Moving Mount Fuji

(by Thilo Gross)

How many dump trucks would you need to move mount Fuji[1]?

This is a classic job-interview question and we can find an answer by a so-called a Fermi estimate: We break the answer into little pieces, each easier than the question as a whole.

The volume of a pyramid is a a third of the volume of the box into which the pyramid would fit. Fuji is a mountain, not a pyramid, but we can use this formula to roughly estimate its volume

$$V =$$

So given this volume we can estimate the approximate mass of mount Fuji

$$M =$$

How many trucks would it need to move this mass?

$$N =$$

[1] You may asume that the mountain is about 4km tall

The top-notch burger joint

(by Thilo Gross)

I would like to open a top-notch burger joint. A really nice expensive restaurant, but only for burgers. If you tried my burgers, you would understand. Of course this quality cannot be mass produced, so it will have to be a small place.

I might actually need your help to figure out how small. Please estimate the number of burgers that I can prepare myself within an hour.

Given the number of burgers that I can make, what is the approximate number of customers that I can serve in an evening?

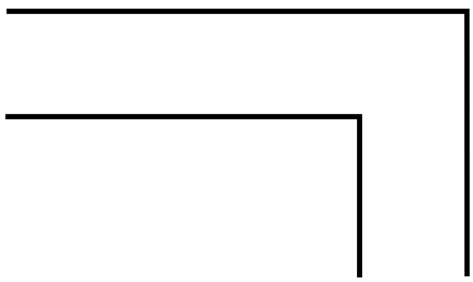
What I really want to know is, given the number of customers, how much space do I need to rent to seat all the customers on tables such that there is still enough space left that they can actually get to the tables.

includeproblems/009-fermichallenge

The Mathematical Removal Company 1

(by Alan Champneys)

The Mathematical Removal Company need to move a series of ladders from a window cleaning business through a corridor that is only 1 metre wide. The corridor consists of two long straight sections joined by a right-angled corner.



To being with, they need to calculate the longest ladder that can be manipulated around the corner, while keeping the ladder precisely horizontal. (You may ignore the thickness of the ladder.) Try drawing some pictures.

Imagine a ladder that is too long. Draw a picture of the point at which it gets stuck. What are the angles involved? Which direction can the ladder be moved?

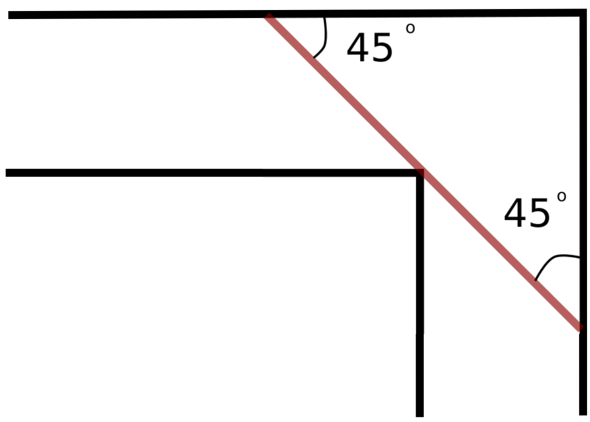
Can you now work out what the angles would need to be if the ladder could be moved in either direction? So, what is the length of the longest ladder?

$$\ell = \qquad\qquad \text{metres}$$

The Mathematical Removal Company 2

(by Alan Champneys)

You should have found something like the following diagram of how the longest possible ladder can fit around the bend.



This gives the maximum length of ladder that can be removed horizontally to

$$\ell_1 = 2\sqrt{2} = \ 2.82 \text{ metres}$$

More realistically, a professional removal worker would tilt the ladder into the third dimension in order to get it around a corner more easily.

So suppose the corridor has a uniform height of 3 metres. What is the maximum length of ladder that can be manipulated around the corner?

$$\ell_2 = \qquad\qquad \text{metres}$$

The Mathematical Removal Company 3

(by Alan Champneys)

The Mathematical Removal Company now want to decide on the optimal shape of box that can contain the most stuff to be removed, and still fits along the same corridor with the 90° bend. Assuming that the box is 3 metres tall so that it fills the entire height of the corridor, what should the cross-section of the box look like if:

the cross-section must be a square?;

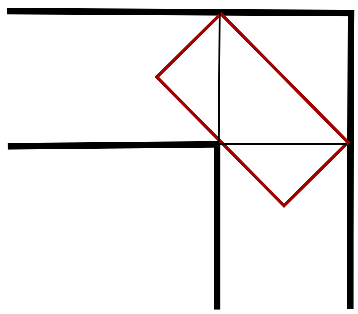
the cross-section can be any rectangle?
[Hint: you can assume that the box can be rotate around the corner, and use the same idea as for the ladder.]

What do you notice about the cross-sectionaarea of A in each case?

The Mathematical Removal Company 4

(by Alan Champneys)

You should have found, like the removers did, that the square box which is just shunted around the corner and the rectangular box that is rotated, have the same cross-sectional area of 1 metre^2. *[Note how the triangular areas of the rectangle that lie outside of the square fit perfectly into area above the rectangle to make the square.]*



One bright spark, the newest member of the Mathematical Removal Company's team, suggested they could carry more stuff if the boxes were semi-circular in cross section. Is she right?

What is the largest semi-circular area that can be rotated around the corner?

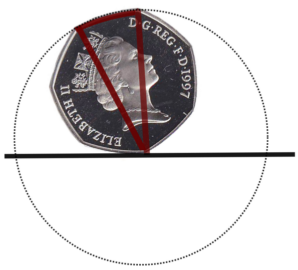
Is this the largest possible area, or can you do even better? What is the largest cross-sectional area you can come up with?

[In fact this challenge is known as the Moving Sofa Problem,
see `https://en.wikipedia.org/wiki/Moving_sofa_problem`
and the solution to for the largest cross-sectional area not known]

Rolling a fifty pence piece 1

(by Alan Champneys)

A fifty pence piece is a seven sided coin. But if you look at it closely, you realise that it is an unusual coin because its sides aren't exactly straight. In fact each side is an arc of a circle whose radius is the distance from that side to the opposite corner.



The fifty pence is designed this way for a specific reason. Suppose such a coin were rolled on a perfectly flat table. Can you draw a sketch of what happens to the highest point of the coin?

What do you notice? Why might this be helpful to the designer of a coin operated machine?

Rolling a fifty pence piece 2

(by Alan Champneys)

You should have found that the fifty pence piece always has a constant height no matter how you roll it! This seems remarkable, given that it is not circular.

Could you design other coins in this way, with edges that are circular arcs with radius the distance from the oppostie corner? Or is a special property of a 7-sided coin?

How about a 5-sided coin? or even a 3-sided coin? Would they also have the same height off the table no matter how you rotate them?

What about an even-sided coin? For example, something with four sides?

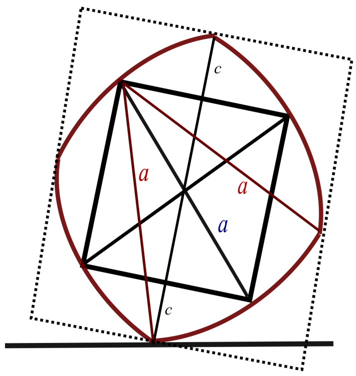
Rolling a fifty pence piece 3

(by Alan Champneys)

You should have found that any odd-sided coin can be constructed in this way so that it has a constant height when rolled.

But things go wrong for an even number of sides. Why?

Consider the four-sided 'square coin' constructed in this way.



