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An approach to the scaling of categorized attributes

BY M. J. R. HEALY
Medical Research Council Clinical Research Centre, Harrow, Middlesex

AND H. GOLDSTEIN
National Children's Bureau, London

SUMMARY

This paper shows that the notion of minimizing the disagreement between parallel assessments of the same quantity unifies a number of current methods of scaling and leads to straightforward extensions to cover new problems. Applications are found in the assessment of physical maturity, the measurement of children's behaviour, and in longitudinal studies with sets of items at several occasions.

Some key words: Canonical correlation; Maturity; Multivariate analysis; Scaling; Scalogram analysis.

1. INTRODUCTION: THE ASSESSMENT OF MATURITY

The notion of 'maturity' plays a rather special role in physical anthropometry. Most measurements on growing children, though possibly difficult to make in a standardized and reproducible manner, do not present problems of definition. Maturity, on the other hand, though its general meaning is fairly clear, does not possess an obvious definition and in particular it is not obvious how it should be measured. What are available in practice are a large number of attributes of the growing child which pass through several well-defined stages or categories as the child's maturity increases (Tanner, 1962, Chapter 4); these include secondary sex characteristics, the teeth and the bones of various joints including the wrist and the knee. For any one of these attributes, it is possible to make unequivocal comparisons between two children: child A is more mature than child B for a specific attribute if he, or she, exhibits a later occurring stage of that attribute. In practice, of course, if child A is more mature than child B for one attribute, he is usually so for many others and it is just this fact that leads us to the notion of a single underlying maturity value for each child, a value which is reflected in the stages of all the different attributes. As an extension of this, we may postulate regional maturities which influence specific sets of attributes and give rise to systematic differences between them. The problem thus arises of assessing a child's maturity value from the stages which he exhibits with respect to a number of attributes, in fact of setting up a maturity scale.

This is a scaling problem of the kind which is of interest in a variety of disciplines; see, for example, Edwards (1957) and Torgerson (1958). Sociologists, for example, have attempted to assess a measure of racial prejudice by administering to each of a number of subjects several sets of questions, each set containing graded categories indicative of increasing degrees of prejudice. Similar methods are used in psychology to assess measures such as intelligence or extroversion.

It seems an obvious first step to attach a numerical score to each stage of each attribute. Every subject will then possess a vector of scores and the problem becomes twofold; that of determining an optimal system of scores and that of deciding how to combine the attribute scores from a particular subject into a single overall score. For the second the only practical proposition seems to be to take the average of the attribute scores, possibly weighted. This approach has been extensively studied by Guttman (1941) under the name of scalogram analysis, and our work may be regarded as a generalization of his, as will appear below. We are left with the problem of determining, indeed of defining, an optimal system of scores. A possible system is that given by allotting a score of one to the first stage of each attribute, two to the second stage, etc. For an application to skeletal maturity see Acheson (1957); in sociology, this system of scoring gives rise to the Likert index (Torgerson, 1958). However, the tacit assumption that all the stages of every attribute are equally spaced may well be inappropriate.

Our proposed definition of an optimal scoring system is motivated by the assumption that all the different attributes are estimating the single underlying variable. We would therefore like the different attribute scores for a particular subject to agree as closely as possible, and we define an optimal system as one which minimizes the total amount of disagreement over a standardizing group of subjects. The consequences of such a definition are developed in the next section.

2. Mathematical formulation

2·1. *General considerations*

Consider a scoring system with h attributes in which the ith attribute has p_i stages and the score allotted to the jth stage of the ith attribute is x_{ij} $(i = 1, \ldots, h;\ j = 1, \ldots, p_i)$; let $n = \Sigma_i p_i$ be the total number of stages for all the attributes. Consider a standardizing group of N subjects and let z_{im} be the score of the mth subject on the ith attribute; if this subject exhibits the jth stage of this attribute, then $z_{im} = x_{ij}$. If we agree to use a weighted mean of the attribute scores as the overall score, then the overall score for the mth subject will be, with $\Sigma_i w_i = 1$,

$$\bar{z}_m = \Sigma_i w_i z_{im}.$$

The attribute scores for the mth subject, $z_{1m}, \ldots, z_{hm}$, will in general not all be equal, and we propose to measure their disagreement by their weighted sum of squares about their mean, $d_m = \Sigma_i w_i (z_{im} - \bar{z}_m)^2$.

The total disagreement for the standardizing group is $D = \Sigma_m d_m$. We adopt a scoring system that makes D a minimum.

Obviously it is essential to minimize D subject to some standardizing constraints on the x_{ij} to avoid a trivial zero situation. We illustrate two possibilities.

