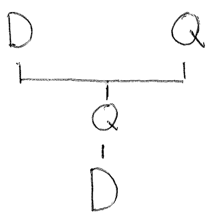
A secret of bees 1

(by Thilo Gross)

Bees have interesting family trees. A male bee, a so-called drone (D) only has one parent, who is a queen (Q). A queen has two parents, a queen and a drone. So a drone has only one parent, and only 2 grandparents. Got it? Continue the family tree that I have started to draw below for at least 3 more generations.



A secret of bees 2

(by Thilo Gross)

The ancestral trees of bees hide a secret. The first step to discover it is to write down the number of queens, drones, and the total number of bees for each generation.

Use the family tree that you have drawn to fill in the next 3 lines of this table

Gen. n	Queens Q_n	Drones D_n	Total T_n
0	0	1	1
1	1	0	1
2	1	1	2
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Do you see a pattern? Can you use it to fill in the remaining lines?

A secret of bees 3

(by Thilo Gross)

Let's write some equations! Suppose in generation n the number of drones is D_n, and the number of queens is Q_n.

We know that every bee has a queen as a parent, and queens have an additional drone parent.

Let's compute the numbers in the generation before n that's the generation $n+1$. According to the above, the number of queens and drones in generation $n+1$ is

$$\begin{aligned} Q_{n+1} &= \\ D_{n+1} &= \end{aligned}$$

Fill in the right-hand-side of these equations, using only D_n and Q_n and whatever arithmetical symbols you need.

A secret of bees 4

(by Thilo Gross)

When you filled the ancestry table you may have discovered the rule

$$Q_{n+1} = Q_n + Q_{n-1}$$

You can check that it holds in your table, but we really want to prove it. On the previous sheet we discovered

$$\begin{aligned} Q_{n+1} &= Q_n + D_n \\ D_{n+1} &= Q_n \end{aligned}$$

can you use these to prove the rule for queens above?

Can you prove a similar rule for drones and the total number of bees?

Don't forget that you can shift the indices, and $T_n = Q_n + B_n$.

A secret of bees 5

(by Thilo Gross)

Using the very convenient formula

$$T_{n+1} = T_n + T_{n-1}$$

we can compute the number of bees for some more generations while we are on it let us also compute the factor by which the number of ancestors increases in every generation. For humans that would obviously be 2 but for bees the ratio is more intriguing (find about 6 digits after the decimal point)

Gen. n	Bees T_n	Ratio T_n/T_{n-1}
$\vdots$	$\vdots$	$\vdots$
10	89	$\vdots$
11	144	1.617977
12	233	1.618056
13		
14		
15		
16		
17		
18		

A secret of bees 6

(by Thilo Gross)

We are getting closer to what the bees are hiding. On the last sheet we discovered the ratio by which the number of bees increases for sufficiently large n. Let's call this ratio f and also compute its inverse

$$\begin{aligned} f &= 1.618034\ldots \\ 1/f &= \end{aligned}$$

Notice anything odd? (Perhaps you would like more digits:
$f = 1.6180339887498948482045868$ what's $1/f$ now?)

If you noted some curious property of f, let's write it as an equation

$$1/f =$$

this time we want a mathematical expression on the right hand side, not digits.

Can you use the equation to compute the exact (!) value of f?

A secret of bees 7

(by Thilo Gross)

Congratulations, you have made it through this set of sheets. We have discovered the secret of bees. The number of ancestors of a bee follows the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

this is the famous Fibonacci sequence. In the long run the number of bees increases by a factor

$$f = \frac{1}{2} + \sqrt{\frac{5}{4}}.$$

in each generation. This is the famous golden ratio. A magical number that that appears all over mathematics. It has other surprising properties, for instance, by some measure, it is the most irrational of all numbers.

Here is a little bonus. On the previous sheet you may have noticed that $1/f = 1 - f$ has a second solution which is actually $-1/f$, this is the golden ratio's little brother g, note

$$\begin{aligned} f &= 1/2 + \sqrt{5/4} \\ g &= 1/2 - \sqrt{5/4} \end{aligned}$$

These two numbers work great as a team for instance they can give you the number of bees in generation n

$$T_n = \frac{f^n + g^n}{\sqrt{5}}$$

you can check that this works exactly, but why it works is a different story.