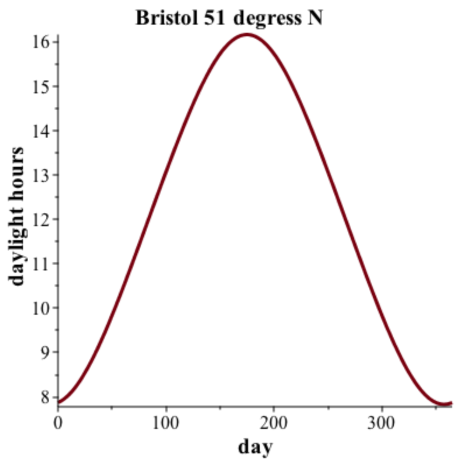
Can you show that this does **not** have a constant height as it rolls? *[Hint: from the above diagram, assume that the smaller inner square has side length 1. Then, can you show that the width a is different from the width $1 + 2c$ as labelled in the diagram?]*

The length of days 1

(by Alan Champneys)

Have you ever thought about what a graph of the number of hours of daylight throughout the year might look like?

Clearly it depends on lattitude. Here is a graph for Bristol which has an angle of lattitude $\phi = 51°$ (north).



Looks like a sine wave, doesn't it? But is it exactly a sine wave? and how would we prove it? Think about what exactly causes the length of the days to change. Discuss in your group.

For the rest of this exercise we are going to try to look test a formula for the daylength anywhere on the Earth.

The length of days 2

(by Alan Champneys)

It turns out that there is a good approximate equation for calculating the length of a day

Sunrise equation

The **sunrise equation** defines an hour angle ω measured in terms of something called the sun's declination angle δ (the apparent angle of the sun to the vertical at noon) and the latitude ϕ.

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

where $-180° < \omega < 0$ corresponds to sunrise, and $0 < 180°$ corresponds to sunset.

The sun's declination angle δ can be defined as 23.45° times a sine wave with amplitude 1 and period of 365 days. That is the angle varies sinusoidally between +23.45o (midsummer) and -23.45° (midwinter) over the course of a year. In the northern hemisphere midsummer is 21st June which, ignoring leap years, is

$$\text{21st June } \equiv d_0 = \qquad\qquad \text{days into the year}$$

But we need to convert days into degrees. Can you therefore find an expression for can the declination angle in terms days d since New Year?

The length of days 3

(by Alan Champneys)

You should get something like

$$\delta \approx 23.45 \cos\left(\frac{360^\circ}{365}[d - 172]\right) \text{ or } 23.45 \sin\left(\frac{360^\circ}{365}[d + 10]\right)$$

Next, given the sunrise equation

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

we need to turn the hour angle ω into a time. The angle is defined as zero at 12 noon and 180° at 12 midnight. Let $\omega > 0$ be the hour angle of sunset. Given that the Earth spins 360° in 24 hours. We can define the sunset and sunrise times in terms of ω as

$$\text{sunrise} = \qquad\qquad\qquad , \quad \text{sunset} =$$

Hence we can now use the sunrise equation to find an experession for the length of a day in terms of the latitude angle ϕ and day number d

$$\text{daylength} = \text{sunset} - \text{sunrise} =$$

Try feeding the numbers for Bristol at different days of the year into this formula to see if you get the right answer.

The length of days 4

(by Alan Champneys)

What happens to the angle ω defined by the sunrise equation, at midsummer when $\phi = 90^\circ - 23.45 = 66.55^\circ$?

What hour does sunrise occur for that location on that day?

Note that $\phi = 66.55^\circ$ precisely defines the arctic circle. What happens to solutions of the sunrise equation at midsummer when $\phi > 66.55^\circ$?

Can you explain this in terms of what you understand happens in summertime inside the arctic circle?

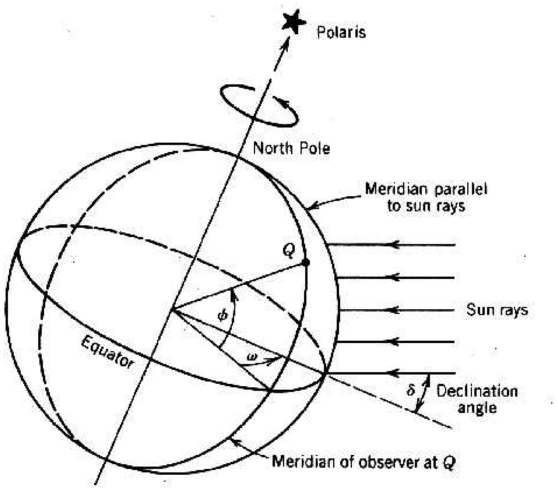
The length of days 5

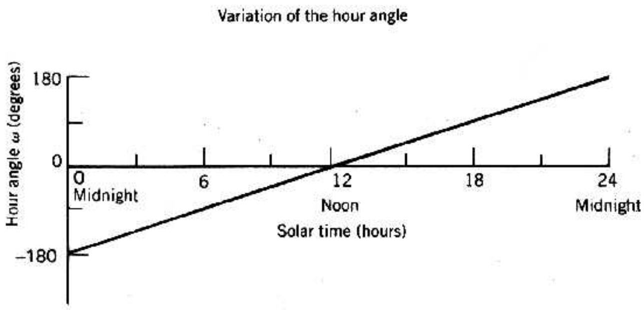
(by Alan Champneys)

Can you derive the sunrise equation?

$$\cos(\omega) = -\tan(\phi)\tan(\delta)$$

The following diagram may help.





Greek geometry 1

(by Ksenia Shalonova)

Practical lessons from ancient geometers. We are going to learn of the contribution from two ancient Greeks, Pythagoras and Thales. Both visited Egypt to gain understanding.

Right-angled triangles

Pythagoras' theorem connects the side lenghts of right-angled trianges. Can you come up with some right-angled triangles with whole numbers for side lengths? This may give you a clue how they made right angles in Ancient Egypt using only a rope.

Suppose you want to make a badminton court in your garden.


- How will you make right angles using a rope? How many people would you ideally need to construct it?
- How many poles would you need to draw a long straight line (you can stick poles in the ground)?
- How will you check in the end that your quadrilateral is a rectangle?

Greek Geometry 2

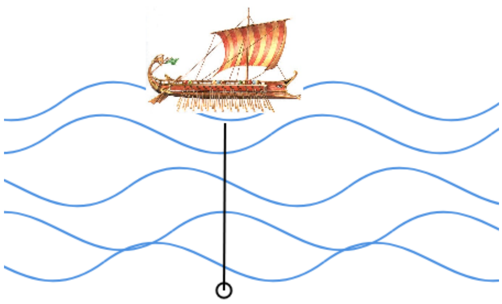
(by Ksenia Shalonova)

Measuring the distance to a ship. Thales used geometry for measuring distances. He measured distances to the ships in navigation, distances to the stars and even worked out the height of the pyramid in Egypt.

Similar triangles

In Ancient Greece they used the properties of the similar triangles to measure distances. In similar triangles, the angles are the same and corresponding sides are proportional. Can you sketch two similar triangles? Can similar triangles be right-angled?

You are standing on the shore and want to calculate distance to the ship. For measuring the distance you are only allowed to use a pole, but you are free to move on the shore.



Greek Geometry 3

(by Ksenia Shalonova)

Measuring heights There are a number of methods for measuring heights,

for example, you can use the length of the shadow (on a sunny day). You can also use a mirror or a hand-drawn triangle.

In Jules Verne's "The Mysterious Island" Captain Harding wanted to find the height of a cliff and for this he used a tall poll. There is only one disadvantage of using this method - you have to lie down on the ground! Can you figure out how he did it by using properties of similar triangles and the line of sight?

