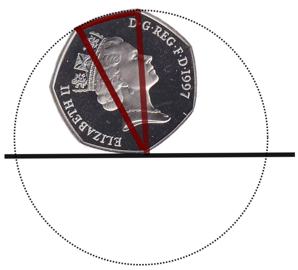
Rolling a fifty pence piece 1

(by Alan Champneys)

A fifty pence piece is a seven sided coin. But if you look at it closely, you realise that it is an unusual coin because its sides aren't exactly straight. In fact each side is an arc of a circle whose radius is the distance from that side to the opposite corner.



The fifty pence is designed this way for a specific reason. Suppose such a coin were rolled on a perfectly flat table. Can you draw a sketch of what happens to the highest point of the coin?

What do you notice? Why might this be helpful to the designer of a coin operated machine?

Solution

Let R be the radius of the arc of each of the seven sides.

Notice that as the coin is rolled, it pivots for a while on one of the corners. While this stage is happening, the highest point of the coin is on the arc of a circle that is a fixed distance R from the pivot point.

The pivoting on this corner ends when the coin starts to roll on the curved edge. During this roll, the highest point of the coin becomes the opposite corner. This is a fixed distance R from the side that it is rolling.

Now, this roll ends when the next corner point touches the table. By symmetry, this coincides with when the oppostite corner ceases to become the highest pooint. The coin now starts to pivot on the next corner and the highest point of the coin is on the opposite edge, which again is a distance R aways.

Hence, as is rolls, the top of the coin is always a fixed distance R above the table.

This is useful to the designers of coin operated machines, because no matter what orientation you put the fifty pence piece into the slot, it will always have the same width. Hence you can design the same kind of slot as you would do for a circular coin.

Rolling a fifty pence piece 2

(by Alan Champneys)

You should have found that the fifty pence piece always has a constant height no matter how you roll it! This seems remarkable, given that it is not circular.

Could you design other coins in this way, with edges that are circular arcs with radius the distance from the oppostie corner? Or is a special property of a 7-sided coin?

How about a 5-sided coin? or even a 3-sided coin? Would they also have the same height off the table no matter how you rotate them?

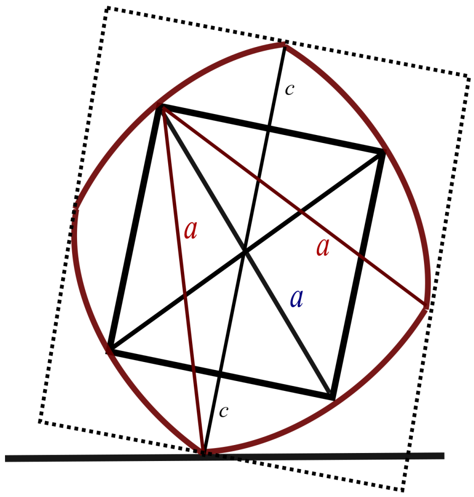
What about an even-sided coin? For example, something with four sides?

Solution

It is straightforward to see by drawing pictures, that the same trick works for any odd number of sides. Even a 3-sided object.

It is important to have an odd number of sides, because we need the corner to be opposite the mid-point of an edge.

Consider the four-sided object constructed in this way, for example.



This doesn't look like its going to have a constant height when we roll it. As we are about to show ...

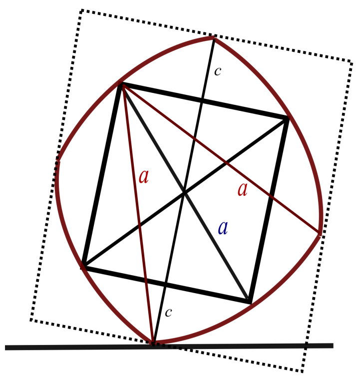
Rolling a fifty pence piece 3

(by Alan Champneys)

You should have found that any odd-sided coin can be constructed in this way so that it has a constant height when rolled.

But things go wrong for an even number of sides. Why?

Consider the four-sided 'square coin' constructed in this way.



Can you show that this does **not** have a constant height as it rolls? *[Hint: from the above diagram, assume that the smaller inner square has side length 1. Then, can you show that the width a is different from the width $1 + 2c$ as labelled in the diagram?]*

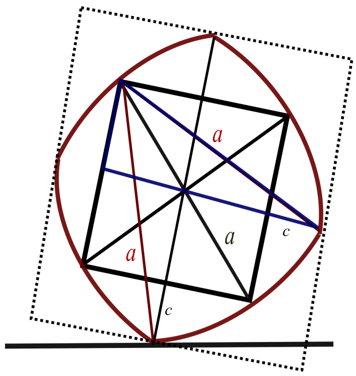
Solution

Consider the square coin as shown below. From the sketch it would appear that $a < 1 + 2c$. Let's try to show this:

If the inner square has side length 1, then from Pythagoras' Theorem we see that

$$a = \sqrt{2} = 1.4142$$

There are a number of ways to calculate the length c. For example,



the blue triangle is right-angled triangle with sides 1/2, $1 + c$ and a. Hence

$$a^2 = (1 + c)^2 + 0.5^2$$

So

$$c = \sqrt{(a^2 - 0.5^2)} - 1 = 0.3228$$

Hence the long side of length

$$1 + 2c = 1.64575$$

And so $1 + 2c > a$ as required.

Incedentally there is a nice Numberfile video on YouTube that considers the extension to 3D

www.youtube.com/watch?v=cUCSSJwO3GU