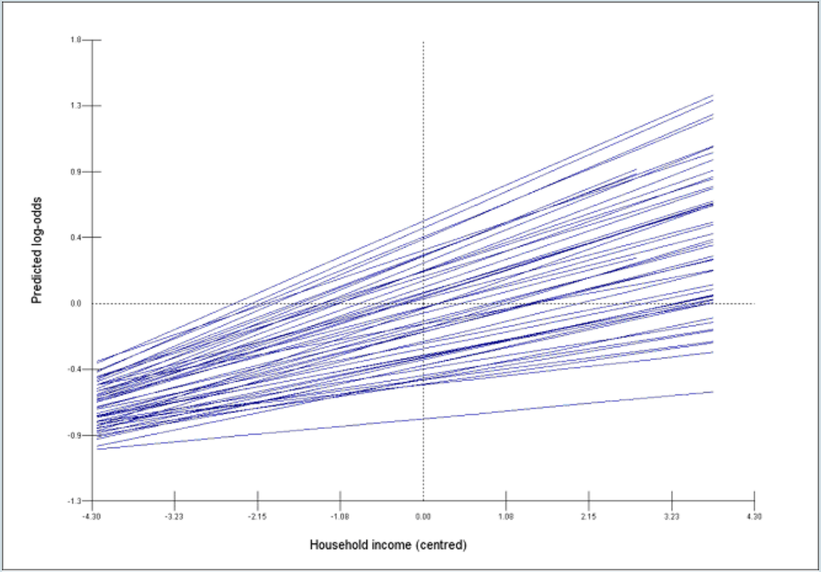
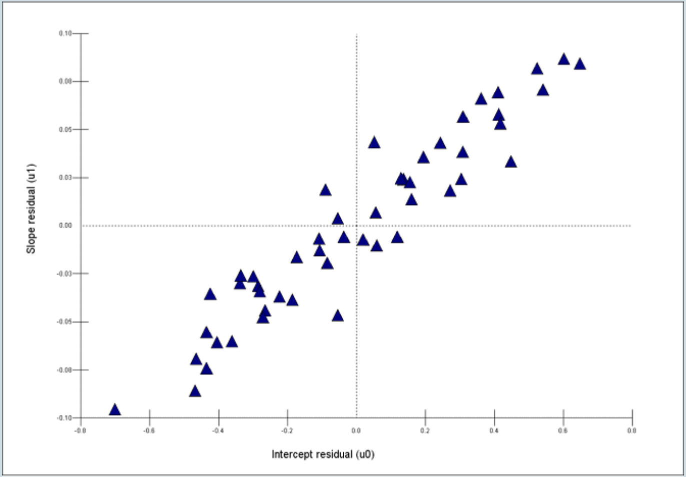
For the income example, Wald $= 9.72$. Comparing with χ^2_2 gives a two-sided p-value of 0.0008

$\implies$ income effect **does** vary across states.

Prediction Lines by State: Random Slopes



Intercept vs. Income Slope Residuals



Bottom left: Washington DC

Top right: Montana and Utah

Level 2 Variance in a Random Slope Model

In a random slope model, the between-group variance is a function of the variable(s) with a random coefficient x:

$$\begin{aligned}\mathrm{var}(u_{0j} + u_{1j}x_{ij}) &= \mathrm{var}(u_{0j}) + 2x_{ij}\mathrm{cov}(u_{0j}, u_{1j}) + x_{ij}^2\mathrm{var}(u_{1j}) \\ &= \sigma_{u0}^2 + 2\sigma_{u01}x_{ij} + \sigma_{u1}^2 x_{ij}^2\end{aligned}$$