Greek Geometry 4

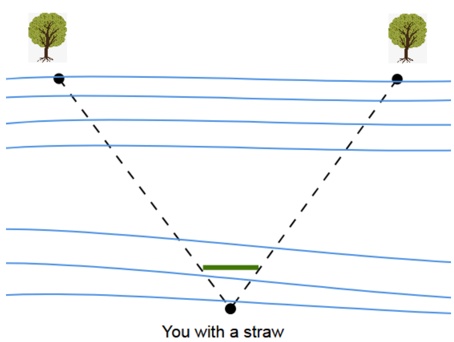
(by Ksenia Shalonova)

Measuring river width with a straw

Thales' Theorems

There are several theorems that are attributed to Thales. For example, the circle is bisected by its diameter or the angle in a semi-circle is right angle. Knowledge of the angles in a circle can help you to solve the next problem.

You are standing on a river shore and want to measure an approximate width of a river. You can use a straw (or a small stick). Try to solve the problem first without looking at the hint!

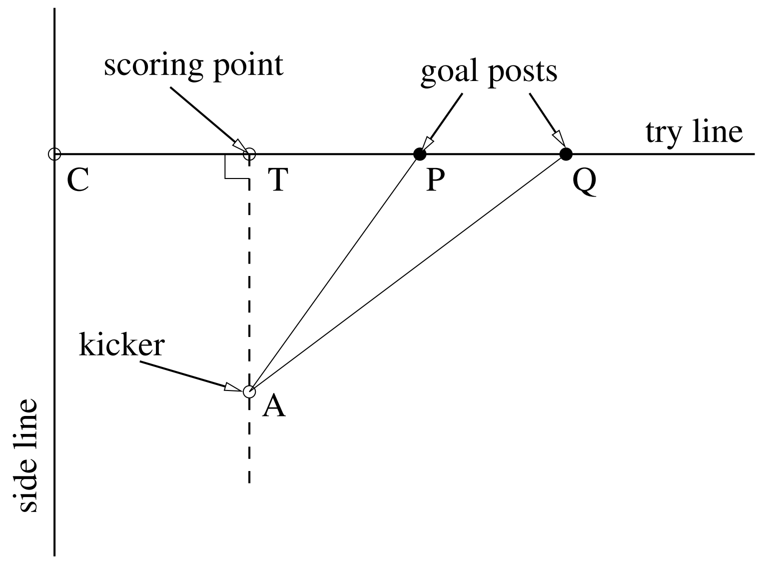


Hint. Notice any two objects on the opposite side of the river (e.g., flowers or trees). Hold a stick horizontally with an outstreched hand where the first object is directly behind the left end of the straw and another object is directly behind the right end of the straw. Reduce the size of the straw - you have to decide by how much. Start moving back until the position of one object is directly behind the left end of the straw and ...

Rugby kicking 1

(by Alan Champneys)

In rugby, when a try is scored, a conversion kick has to be taken form a perpendicular line that intersects the try line where the try was scored (the dashed line in the diagram). The kicker is trying to get the ball through the posts. The kicker is free to choose the point A on the dashed line. But where should A be chosen to maximise the angle <PAQ between the goalposts?



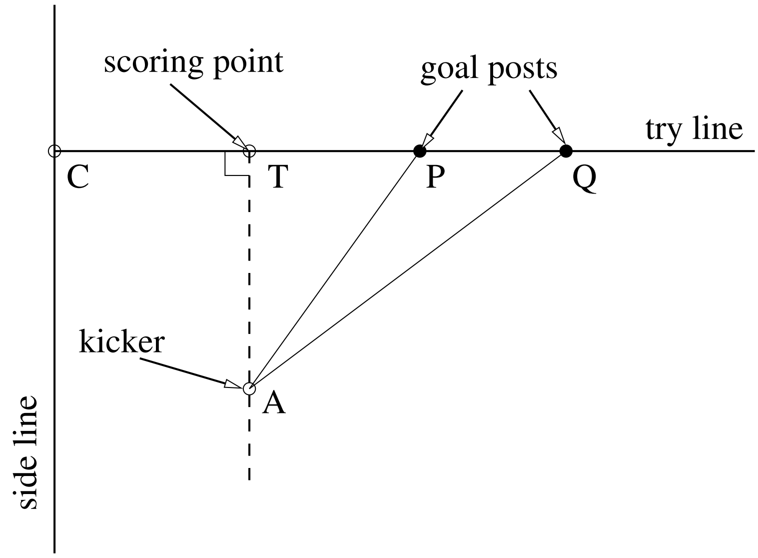
What happens to the angle <PAQ if A is chosen just next to T? What happens to the angle if A is chosen to be at the far end of the pitch?

We are going to find a formula for chosing the optimal point A.

Rugby kicking 2

(by Alan Champneys)

Let's start with some numbers. For a typical rugby pitch, the width is 70 m and the goal width PQ is 5.6 m. Given these dimensions (assuming the goal is in the middle of the try line) calculate the distances CP.



As an example, let's suppose the try is scored at 10 m from the side line. That is, CT= 10 m. Calculate the distances TP and TQ.

Suppose the kicker chooses to kick from 10 m away from the try line. That is, TA= 10 m. Use trigonometry to calculate

$$\tan(< \mathrm{TAP}) = \qquad\qquad \text{and} \quad \tan(< \mathrm{TAQ}) =$$

Hence, calculate the angle <PAQ

$$\text{for TA= 10 m:} \qquad < \mathrm{PAQ} =$$

Rugby kicking 3

(by Alan Champneys)

Now suppose the kicker instead stands 30 m away from T (with again the try being scored at TC= 10 m).

Repeating the calculation for TA= 30 m:

$\tan(< \mathrm{TAP}) =$ and $\tan(< \mathrm{TAQ}) =$

Hence

for TA= 10 m: $< \mathrm{PAQ} =$

Which is the better position to kick from; 10 m or 30 m?

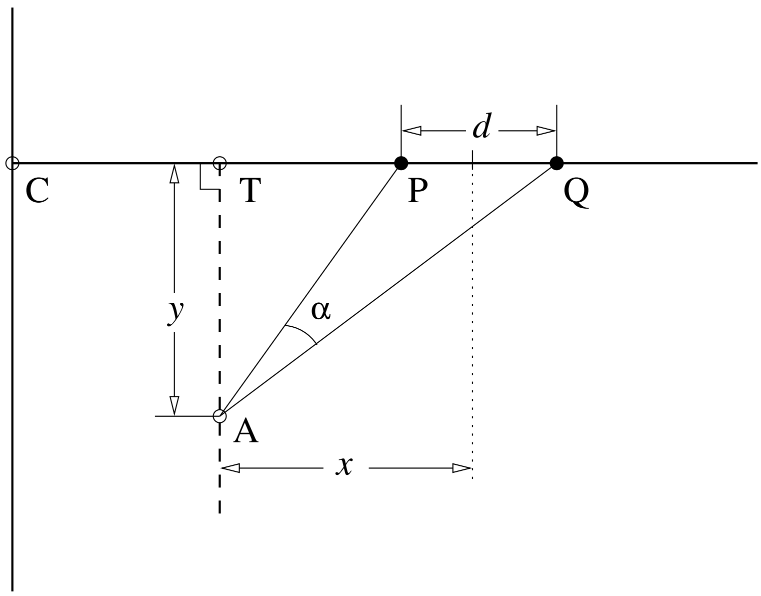
What about if they stood right at the far end of the pitch, TA= 100 m?

Can you see that there must be an intermediate distance (between 0 and 100 m) such that the angle <PAQ is maximised?

Rugby kicking 4

(by Alan Champneys)

Now we are going to try to generalise using algebra.



Suppose that the post width d and the distance x of T from the centre line of the pitch are fixed. We want to find the optimum value of y, which maximises α.

