Single-level Models for Binary Responses

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y_i response for individual i ($i = 1, \ldots, n$), coded 0 or 1

Denote by r the number in the sample with $y = 1$

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$\mathrm{E}(y) = \pi = \Pr(y = 1)$ estimated by $\hat{\pi} = \frac{r}{n}$

$\mathrm{var}(y) = \pi(1 - \pi)$ estimated by $\hat{\pi}(1 - \hat{\pi})$

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y follows a Bernoulli distribution (special case of binomial distribution)

Linear Probability Model

Model for the mean of y

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or expressed as a model for y_i

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Estimate using ordinary least squares (as for continuous y).

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- Relationship between π and x may be nonlinear, although linearity assumption often reasonable for π between 0.2 and 0.8
- Possible to get predicted probabilities outside [0,1]. Again, this is unlikely if π lies between 0.2 and 0.8 for all x (or combinations of values on a set of x variables)

The Generalised Linear Model

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usually written as

$$F^{-1}(\pi_i) = \beta_0 + \beta_1 x_i$$

where F^{-1} is called the link function

The Logit/Logistic Model

Write $z = \beta_0 + \beta_1 x$

Logistic transformation of z

$$\pi = F(z) = \frac{\exp(z)}{1 + \exp(z)} = \frac{e^z}{1 + e^z}$$

π always lies between 0 and 1

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Logit model

$$\log\left(\frac{\pi}{1-\pi}\right) = z = \beta_0 + \beta_1 x$$

where $\pi/(1-\pi)$ is the odds that $y = 1$ and $\log[\pi/(1-\pi)]$ is the log-odds or logit

Interpretation of Logit Coefficients

Take exponentials of each side of logit model:

$$\frac{\pi}{1-\pi} = \exp(\beta_0 + \beta_1 x) = \exp(\beta_0).\exp(\beta_1 x) \tag{1}$$

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Interpret $\exp(\beta_1)$ as an odds ratio, comparing the odds for two individuals with x-values spaced 1 unit apart.

Example: State Differences in US Voting

$y = 1$ if intends to vote Bush in 2004 election, 0 otherwise

Variable	$\hat{\beta}$	$se(\hat{\beta})$	$\exp(\hat{\beta})$
Constant	-0.34	0.05	0.71
State (ref=California)			
New York	-0.19	0.08	0.83
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- Odds of voting Bush in California = 0.71
- Odds of voting Bush in New York are 0.83 times odds in California (17% lower)
- Odds of voting Bush in Texas are twice the odds in California (100% higher)

Other Link Functions

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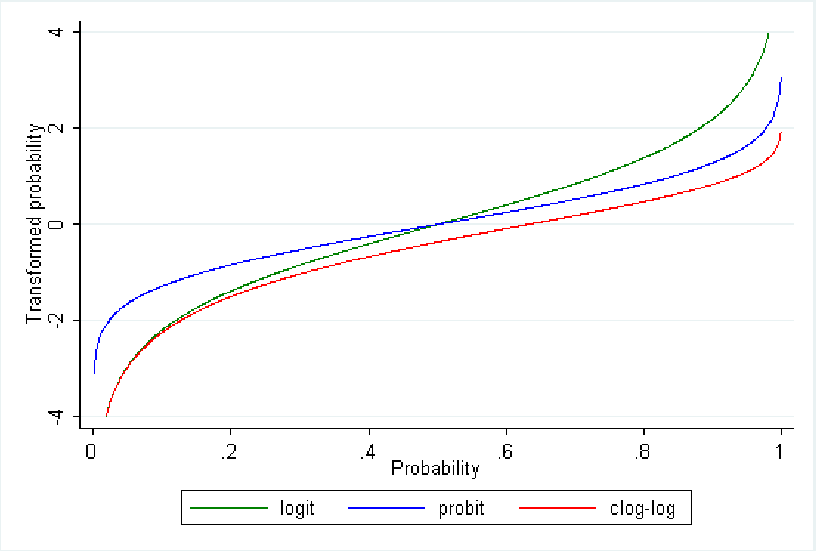
- Logit(0.5) = probit(0.5) but move further apart as π gets close to 0 or 1
- Logit and clog-log almost indistinguishable for small π

Logit, probit and clog-log transformations of π



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- $e_i^* \sim N(0, 1) \rightarrow$ probit model
- $e_i^* \sim$ standard logistic (with variance $\simeq 3.29$) $\rightarrow$ logit model

Relationship between probit and logit

The residual in the threshold model has fixed variance, but the value it is fixed at depends on the chosen distribution:

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Standard normal	1	probit
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Increasing the variance (and therefore scale of y^*) increases the magnitude of the coefficients.

$$\hat{\beta}_{\text{logit}} \simeq \sqrt{3.29}\ \hat{\beta}_{\text{probit}} = 1.8\ \hat{\beta}_{\text{probit}}$$

US Voting: Logit and Probit Coefficients

Compare coefficients from logit and probit models of voting Bush.

Variable	$\hat{\beta}_{\text{probit}}$	$\hat{\beta}_{\text{logit}}$	$\hat{\beta}_{\text{logit}}/\hat{\beta}_{\text{probit}}$
Constant	−0.21	−0.34	1.62
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New York	−0.12	−0.19	1.58
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But, for this simple model, predicted probabilities of voting Bush will be exactly the same for logit and probit models.