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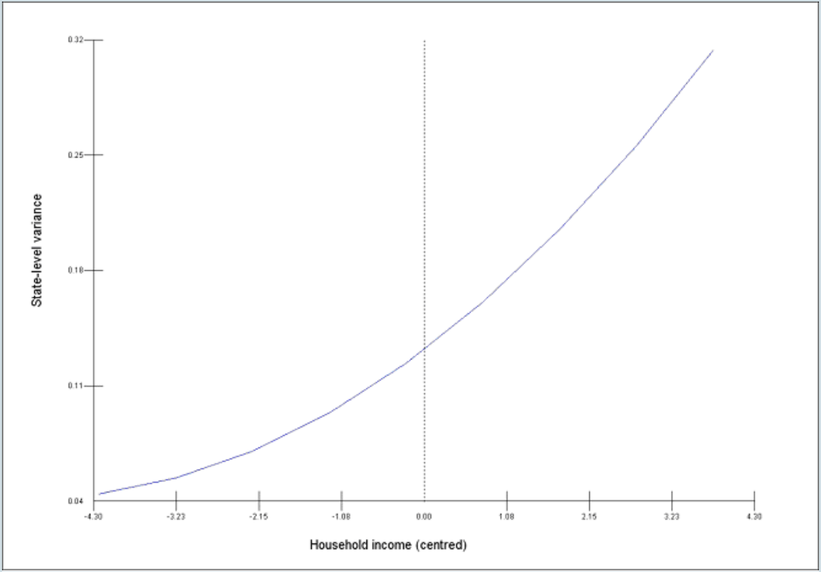
Between-state variance in log-odds of voting Bush

$$0.132 + 0.036\ \textbf{Income} + 0.003\ \textbf{Income}^2$$

Between-State Variance by Income



Adding a Level 2 x: Contextual Effects

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A random intercept logit model with a level 1 variable x_{1ij} and a level 2 variable x_{2j} is:

$$\log\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2j} + u_j$$

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Especially important to use a multilevel model if interested in contextual effects as $\text{se}(\hat{\beta}_2)$ may be severely estimated if a single-level model is used.

Individual and Contextual Effects of Religiosity

Individual religiosity measured by dummy variable for frequency of attendance at religious services (1=weekly or more, 0=other)

State religiosity is proportion of respondents in state who attend a service weekly or more.

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	No contextual effect		Contextual effect	
Variable	Est.	se	Est.	se
Individual religiosity	0.556	0.037	0.543	0.037
State religiosity	-	-	2.151	0.350
Between-state variance	0.083	0.022	0.030	0.010

(Model also includes age, sex, income and marital status.)

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The null hypothesis for a test of a cross-level interaction is $\mathrm{H}_0 : \beta_3 = 0$.

Example of Cross-Level Interaction

Suppose we believe that the effect of individual age on the probability of voting Bush might depend on the conservatism of their state of residence, which we measure by state religiosity.

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Z-ratio for interaction coefficient is $0.043/0.013 = 3.31$ which is highly significant $\implies$ effect of age depends on state religiosity.

Effect of Age by State Religiosity

Age effects on log-odds of voting Bush

Proportion attending services weekly	Age Effect
0.16	$0.012 - (0.043 \times 0.16) = 0.005$
0.30	$0.012 - (0.043 \times 0.30) = -0.0009$
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$\Longrightarrow$ Difference between young and old respondents in voting intentions is greatest in most religious states.

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  - Markov chain Monte Carlo (MCMC) methods (WinBUGS, MLwiN)
- In some situations, different procedures can lead to quite different results

Comparison of Quasi-Likelihood Methods

Rodríguez and Goldman (2001, *J. Roy. Stat. Soc.*) simulated a 3-level data structure with 2449 births (level 1) from 1558 mothers (level 2) in 161 communities (level 3), and one predictor at each level.

Comparison of Quasi-Likelihood Methods

Rodríguez and Goldman (2001, *J. Roy. Stat. Soc.*) simulated a 3-level data structure with 2449 births (level 1) from 1558 mothers (level 2) in 161 communities (level 3), and one predictor at each level.

Results from 100 simulations

Parameter	True value	MQL1	MQL2	PQL2
Child-level x	1	0.74	0.85	0.96
Family-level x	1	0.74	0.86	0.96
Community-level x	1	0.77	0.91	0.96
Random effect st. dev.				
Family	1	0.10	0.28	0.73
Community	1	0.73	0.76	0.93

Comparison of Estimation Procedures

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Random effect standard deviations

	PQL2	PQL1-B	ML	MCMC
Family	1.75	2.69	2.32	2.60
Community	0.84	1.06	1.02	1.13

PQL-B is PQL with a bias correction; ML is maximum likelihood; MCMC is Markov chain Monte Carlo (Gibbs sampling)

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- MCMC methods are flexible and becoming increasingly computationally feasible; the recommended method in MLwiN