Previously we calculated the angle PAQ $= \alpha$ for some given values of d, x and y. Repeat the calculation to find a general expression for

$$\alpha = \arctan(\qquad\qquad\qquad\qquad) - \arctan(\qquad\qquad\qquad\qquad).$$

Rugby kicking 5

(by Alan Champneys)

The function from the previous part can be written

$$\alpha = \arctan\left(\frac{2x+d}{2y}\right) - \arctan\left(\frac{2x-d}{2y}\right)$$

Taking the realistic value for the goal width $d = 5.6$ m, and taking $x = 25$ m (the same position of T used in parts 2 and 3) find a table of values of α for $y = 0$ m, 5 m, 10 m, 15 m, etc. up to 50 m:

y (m)	0	5	10	15	20	...
α (°)						

Plot these values on a graph of α versus y. Estimate the value of y for which α is maximised.

An A-level extension

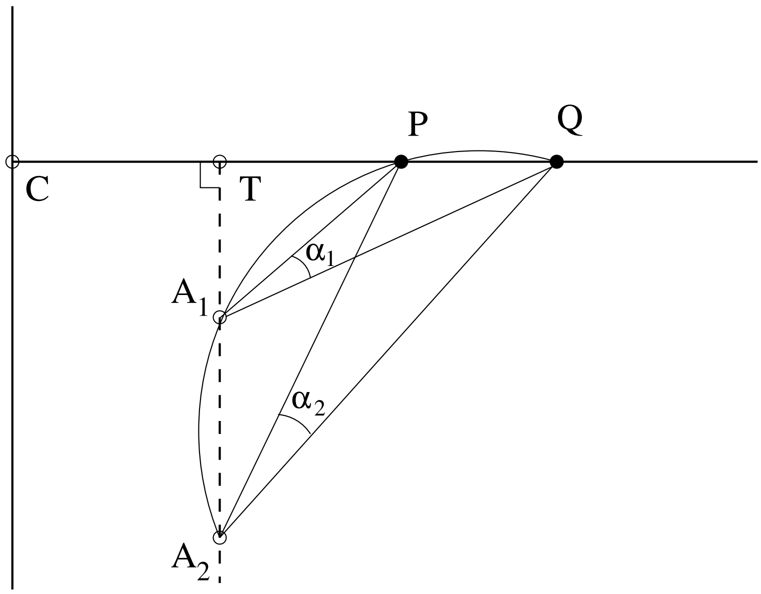
Use calculus to find the maximum of α as a function of y

But we are going to find another way ...

Rugby kicking 6

(by Alan Champneys)

We are now going to use geometry. Consider the two possible kicking positions A_1 and A_2, depicted in the diagram below.



Note that A_1 and A_2 lie on the same circle through the posts P and Q. Hence what can you say about the angles α_1 and α_2? Why?

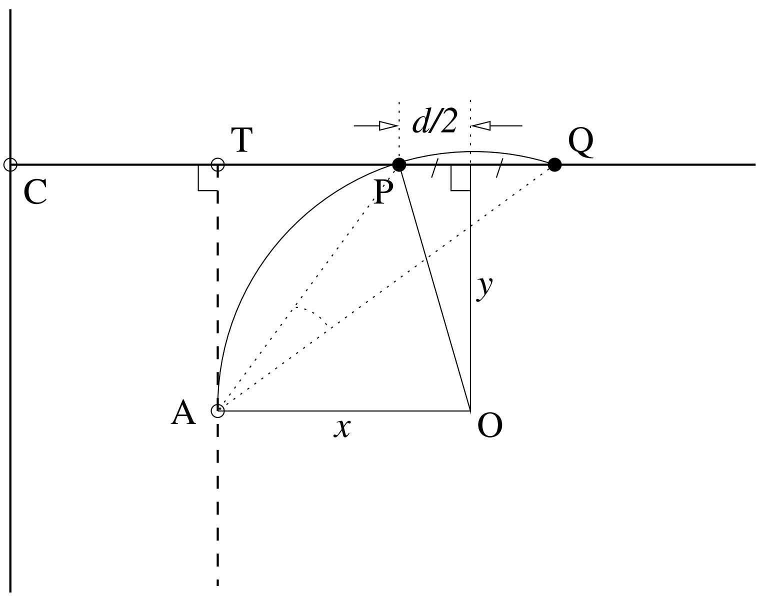
What happens to the angles α_1 and α_2 if you make the circle larger or smaller?

Can you draw the circle on which the optimum kicking point on the dashed line must lie?

Rugby kicking 7

(by Alan Champneys)

Now we are going to use this touching circle to calculate the optimal distance y. Consider this diagram



What is the distance OP?

Hence use Pythagoras' Theorem to find an expression for y.

$$y =$$

Congratulations you have found the optimal kicking distance!

What does curve of y against x look like? Can you sketch it?
What does it mean?

The payday loan 1

(by Alan Champneys)

Loans-R-Us, a so-called pay-day loan shop, is offering three different interest rates

(a) 3500% per year, (b) 35% per month, (c) 1% per day,

Interest is charged at the beginning of each period. So that if I borrow £1 for just one hour, under interest rate (a) I would pay pay £36, under (b) I would pay .35, and under (c) I pay £1.01.

Each time interest is charged, it is calculated on the whole amount owed, including any unpaid interest.

Calculate the total amount I would owe under each interest rate, if I were to borrow £1 for a whole year, under each interest rate scheme, assuming that I don't pay anything back until the end of the year.

Which interest rate should I choose?

The payday loan 2

(by Alan Champneys)

You should have found that the amounts you pay back under each rate are:

$$\text{(a) } 3500\% \text{per year} \text{Total} = £36, \quad (1)$$
$$\text{(b) } 35\% \text{per month} \text{Total} = £(1+0.35)^{12} = £36.64, \quad (2)$$
$$\text{(c) } 1\% \text{per day} \text{Total} = £(1+0.01)^{365} = £37.78. \quad (3)$$

So you should choose option (c) even though it looks like the worst deal.

This is because of the exponential growth of compound interest, and shows how so-called "pay day loan" companies prey on the poor and vulnerable to make huge profits. It is also why, by law, all loans must specify an equivalent *Anuual Percentage Rate* (APR). This calculation shows that the APR of a loan that is advertised as '1% per day' is actually 3678 % , which does not sound so appealing

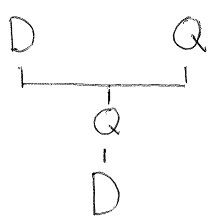
But it gets worse. Suppose I choose rate (c) and borrow £1 for 5 years. How much would I owe at the end of 5 years?

What if I were to only pay back after 10 years? where could I find such money from?

A secret of bees 1

(by Thilo Gross)

Bees have interesting family trees. A male bee, a so-called drone (D) only has one parent, who is a queen (Q). A queen has two parents, a queen and a drone. So a drone has only one parent, and only 2 grandparents. Got it? Continue the family tree that I have started to draw below for at least 3 more generations.



A secret of bees 2

(by Thilo Gross)

The ancestral trees of bees hide a secret. The first step to discover it is to write down the number of queens, drones, and the total number of bees for each generation.

Use the family tree that you have drawn to fill in the next 3 lines of this table

Gen. n	Queens Q_n	Drones D_n	Total T_n
0	0	1	1
1	1	0	1
2	1	1	2
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Do you see a pattern? Can you use it to fill in the remaining lines?

A secret of bees 3

(by Thilo Gross)

Let's write some equations! Suppose in generation n the number of drones is D_n, and the number of queens is Q_n.

We know that every bee has a queen as a parent, and queens have an additional drone parent.

Let's compute the numbers in the generation before n that's the generation $n+1$. According to the above, the number of queens and drones in generation $n+1$ is

$$\begin{aligned} Q_{n+1} &= \\ D_{n+1} &= \end{aligned}$$

Fill in the right-hand-side of these equations, using only D_n and Q_n and whatever arithmetical symbols you need.

