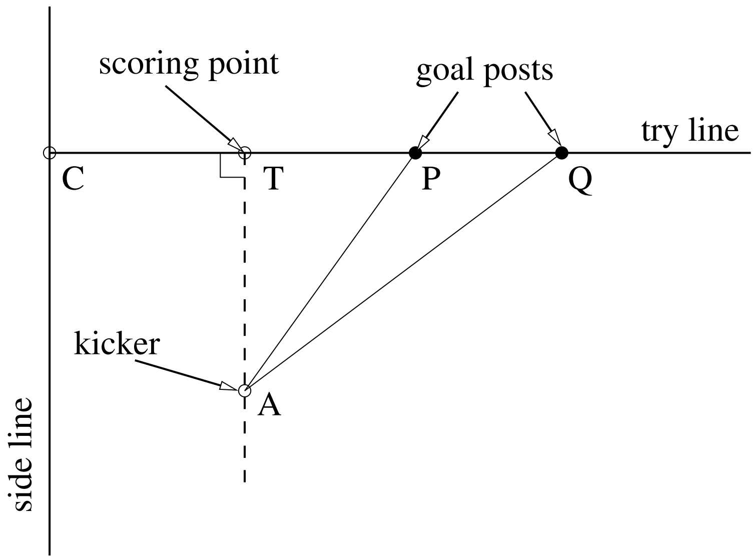
Rugby kicking 1

(by Alan Champneys)

In rugby, when a try is scored, a conversion kick has to be taken form a perpendicular line that intersects the try line where the try was scored (the dashed line in the diagram). The kicker is trying to get the ball through the posts. The kicker is free to choose the point A on the dashed line. But where should A be chosen to maximise the angle <PAQ between the goalposts?



What happens to the angle <PAQ if A is chosen just next to T? What happens to the angle if A is chosen to be at the far end of the pitch?

We are going to find a formula for chosing the optimal point A.

Solution

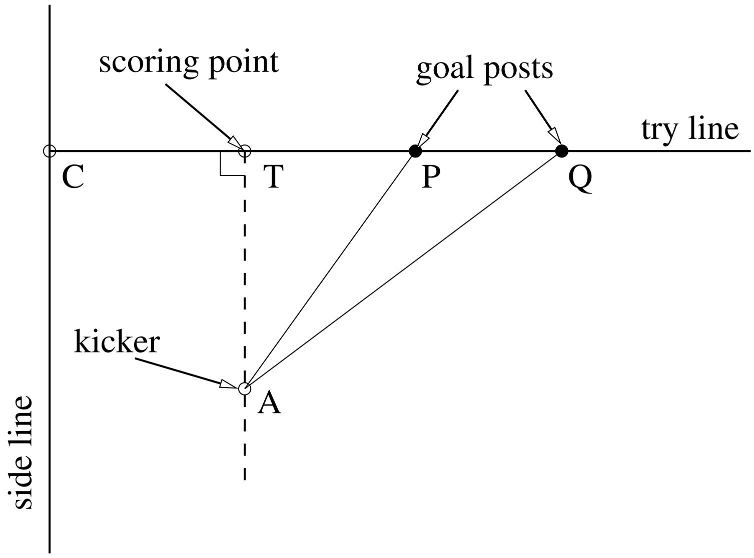
We suggest you find a video of a rugby kicker to introduce this problem.

You should also stress that (for the time being) we are doing only the 2D problem and are ignoring the height of the kick.

Rugby kicking 2

(by Alan Champneys)

Let's start with some numbers. For a typical rugby pitch, the width is 70 m and the goal width PQ is 5.6 m. Given these dimensions (assuming the goal is in the middle of the try line) calculate the distances CP.



As an example, let's suppose the try is scored at 10 m from the side line. That is, CT= 10 m. Calculate the distances TP and TQ.