2·2. *Quadratic constraint*

In the case of a maturity scale the standardizing group will presumably cover the whole range of maturity and it is elementary that we wish their scores to differ so as to reflect their maturity differences. Likewise, in other applications where the standardizing group is representative of a population of individuals, we wish subject scores to differ along the underlying dimension. We may therefore constrain the variance of the N subject overall scores to be different from zero by imposing, say,

$$\Sigma_m(\bar{z}_m - \tilde{z})^2 = 1, \quad \tilde{z} = \Sigma_m \bar{z}_m/N. \tag{2·1}$$

It is clear that we can add a constant to all the x_{ij} without affecting the d_m, and without loss of generality we impose the further constraint $\tilde{z} = 0$, so that (2·1) becomes simply $\Sigma_m \bar{z}_m^2 = 1$. The algebra of the minimization is straightforward.

We are to minimize $D = x^T A x$ subject to $x^T Z x = 1$ and $x^T S 1 = 0$. Here the vector x is $n \times 1$ with elements x_{ij}. The matrix A is $n \times n$ and symmetric. Its diagonal elements are $N_{ij} w_i(1-w_i)$, where N_{ij} is the number of subjects in stage j of attribute i. Its off diagonal elements are $-N_{ijkl} w_i w_k$, where N_{ijkl} is the number of subjects exhibiting both stage j of attribute i and stage l of attribute k. Since $N_{ijkl} = 0$ if $i = k$ and $j \neq l$, A has diagonal blocks on its main diagonal. The matrix Z is $n \times n$ and symmetric with diagonal elements $N_{ij} w_i^2$ and off-diagonal elements $N_{ijkl} w_i w_k$. The matrix $S = A + Z$ is $n \times n$ diagonal with elements $N_{ij} w_i$. Finally the vector 1 is $n \times 1$ with all elements equal to one.

By using Lagrange multipliers, this leads to

$$2Ax - 2\lambda Zx - \mu S1 = 0. \tag{2·2}$$

Multiplying on the left by 1^T and noting that $1^T A = 0$, $1^T Sx = 0$, we find that $\mu 1^T S 1 = 0$, so that $\mu = 0$. Thus (2·2) becomes

$$(A - \lambda Z)x = 0. \tag{2·3}$$

Premultiplication by x^T gives $D = x^T A x = \lambda x^T Z x = \lambda$, so that we appear to need the smallest latent root of (2·3); but there is in fact a zero root, with latent vector 1, which corresponds to the trivial zero solution. The optimal scores by our criterion will thus be given by the elements of the latent vector corresponding to the smallest nonzero root of

$$|A - \lambda Z| = 0. \tag{2·4}$$

If we take all the w_i to be equal, this is essentially Guttman's solution (Torgerson, 1958, pp. 338–45).

The next smallest root of (2·4) gives a system of scores which minimizes D subject to the extra constraint of being uncorrelated with the first system over the standardizing group. This gives the possibility of defining differential regional maturity measures as discussed in § 1.

2·3 *Linear constraint*

Since the stages for each attribute are ordered, we can envisage subjects whose overall maturity is the lowest possible, namely all attributes at their earliest stages, and the highest possible, namely all attributes at their latest stages. It is a distinguishing feature of physical maturity that all normal individuals pass from one of these situations, postconception, to the other, adulthood. Under these circumstances we are led to think of a subject's overall maturity as a proportion of the maximum possible, and so of a maturity scale that extends from 0 to 1. With the notation of the previous section, the extreme scores will be $\Sigma_i w_i x_{i1}$ and $\Sigma_i w_i x_{ip_i}$, respectively, and we can avoid the trivial solution by constraining these to be different, say equal to 0 and 1. In matrix form these constraints are

$$x^T q = 0, \quad x^T r = 1, \tag{2·5}$$

where q and r are $n \times 1$ with all elements equal to 0 except for those corresponding to stages 1 and p_i, respectively, which are equal to 1.

The minimization of D now leads to $2Ax-\lambda q-\mu r=0$, and these n linear equations together with (2·5) can be solved for the $n+2$ unknowns x, λ and μ. We find that $2D=\mu=-\lambda$.

As before we can obtain further auxiliary scoring systems. We may, for instance, obtain scores y which minimize D subject to (2·5) and subject also to being uncorrelated with the first system over the standardizing group. Then

$$y^{\mathrm{T}}Zx=0, \tag{2·6}$$

and we get the $n+3$ equations given by (2·5), (2·6) and $2Ay-\lambda q-\mu r-\nu Zx=0$.

Neither of the techniques we have described forces the scores allotted to the stages of a single attribute to increase monotonically. Methods with this property could be developed from linear programming concepts (Bradley, Kalli & Coons, 1962), but in our view these would be inappropriate. We would rather regard any serious failure of monotonicity as indicating a defect in the definitions of the stages of the corresponding attribute.

3. Example

Physical maturity data are usually rather extensive (Tanner *et al.* 1975), and in the interests of compactness we give a behavioural example using data taken from the National Child Development Study (Davie, Butler & Goldstein, 1972). The subjects were 12232 mothers of 11-year old children and the attributes were questions designed to measure 'antisocial behaviour'.

(i) Does the child destroy its own or others' belongings?
(ii) Does the child fight with other children?
(iii) Is the child disobedient at home?