A secret of bees 4

(by Thilo Gross)

When you filled the ancestry table you may have discovered the rule

$$Q_{n+1} = Q_n + Q_{n-1}$$

You can check that it holds in your table, but we really want to prove it. On the previous sheet we discovered

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

can you use these to prove the rule for queens above?

Can you prove a similar rule for drones and the total number of bees?

Don't forget that you can shift the indices, and $T_n = Q_n + B_n$.

A secret of bees 5

(by Thilo Gross)

Using the very convenient formula

$$T_{n+1} = T_n + T_{n-1}$$

we can compute the number of bees for some more generations while we are on it let us also compute the factor by which the number of ancestors increases in every generation. For humans that would obviously be 2 but for bees the ratio is more intriguing (find about 6 digits after the decimal point)

Gen. n	Bees T_n	Ratio T_n/T_{n-1}
$\vdots$	$\vdots$	$\vdots$
10	89	$\vdots$
11	144	1.617977
12	233	1.618056
13		
14		
15		
16		
17		
18		

A secret of bees 6

(by Thilo Gross)

We are getting closer to what the bees are hiding. On the last sheet we discovered the ratio by which the number of bees increases for sufficiently large n. Let's call this ratio f and also compute its inverse

$$\begin{aligned} f &= 1.618034\ldots \\ 1/f &= \end{aligned}$$

Notice anything odd? (Perhaps you would like more digits:
$f = 1.6180339887498948482045868$ what's $1/f$ now?)

If you noted some curious property of f, let's write it as an equation

$$1/f =$$

this time we want a mathematical expression on the right hand side, not digits.

Can you use the equation to compute the exact (!) value of f?

A secret of bees 7

(by Thilo Gross)

Congratulations, you have made it through this set of sheets. We have discovered the secret of bees. The number of ancestors of a bee follows the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

this is the famous Fibonacci sequence. In the long run the number of bees increases by a factor

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}.$$

in each generation. This is the famous golden ratio. A magical number that that appears all over mathematics. It has other surprising properties, for instance, by some measure, it is the most irrational of all numbers.

Here is a little bonus. On the previous sheet you may have noticed that $1/f = 1 - f$ has a second solution which is actually $-1/f$, this is the golden ratio's little brother g, note

$$\begin{aligned} f &= 1/2 + \sqrt{5/4} \\ g &= 1/2 - \sqrt{5/4} \end{aligned}$$

These two numbers work great as a team for instance they can give you the number of bees in generation n

$$T_n = \frac{f^n + g^n}{\sqrt{5}}$$

you can check that this works exactly, but why it works is a different story.

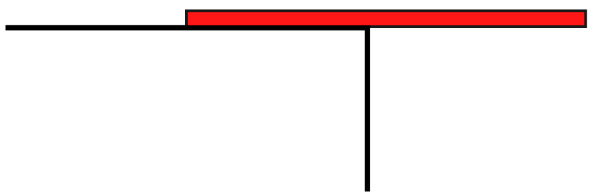
The one-sided arch 1

(by Alan Champneys)

Your task is to construct the longest possible overhanging arch to reach out over the sea, from a rigid clifftop, using only a very large collection of identical planks made of uniform material and length 2 metres. No nails, no glue, just planks, balanced one on top of the other.

We shall build up a solution bit by bit.

Using just one plank you can reach out precisely 1 metre with the plank teeting on the very edge of the cliff.



Can you think of a way of making a longer overhanging structure using two planks? What about three planks? or four? What is the longest overhang you can make?

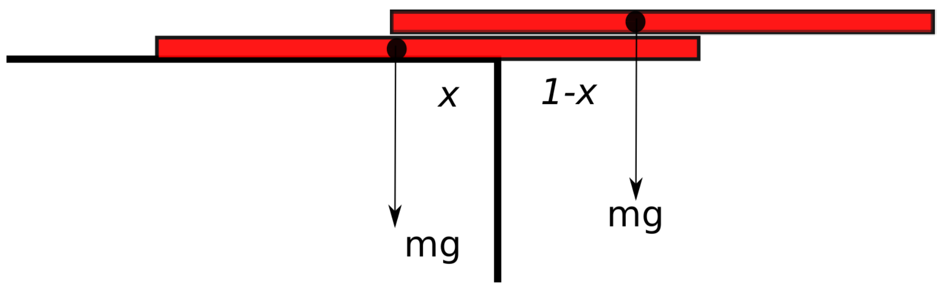
Start drawing some pictures before doing any maths.

The one-sided arch 2

(by Alan Champneys)

We are going to constrain things by saying that you can only place one plank on top of any other one and that all planks must point in the same direction.

Under this constraint, how far out can you place two planks, one on top of the other so that the top plank is teeting on the edge of bottom plank, and the bottom one is teeting on the edge of the cliff? *[The following diagram might help, where m is the mass of each plank]*

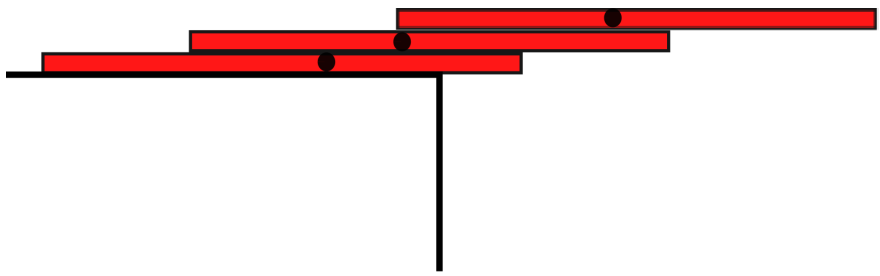


The one-sided arch 3

(by Alan Champneys)

You should have found that if $x = 0.5$ metres then the two planks just teeter on the cliff edge, with a total overhang of $1 + x = 1.5$ metres.

Now we are going to proceed using the same approach, essentially taking the solution we already have for two planks, which we know is just teeters on the edge of the cliff, and add one more plank underneath, with the upper two just teetering on it. Take a look at the diagram



in which the centres of mass of each plank is marked by a large black blob.

Can you calculate the maximum overhang of the lower plank, so that the three planks just teeter on the edge of the cliff?

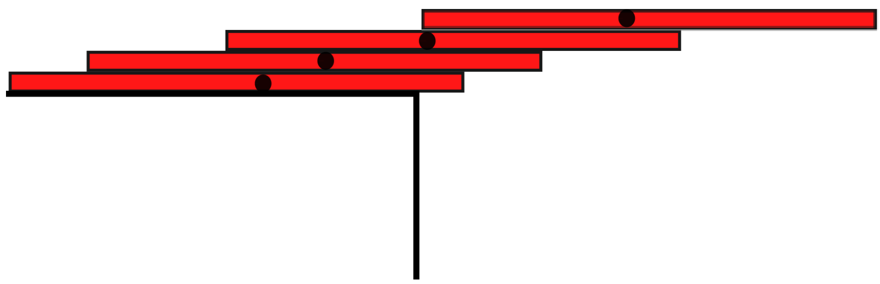
The one-sided arch 4

(by Alan Champneys)

You should have found the answer that the new piece of overhang is $1/3$ metre, so that

$$\text{Total overhang} = 1 + 1/2 + 1/3 = 1.83333\,\text{metres}$$

Now lets repeat this process by slotting a fourth plank underneath. Can you guess what the total overhang is now? How do you prove this?



Can you generalise this result to an arbitrary number of planks, N?

The one-side arch 5

(by Alan Champneys)

So you should have found that with a total of N planks the overhang is equal to

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/N \text{ metres}$$

Note how the additional overhang gets smaller and smaller each time.

But what is the maximum length of overhang I can produce via this method if I keep taking more and more planks?

Mathematically, what we want to know is what is the sum as N tends to infinity of

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$$

This is known as the Harmonic Series. Try summing this on a calculator.