Suppose the kicker chooses to kick from 10 m away from the try line. That is, TA= 10 m. Use trigonometry to calculate

$$\tan(< \text{TAP}) = \qquad\qquad \text{and} \quad \tan(< \text{TAQ}) =$$

Hence, calculate the angle <PAQ

$$\text{for TA}= 10\text{ m:} \qquad < \text{PAQ} =$$

Solution

We have that

$$CP = \frac{\text{pitch width}}{2} - \frac{\text{post width}}{2} = 35 - 2.8 \; = 32.2 \text{ m}$$
$$CQ = \frac{\text{pitch width}}{2} + \frac{\text{post width}}{2} = 35 - 2.8 \; = 37.8 \text{ m}$$

Hence

$$\tan(< \text{TAP}) = \frac{32.2 - 10}{10} = 2.22 \;\text{ and }\; \tan(< \text{TAQ}) = \frac{37.8 - 10}{10} = 2.78$$

and so

$$< \text{PAQ} = \arctan(2.78) - \arctan(2.22) = 70.216 - 65.751 = 4.465^\circ$$

Rugby kicking 3

(by Alan Champneys)

Now suppose the kicker instead stands 30 m away from T (with again the try being scored at TC$=10$ m).

Repeating the calculation for TA$=30$ m:

$\tan(< \text{TAP}) =$ and $\tan(< \text{TAQ}) =$

Hence

for TA$=10$ m: $< \text{PAQ} =$

Which is the better position to kick from; 10 m or 30 m?

What about if they stood right at the far end of the pitch, TA$=100$ m?

Can you see that there must be an intermediate distance (between 0 and 100 m) such that the angle <PAQ is maximised?

Solution

Now for TA $= 30$ m, we have

$$\tan(< \mathrm{TAP}) = \frac{22.2}{30} = 0.740 \text{ and } \tan(< \mathrm{TAQ}) = \frac{27.8}{30} = 0.927$$

and so

$$< \mathrm{PAQ} = \arctan(0.927) - \arctan(0.74) = 42.820 - 36.501 = 6.319^\circ$$

This is better (a wider angle) than the TA $= 10$ m kick.

If we start on the try line we find (trivially) $< \mathrm{PAQ} = 0$, so you might think that the angle keeps getting bigger as we increase TA.

But, if we kick from 100 meters (the far end of the pitch), we find

$$< \mathrm{PAQ} = \arctan(0.278) - \arctan(0.222) = 15.536 - 12.517 = 3.019^\circ$$

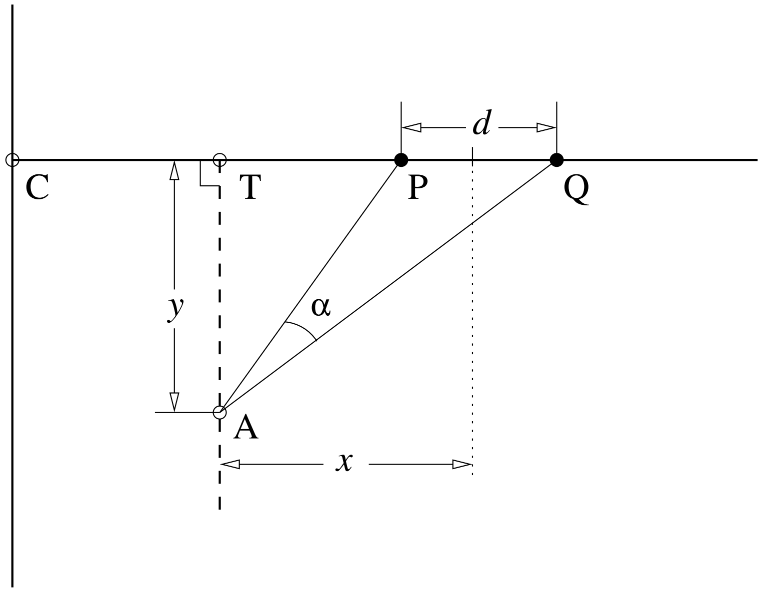
which is worse than TA$= 30$ m.