Predicted Probabilities of Voting Bush

Logit model

$$\hat{\pi} = \frac{\exp(-0.34 - 0.19 \; NY + 0.69 \; TEX)}{1 + \exp(-0.34 - 0.19 \; NY + 0.69 \; TEX)}$$

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California	(NY = TEX = 0)	$\hat{\pi} = 0.71/(1 + 0.71) = 0.42$
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Calculate using NORMDIST in Excel or CDFNORM in SPSS

Significance Testing

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Likelihood ratio test

Compare difference in deviance (-2 log-likelihood) between two nested models with χ^2_q, where $q =$ difference in number of parameters. Not available in MLwiN for discrete response models.

Confidence Intervals for β and Odds Ratios

95% CI for β

$$(\hat{\beta} - 1.96\ se(\hat{\beta}),\ \hat{\beta} + 1.96\ se(\hat{\beta}))$$

E.g. 95% CI for NY-California difference in log-odds of voting Bush is $-0.19 \pm (1.96 \times 0.08) = -0.35$ to -0.03

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95% CI for odds ratio, $\exp(\beta)$

Take exponent of lower and upper limits of 95% interval for β.

E.g. 95% CI for NY vs California odds ratio is $\exp(-0.35)$ to $\exp(-0.03) = 0.70$ to 0.97

Adding Further Predictors to Logit Model

Variable	$\hat{\beta}$	$se(\hat{\beta})$	Wald	$\exp(\hat{\beta})$
Constant	-0.42	0.11	-	0.66
State (ref=California)				
New York	-0.18	0.08	4.4	0.84
Texas	0.71	0.08	76.7	2.03
Female	-0.27	0.07	15.9	0.76
Age (years)	0.005	0.002	5.3	1.005

Note: Wald for joint test of state effects is 111.6 on 2 df ($\chi^2_{2;0.05} = 5.99$). All other tests on 1 df ($\chi^2_{1;0.05} = 3.84$).

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- Odds of voting Bush are 25% lower for women than for men (controlling for state and age)
- Odds of voting Bush increased by 0.5% for each 1-year increase in age (controlling for state and gender)

Predicted Probabilities from Logit Model

Variable	Value	$\hat{\pi}$
State	California	0.42
	New York	0.37
	Texas	0.59
Sex	Male	0.49
	Female	0.42
Age (years)	20	0.42
	30	0.43
	40	0.45

- Values of each variable varied in turn, holding others at sample mean (proportions for categorical variables)
- E.g. $\hat{\pi}$ for State calculated for each possible set of values for the State dummies, with Female=0.54 and Age=46.7 years

Interaction Effects

Suppose we believe the effect of age on voting intentions differs across states, then fit:

$$\begin{aligned}\log\left(\frac{\pi}{1-\pi}\right) &= \beta_0 + \beta_1 \text{ NY} + \beta_2 \text{ TEX} + \beta_3 \text{ FEMALE} + \beta_4 \text{ AGE} \\ &+ \beta_5 \text{ NY} \cdot \text{AGE} + \beta_6 \text{ TEX} \cdot \text{AGE}\end{aligned}$$

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State	Age effect	Estimated age effect
California	β_4	0.005
New York	$\beta_4 + \beta_5$	0.005+0.010
Texas	$\beta_4 + \beta_6$	0.005−0.012

Interaction Effects

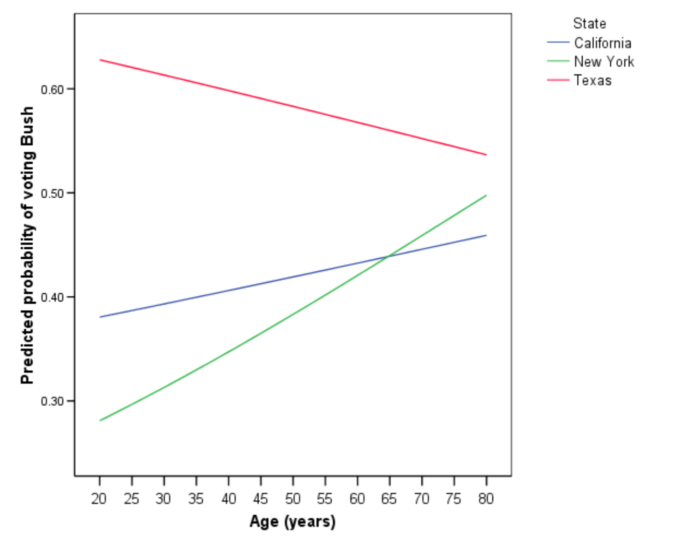
Suppose we believe the effect of age on voting intentions differs across states, then fit:

$$\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 \text{ NY} + \beta_2 \text{ TEX} + \beta_3 \text{ FEMALE} + \beta_4 \text{ AGE} + \beta_5 \text{ NY} \cdot \text{AGE} + \beta_6 \text{ TEX} \cdot \text{AGE}$$

State	Age effect	Estimated age effect
California	β_4	0.005
New York	$\beta_4 + \beta_5$	0.005+0.010
Texas	$\beta_4 + \beta_6$	0.005−0.012

Test of $H_0 : \beta_5 = \beta_6 = 0$ gives Wald$= 15.3$ on 2 df, so strong evidence of interaction effect

$\hat{\pi}$ by Age and State



Note: Female fixed at 0.54 (sample proportion)