What do you think the limit is a $N \to \infty$?

How would you prove this?

The weightless girl 1

(by Alan Champneys)

Some people say gravity is caused by the Earth's rotation. But a little thought shows that centrifugal force acts outwards not towards the ground. (Think of a wet dog spinning its body to throw off excess water).

Lauren, a girl of mass M, is standing on the equator where acceleration due Centrifugal force will make experience a weight that is slightly less than Mg. But how much less?

Some useful facts

- Centrifugal force $= mr\omega^2$, where r is distance to spin axis and ω is rotation speed in radians/s ($2\pi\times$ revs/s)
- The radius of the earth is 6371 km. The earth spins one revolution per 24 hours. g =9.81.

First, what is the earth's rotation speed in radians per second?

$\omega =$ revs/hour = rad/s

Hence calculate Laruen's centrifugal force as a function of her mass M. What percentage of her gravitational weight is lost due to centrifugal force?

centrifugal force = $\times M =$ % of Mg

The weightless girl 2

(by Alan Champneys)

So, the centrifugal force experienced by Lauren, standing at the equator, is about 291 times weaker than gravity.

Now imagine that the Earth was spinning faster. Centrifugal force increases with rotation speed, but gravity is unaffected. But how much faster would it to spin in order for Lauren to feel weightless. That is, is there a critical rotation speed ω_c for which centrifugal force exactly balances gravity.

The critical angular velocity (in radians per second) is

$$\omega_c = \qquad\qquad\qquad\qquad \text{rad/s}$$

which is   times faster than the earth currently spins

How long would a day last on such a fast rotating Earth?

$$\text{day length} = \qquad\qquad\qquad \text{hours}$$

Interstellar flight 1

(by Thilo Gross)

I want to establish a base in another solar system and I need your help planning it. We will do this in 7 steps. First we need to figure out how long the trip to the destination will take, then we can think about what to bring and whom to take along.

Our target will be the closest star (well, second closest actually), Proxima Centauri. Proxima is only 4.243 light years away, which means light from the sun needs 4.243 years to get to Proxima.

But light is pretty fast, $300,000$ km/s, so how far is Proxima actually away in m?

$$d =$$

Interstellar flight 2

(by Thilo Gross)

I want the spaceship to constantly accelerate for half the way, and constantly decelerate for the second half of the way.

The spaceship's engines can accelerate it at

$$a = 10\mathrm{m/s}^2$$

So if we accelerate for $t = 100\mathrm{s}$ what will be our velocity?

$$v(100\mathrm{s}) =$$

Can you write an equation for the velocity after accelerating for time t?

$$v(t) =$$

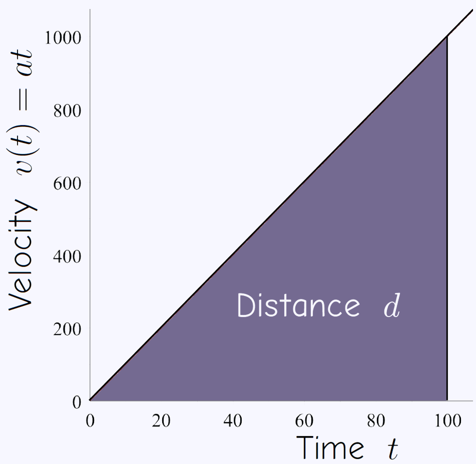
Interstellar flight 3

(by Thilo Gross)

After half the way we need to turn the ship around and fire our engines in the direction of Proxima to start decelerating. How long will it actually take to reach the halfway point? To work this out, we use the following trick:

Distance covered

If you plot your speed over time, the area under the curve is the distance covered up to a certain time. For constant acceleration this becomes a triangle, so we can use our knowledge of the area of triangles to find the distance covered.



Can you write an equation for d, the distance covered, as a function of acceleration a and time t?

$$d =$$

Interstellar flight 4

(by Thilo Gross)

Use the equation from the previous sheet to derive a formula for the time t it takes to go a distance, d, at a constant acceleration a

$$t =$$

How, long in years do we need to get to the half way point?

$$t_{\text{halfway}} =$$

So how long, in years, does the whole journey to Proxima take? (its shorter than you would think!)

$$t_{\text{Proxima}} =$$

Interstellar flight 5

(by Thilo Gross)

We could go on and compute the amount of fuel we need. But let's focus on something more important: food. Water and air can be recycled readily, but humans need to consume biomass to survive. Suppose we want to bring 1,000 people to Proxima, how much food do we need to bring for the journey and to survive the first year there? (Hint: You can start by thinking about how much food you eat per day)

An alternative to bringing food is growing biomass on board the ship. Adding vitamins as needed is relatively easy and because we will have to put up with nuclear power anyway, we can generate enough light to keep things growing.

The highest biomass production is achieved with algal growth tanks, which produce about 4g per day per litre of tank volume. Suppose that for our spaceship we can use tank units which weigh 300kg and each produce 1kg of dry biomass per day.

How long does a journey have to be so that the tank units are more efficient than just bringing the food? And, how many units do you want to install in our spaceship?

Interstellar flight 6

(by Thilo Gross)

We should also think about who we should bring. For instance our algal growth tanks need maintenance every 100 days and it takes a biosphere technician 6 hours to carry out the maintenance procedures on one tank. So how many biosphere technicians do we need?

What about physicians? Could one doctor look after all us?

If it is fun, go one, who else would you want to bring to keep the ship going and get a colony on a distant planet started?

Interstellar flight 7

(by Thilo Gross)

Congratulations you have made it through this set of exercises. Good work!

Of course if you enjoyed it you can always go on. There are plenty of questions related to the planned colony that benefit from mathematical considerations. For example how much energy do we need to produce to keep the spaceship going? How many people do we expect to die on the way to the destination? How many will be born? What else would we want to bring. Say, we want to build concrete domes as living space, how much concrete would be required? Or would it be more efficient to manufacture the concrete at the destination? In that case how long would it take till everybody would have a place to live on the surface of the planet?

Ball bouncing 1

(by Alan Champneys)

What happens if you put a tennis ball on top of a basketball and then drop them both? The result is surprising. To explore this, start by finding out how to mathematically model bouncing balls:

Newton's law of bouncing*

$$[\text{ speed afterwards }] = e \cdot [\text{ speed before }]$$

$e =$ 'coefficient of bouncing' (a property of the ball): $0 \leq e \leq 1$
$e = 1$ – a perfectly elastic ball; $e = 0$ – a squashy tomato;
$e = 0.8$ – a reasonable value for a well pumped ball

Suppose a basketball has $e = 0.8$ and is travelling at $v^{\mathrm{b}} = -5$ m/s as it hits the floor (here, negative velocity signifies going down, and a and b indicate before and after). How fast will it be going upwards as it lifts off?

$$v^{\mathrm{a}} = -ev^{\mathrm{b}} =$$

Can you think of any physical situation in which $e > 1$?

* actually Newton's law of restitution.