Hence there must be an optimum distance somewhere between TA$= 10$ m and 100 m

Rugby kicking 4

(by Alan Champneys)

Now we are going to try to generalise using algebra.



Suppose that the post width d and the distance x of T from the centre line of the pitch are fixed. We want to find the optimum value of y, which maximises α.

Previously we calculated the angle PAQ $= \alpha$ for some given values of d, x and y. Repeat the calculation to find a general expression for

$$\alpha = \arctan(\qquad\qquad\qquad\qquad) - \arctan(\qquad\qquad\qquad\qquad).$$

Solution

Proceeding as before ,we find

$$\tan(< \mathrm{TAP}) = \frac{x-(d/2)}{y} \quad \text{and} \quad \tan(< \mathrm{TAQ}) = \frac{x+(d/2)}{y}$$

and so

$$\alpha = \arctan\left(\frac{x+(d/2)}{y}\right) - \arctan\left(\frac{x-(d/2)}{y}\right)$$

Rugby kicking 5

(by Alan Champneys)

The function from the previous part can be written

$$\alpha = \arctan\left(\frac{2x+d}{2y}\right) - \arctan\left(\frac{2x-d}{2y}\right)$$

Taking the realistic value for the goal width $d = 5.6\,\mathrm{m}$, and taking $x = 25$ m (the same position of T used in parts 2 and 3) find a table of values of α for $y = 0\,\mathrm{m}$, $5\,\mathrm{m}$, $10\,\mathrm{m}$, $15\,\mathrm{m}$, etc. up to $50\,\mathrm{m}$:

y (m)	0	5	10	15	20	...
α (°)						

Plot these values on a graph of α versus y. Estimate the value of y for which α is maximised.

An A-level extension

Use calculus to find the maximum of α as a function of y

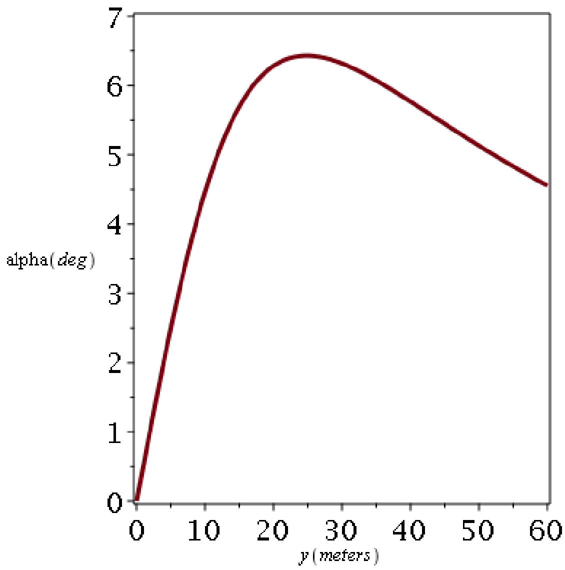
But we are going to find another way ...

Solution

Using $d = 5.6$ and $x = 25$ we find

y (m)	0	5	10	15	20	25	30	35	40	45	50
α (°)	0	2.52	4.41	5.70	6.28	6.43	6.31	6.07	5.77	5.43	

See the graph:



For A-level students we can differentiate $\alpha(y)$ to find

$$\begin{aligned}\frac{d\alpha}{dy} &= \frac{x-p}{(x-p)^2+y^2} - \frac{x+p}{(p+x)^2+y^2} \\ &= -\frac{2p(x^2-y^2-p^2}{[(p+x)^2+y^2][(x-p)^2+y^2]}\end{aligned}$$

where $p = d/2$ is half of the goal width.