Ball bouncing 2

(by Alan Champneys)

Let's check if our assumption that the ball hits the floor at 5 m/s is about right. A very useful principle is

Conservation of energy

$$\text{kinetic energy} = \tfrac{1}{2}Mv^2; \qquad \text{potential energy} = Mgh$$

$$M = \text{mass of object}; \qquad v = \text{its speed};$$
$$h = \text{height above floor}; \qquad g = 9.81 \text{ m/s}^2.$$

Make reasonable assumptions about M and the height h_b from which the ball is dropped. The kinetic energy of the ball when it hits the floor must equal the potential energy that it had in the beginning, so with how much energy does the ball hit the floor? (Hint: 1 Joule = 1000 gm^2/s^2)

$$E_\mathrm{b} =$$

What velocity does the balls kinetic energy correspond to? Was our assumption of -5 m/s approximately right?

$$v^\mathrm{b} =$$

Ball bouncing 3

(by Alan Champneys)

We noticed on the first sheet that some velocity is lost as the ball hits the floor. How fast is your ball on lift off? And what is its kinetic energy?

$$v^{\mathrm{a}} =$$

$$E_{\mathrm{a}} =$$

As the ball rises again its kinetic energy gets converted into potential energy. Compute the height, h_{a}, that it can still reach after the bounce.

$$h_{\mathrm{a}} =$$

Now here is a challenge: Can you find a general formula for $h_{\mathrm{a}}/h_{\mathrm{b}}$?

$$\frac{h_{\mathrm{a}}}{h_{\mathrm{b}}} =$$

Ball bouncing 4

(by Alan Champneys)

Now let's return to the case where a tennis ball and a basketball are dropped together. For simplicity assume that both balls reach a velocity of -5m /s before the basketball hits the floor.

Here is a trick: Assume basketball first bounces off the floor, then, rising at velocity v_1^{a}. An instant later it collides with the tennis ball, which is still moving downward at velocity v_2^{b}. What is the relative speed at which the balls now approach each other? (it's not 10 m/s)

$$v_1^{\mathrm{a}} - v_2^{\mathrm{b}} =$$

Newton's law of bouncing for 2 objects

$$[\text{ speed of separation }] = e \cdot [\text{ speed of approach }]$$

Using the law above, and $e = 0.8$ for the collision between the two balls, find the speed at which they separate.

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} =$$

Here we have used the c to indicate the velocities after the balls collide.

Ball bouncing 5

(by Alan Champneys)

To solve for the speed v_1^{c} of the basketball and v_2^{c} of the tennis ball after the collision we have to use another principle:

Conservation of momentum

$$\text{Total momentum} = \sum[\text{mass} \cdot \text{velocity of balls}]$$

where $\sum$ means 'sum over all balls' and
Total momentum before collision = Total momentum after collision

Make a reasonable assumption about the masses of the basketball, M_1, and the tennis ball, M_2, and find the total momentum just before the basketball and tennis ball collide (after the basketball lifts off from the floor)

$$P = M_1 v_1^{\mathrm{a}} + M_2 v_2^{\mathrm{b}} =$$

Since the momentum is conserved in the collision

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}}$$

is also true.

Ball bouncing 6

(by Alan Champneys)

We just discovered

$$P = M_1 v_1^{\mathrm{c}} + M_2 v_2^{\mathrm{c}},$$

where we know P, M_1, and M_2 already. So there are only two unknowns, the velocities after the collision. We also know from sheet 4 that

$$v_2^{\mathrm{c}} - v_1^{\mathrm{c}} = 7.2 \text{ m/s}$$

We have two equations for two unknowns! Substitute all the values and compute the velocity of the tennis ball after the collision

$$v_2^{\mathrm{c}} =$$

This velocity should take the tennis ball to a height h_{c}, higher than the height from which it was released. Use your insights from sheet 3 to determine the factor $h_{\mathrm{c}}/h_{\mathrm{b}}$. Furthermore, by what factor, $h_{\mathrm{c}}/h_{\mathrm{a}}$ is this bounce higher than the height that would be reached if we dropped the tennis ball without the basketball?

Ball bouncing 7

(by Alan Champneys)

Congratulations you have explained what happens in the two-ball bounce.

Of course there are some extensions to think about. What would happen if we used two tennis balls instead of a tennis ball and a basketball? What about if the tennis ball and the basketball were reversed? What about three balls on top of each other (of successively decreasing mass)?

Very similar principles apply also in horizontal collisions, this is important for instance in nuclear physics where particles become very fast due to collisions with heavier particles, or in traffic accidents, where heavy vehicles can transfer a lot of energy to lighter ones.

Sports betting 1

(by Filippo Simini)

In sporting bets, the return of a winning bet is calculated multiplying the stake by the 'odds multiplier'.

The following table shows the odds multipliers of two bookmakers for the same game, which has two possible outcomes: victory of the home team (Win) or defeat of the home team (Lose).

	Bookmaker 1	Bookmaker 2
Win	$w_1 = 3$	$w_2 = 2.5$
Lose	$l_1 = 1.5$	$l_2 = 2$

For example, a bet of £10 with Bookmaker 1 on the victory of the home team would return $£10w_1 = £30$ if the home team wins (and zero if the home team lose). Note that odd ratios are always larger than 1: $l_1, l_2, w_1, w_2 > 1$.

Suppose you have £1 to bet and you think the home team will lose the game, which Bookmaker should you pick to maximise your return?
How much will you gain if you win, and what will be your loss if you lose?

(Note that here we are ignoring any betting fee).

Sports betting 2

(by Filippo Simini)

Consider the odds ratios of part 1:

	Bookmaker 1	Bookmaker 2
Win	w_1	w_2
Lose	l_1	l_2

where $w_1 > w_2$ and $l_2 > l_1$.

Suppose you have £1 to bet, but you are not sure of the outcome of the game, so you decide to bet on both outcomes.
Assume you bet £x on the victory of the home team and £$y = (1 - x)$ on the defeat of the home team.

What would be your total gain (or loss) if the home team wins? And if it loses?

Sports betting 3

(by Filippo Simini)

In gambling, a "Dutch book" is a set of odds and bets which guarantees a profit, regardless of the outcome of the gamble.

Assuming you bet £x on the victory of the home team and £$(1-x)$ on the defeat of the home team, your have

$$R_w = w_1 x - 1$$

if the home team wins, and

$$R_l = l_2(1-x) - 1$$

if the home team loses.

In this situation, is it possible to create a Dutch book and gain some money irrespective of the outcome of the game?

Under which conditions on the odds ratios w_1 and l_1 would it be possible to find a value of x which always guarantees a profit?

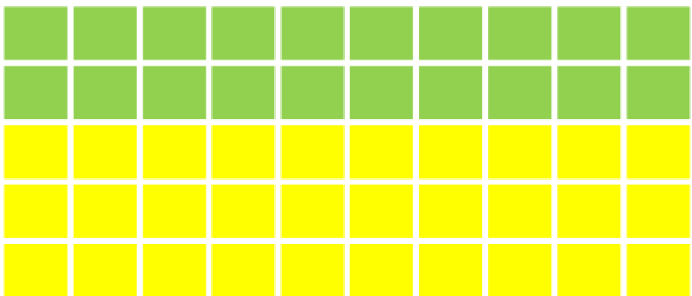
It is important to realise that bookmakers have lots of tricks that they always make a profit. While in the short term, some gamblers can win money. All gamblers lose money in the long term. Gambling is addictive and can lead to misery!

Simpson's paradox 1

(by Filippo Simini)

"Gerrymandering" consists of the manipulation of the boundaries of constituencies in order to alter the electoral results in a non-proportional system.

For example, consider the following region where each square represents a precinct: green squares vote for party A and yellow squares vote for party B. Out of the total 50 precincts, 20 (40%) vote for party A, and 30 (60%) for party B.



Can you draw 5 constituencies of equal size (10 neighboring precincts each) so that party A and B win in proportion to their overall voting?

Can you draw other 5 constituencies of equal size (10 neighboring precincts each) so that party A, the minority party, wins in the majority of constituencies?

Simpson's paradox 2

(by Filippo Simini)

Gerrymandering is a real life example of what's know as Simpson's paradox, where a trend appearing in different groups disappears when the groups are combined. Another example is the following:

A school has two classes. Class 1 has 25 students, 11 males and 14 females, and class 2 has 23 students, 13 males and 10 females.

Both classes take the same Maths test.
Overall, female students did better than male students: 12 out of 24 females passed the test (success rate 0.5), while just 11 of the 24 males passed it (success rate 0.458).

However, in each class, males had a higher success rate than females!
How is this possible?