To find an extremum we set this derivative to zero. Note the denominator is positive definite, so this is equivalent to setting the numerator to zero. Thus to get the extremal value we set

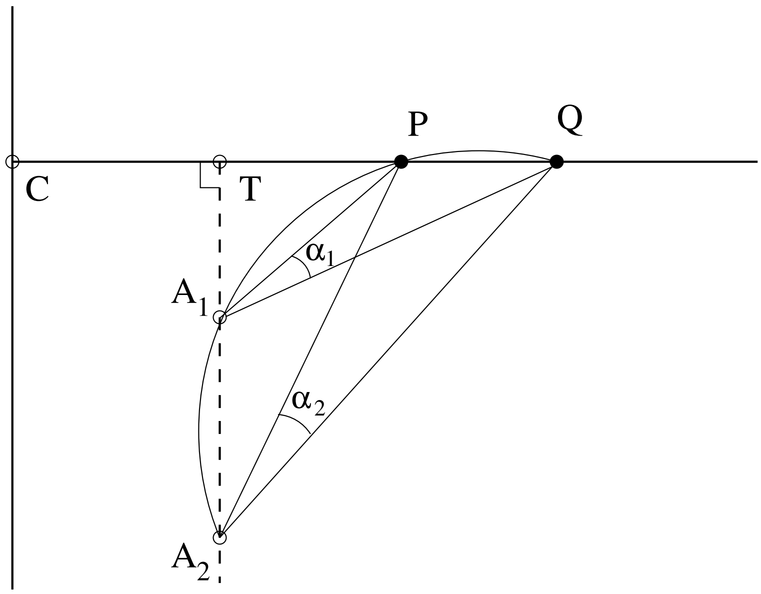
$$x^2 - y^2 - p^2 = 0 \qquad \text{hence} \quad y = \sqrt{x^2-p^2}$$

Is it obvious that this extremum has to be a maximum?

Rugby kicking 6

(by Alan Champneys)

We are now going to use geometry. Consider the two possible kicking positions A_1 and A_2, depicted in the diagram below.



Note that A_1 and A_2 lie on the same circle through the posts P and Q. Hence what can you say about the angles α_1 and α_2? Why?

What happens to the angles α_1 and α_2 if you make the circle larger or smaller?

Can you draw the circle on which the optimum kicking point on the dashed line must lie?

Solution

One of the Circle Theorems states

"All angles inscribed in a circle and subtended by the same chord are equal."

Hence $\alpha_1 = \alpha_2$

Making the circle larger clearly makes α smaller. *Suggest students see this by drawing different circles using a compass and measuring the required angles.*

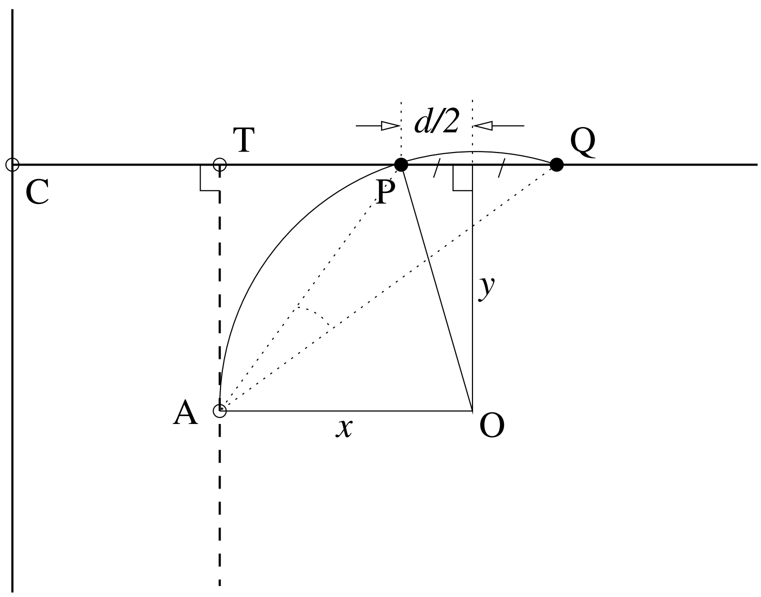
Hence the optimal distance y which makes α the largest possible, is to make the circle as small as possible.

That is, we **choose the circle that just touches the line** on which the kicker must kick.

Rugby kicking 7

(by Alan Champneys)

Now we are going to use this touching circle to calculate the optimal distance y. Consider this diagram



What is the distance OP?

Hence use Pythagoras' Theorem to find an expression for y.

$$y =$$

Congratulations you have found the optimal kicking distance!

What does curve of y against x look like? Can you sketch it?
What does it mean?

Solution

OP is a radius of the circle. Hence OP=x.

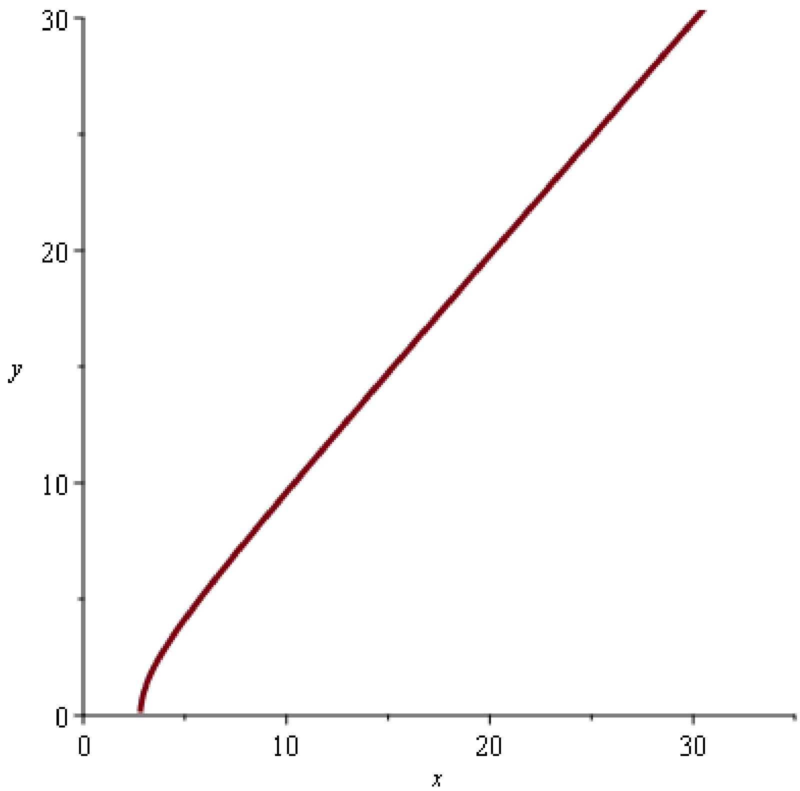
Consider now the triangle OPM where M is the midpoint between the posts. Applying Pythagoras' Theorem we get

$$x^2 = y^2 + (d/2)^2$$

Or,

$$y = \sqrt{x^2 - (d/2)^2}$$

Note this is a rectangular hyperbola. You might like to suggest the students try plotting it.



Further information

There are many possible more realistic extensions to this problem ....

What if we had to take the height of the posts into consideration? The kick must clear a bar that is $3\,m$ off the ground. Does this affect the optimal y? How would you optimise the height at which the kicker should kick?

What if $-d/2 < x < d/2$; that is, if the kick is in front of the posts? What would be optimal distance y then?

What if we take wind into account? What if we took the statistics of a particular kicker's past performance?

For more detailed information. Here are two reports that can be found online that provide further mathematical insight:

`https://epubs.siam.org/doi/pdf/10.1137/130913225`
`https://www.qedcat.com/articles/rugby.pdf`

In fact, these days, the science of sport is a lucrative business.