Can you find a set of results such that male students have a higher success rate than female students in each class, but a lower success rate overall?

	Class 1	Class 2	Total
Males	**? / 11**	**? / 13**	11 / 24
Females	? / 14	? / 10	**12 / 24**

Simpson's paradox 3

(by Filippo Simini)

Simpson's paradox has relevant implications on our ability to understand the results of scientific experiments, for example in medical studies.

Consider the previous example, with the same outcomes, but instead of class tests assume that the results describe two independent investigations (test 1 and 2) on the effectiveness of two drugs, A and B. The numerators now correspond to the number of patients successfully treated using the drug.

	Test 1	Test 2	Total
Drug A	**9 / 11**	**2 / 13**	11 / 24
Drug B	11 / 14	1 / 10	**12 / 24**

According to both independent experimentations (tests), drug A is more effective than B. However, when the results of the tests are combined, we reach the opposite conclusion: drug B works better than A.

What should be trusted, the unanimous conclusions of the independent tests, or the reverse indication of the aggregate data?

Simpson's paradox 4

(by Filippo Simini)

Suppose we test the drugs on groups that are ten times bigger than the previous ones. For example, in Test 1 drug A is now tested on 110 individuals instead of 11.
We also assume that the success rates do not depend on the group sizes.

	Test 1	Test 2	Total
Drug A	? / 110	? / 130	? / 240
Drug B	? / 140	? / 100	? / 240

Does increasing the number of participants in all groups resolve the paradox?

Simpson's paradox 5

(by Filippo Simini)

Suppose we test the drugs on groups of equal sizes, of 100 patients each:

	Test 1	Test 2	Total
Drug A	? / 100	? / 100	? / 200
Drug B	? / 100	? / 100	? / 200

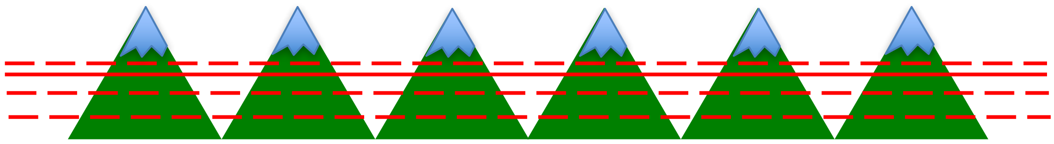
Assume again that the success rates do not depend on the group sizes.

Does considering groups of equal sizes resolve the paradox?

Railway line 1

(by Filippo Simini)

Design the path of a high-speed railway line across a mountain region in order to minimise the total construction cost. The construction costs depend on the excavation of tunnels and the construction of viaducts. The high speed trains should avoid slopes, so the railway line must run horizontally. Trains should also avoid sharp corners and travel as much as possible on a straight line. For simplicity, we can consider the problem in one dimension, where the mountains are a series of triangles placed next to each other and we can assume that the railway line is a straight horizontal line. The red lines in the figure below show possible railway lines under these assumptions.



The cost of excavation of a tunnel can be assumed to be proportional to the length of the tunnel. For the cost of building a viaduct, we can consider two scenarios. In the first scenario, the cost is proportional to the viaduct length, as in the case of tunnels.

Can you find the optimal height of the railway line in this first scenario?

Railway line 2

(by Filippo Simini)

In the second scenario the cost to build a viaduct depends both on the length of the viaduct and on the height of its pillars, because more material is needed to build higher viaducts. So in this case we can assume that the cost is proportional to the area between the viaduct and the mountain.

Can you find the optimal height of the railway line in this second scenario?

Comparative advantage 1

(by Filippo Simini)

The manager of a furniture company that manufactures tables hires you to increase productivity.

Four table legs and one table top are needed to produce one table.
On average, after a day worth of effort one worker produces $l = 10$ legs or $t = 5$ table tops.

Suppose that n out of the N workers of the company are assigned to the production of legs, and the remaining $(N - n)$ to the production of tops.

How many full tables will be produced in one day, on average, as a function of n ?

Comparative advantage 2

(by Filippo Simini)

To maximise the production, workers should be assigned to the production of legs or tops in order to manufacture 4 legs in the time needed to complete 1 table top.

How many workers should produce legs and how many should produce tops, in order to maximise the production of tables ?

Comparative advantage 3

(by Filippo Simini)

We found that the average number of tables produced in one day is equal to $P = N5/3$. Is it possible to further increase the production of tables?

Previously we assumed that the productivity of each worker is exactly equal to the average productivity. This may not be true in general, as it is common to find more and less productive workers.

Suppose that you find out that in the company there are some workers that are faster than the average at producing table tops, and some other workers that are faster at producing legs.
Considering this information, you are able to form two groups:
Group 1 has $2/3$ of the workers and they produce 12 legs and 4 tops per day, on average; Group 2 has $1/3$ of the workers and they produce 6 legs and 7 tops per day, on average.

Average number of units produced by one worker per day			
	Group 1	Group 2	All
fraction of workers	$2/3$	$1/3$	1
legs	12	6	10
tops	4	7	5

How can you use this new information to increase the productivity ?

Comparative advantage 4

(by Filippo Simini)

Suppose instead that you were not able to split workers into two groups such that each group is more efficient than the other at producing one item, tops or legs.
This can happen when there are some workers that are more efficient than others at both tasks, while all other workers are less efficient than the average at both tasks.

Dividing the workers into two groups of equal size according to their efficiency, you are able to form the following two groups:
Group 1 is formed by the most efficient workers that produce 14 legs and 6 tops per day, on average; Group 2 comprises the least efficient workers that produce 6 legs and 4 tops per day, on average.

Average number of units produced by one worker per day			
	Group 1	Group 2	All
fraction of workers	1/2	1/2	1
legs	14	6	10
tops	6	4	5

How should the work be divided among the two groups in order to have balanced production, such that one table top and four legs are produced at the same rate ?

Comparative advantage 5

(by Filippo Simini)

We found that the production is balanced (i.e. tops and 4-legs are produced at the same rate) if x workers of group 1 are assigned to produce legs, and $y_{eq}(x)$ workers of group 2 are assigned to produce legs, where

$$y_{eq}(x) = \frac{t_1 + t_2}{l_2 + t_2} - x\frac{l_1 + t_1}{l_2 + t_2}$$

In this case the productivity per worker is

$$P_3(x) = 1/2\,[l_1 x + l_2 y_{eq}(x)]$$

Is it still possible to divide the production among the two groups in order to increase the productivity with respect to the solution found in part 2 ?

Watering a sports field

(by Martin Homer)

This is an open-ended challenge, there are no right or wrong answers, it can be used as a brainstorming exercise or carried out over a prolonged period of time, like a project.

During a long, hot summer, cricket fields need regular watering. To make sure the grass is in optimal condition, your local club wants to use an automatic pop-up sprinkler system. But how should the sprinklers be laid out to ensure the best possible watering for least cost and inconvenience?

Ideally, every bit of the field would be watered equally, and the club particularly wants to avoid over- and under-watering the grass. Wasting water would be bad, too.

Most pop-up sprinklers can water either a whole circle, or a segment of a circle. You can assume that the club opts for a sprinkler with an 11m coverage radius.

Car Parking

(by Martin Homer)

This is an open-ended challenge, there are no right or wrong answers, it can be used as a brainstorming exercise or carried out over a prolonged period of time, like a project.

Land in cities is in short supply, and so the value of parking spaces is very high; perhaps as much as £100,000 per space in central London. So when making a new car park, it is important to design them so as to fit in as many parking spaces as possible.

Your challenge is to find a way of fitting as many parking spaces as possible into a given area, whilst maintaining the overall usability of the car park.

Two areas to try are shown below. You can move the entrances and exits, provided they stay on the same side of the car park, as shown in the figure.