Each attribute had three ordered categories, the replies 'never', 'sometimes' and 'frequently'. We use equal weights for each attribute, $w_i=\frac{1}{3}$ $(i=1,2,3)$. The lower triangle of the matrix Z is shown in Table 1. The scores corresponding to quadratic and linear constraints are shown in Table 2, rescaled so that each 'never' response scores 0 while a response of 'frequently' to all three questions scores 100. It will be seen that the two systems

Table 1. *The lower triangle of the matrix* 9*Z for example of* 11-*year scores*

1st column:	11440, 0, 0, 5923, 5134, 383, 5957, 5254, 229
2nd column:	667, 0, 143, 440, 84, 135, 468, 64
3rd column:	125, 22, 62, 41, 18, 70, 37
4th column:	6088, 0, 0, 3896, 2111, 81
5th column:	5636, 0, 2084, 3387, 165
6th column:	508, 130, 294, 84
7th column:	6110, 0, 0
8th column:	5792, 0
9th column:	330

Thus 11440 mothers answered 'never' to question one and of these 383 answered 'frequently' to question two.

are rather different; in particular, the linear constraint causes the 'frequently' response to the first question to be given a relatively high score, while it effectively equates the 'never' and 'sometimes' responses on all three questions.

Table 2. *Scaled 11-year scores for antisocial behaviour attributes*

		Constraints	
Attribute	Category	Quadratic	Linear
1	Never	0	0
	Sometimes	19·7	2·7
	Frequently	39·4	56·7
2	Never	0	0
	Sometimes	9·2	1·7
	Frequently	27·6	17·8
3	Never	0	0
	Sometimes	9·7	1·8
	Frequently	33·0	25·5

4. Relating two sets of attributes

4·1. *General considerations*

We have established a scoring system for a single variable by minimizing the disagreement within a single set of scores. The underlying principle can be used in a number of other contexts. Suppose, for example, that we have two sets of attributes, perhaps one relating to a number of behavioural attributes and the other to a number of social attributes. We may wish to set up a scoring system with the property that the two overall scores for each subject agree as closely as possible.

Suppose that there are h_1 and h_2 attributes in the two sets; denote the two scoring systems by vectors x_1 and x_2 of lengths n_1 and n_2 and the associated weights by vectors w_1 and w_2. If the attribute scores for the mth subject are z_{1im} and z_{2jm} $(i = 1, \ldots, h_1; j = 1, \ldots, h_2)$ we define the two overall scores to be

$$\bar{z}_{1m} = \Sigma_i w_{1i} z_{1im}, \quad \bar{z}_{2m} = \Sigma_i w_{2i} z_{2im},$$

and the disagreement to be simply $d_m = (\bar{z}_{1m} - \bar{z}_{2m})^2$. We now choose the scoring systems to minimize $D = \Sigma_m d_m$. We have

$$D = x_1^{\mathrm{T}} Z_{11} x_1 + x_2^{\mathrm{T}} Z_{22} x_2 - 2x_1^{\mathrm{T}} Z_{12} x_2.$$

Here Z_{11} and Z_{22} are equivalent to the matrix Z of § 2·2, while Z_{12} is an $n_1 \times n_2$ matrix with typical element $w_{1i} w_{2k} N_{kl}^{ij}$, the number of subjects exhibiting stage j of attribute i in the first set of attributes and simultaneously stage l of attribute k in the second set. Again there is a trivial solution in which all the x_1's and x_2's are equal and we must impose a constraint to avoid this.

4·2. *Quadratic constraints*

We may constrain the variances of both of the overall scores to be nonzero, say $\Sigma_m(\bar{z}_{km} - \tilde{z}_k)^2 = 1$, with $\tilde{z}_k = \Sigma_m \bar{z}_{km}/N$ $(k = 1, 2)$. We can still impose arbitrary origins on the scores for each attribute, and we do this so that $\Sigma_j N_{kij} x_{kij} = 0$, for all i $(k = 1, 2)$, noting that this gives $\tilde{z}_1 = \tilde{z}_2 = 0$. In matrix form the constraints are

$$x_1^T Z_{11} x_1 = 1, \quad x_2^T Z_{22} x_2 = 1, \tag{4·1}$$

$$x_1^T S_1 Q = 0, \quad x_2^T S_2 R = 0, \tag{4·2}$$

where S_1 and S_2 are diagonal with elements $w_{1i} N_{1ij}$ and $w_{2i} N_{2ij}$, and where Q and R are of sizes $n_1 \times h_1$ and $n_2 \times h_2$, with unity in those elements of the ith column that correspond to stages of the ith attribute and zeros elsewhere. The problem becomes that of maximizing subject to (4·1) the quantity $E = x_1^T Z_{12} x_2$ which amounts to the correlation between the two overall scores. This is the canonical correlation problem, the scores being given by the canonical vectors corresponding to the largest correlation. When each set contains only one attribute, the problem is that of obtaining optimal additive scores from a contingency table (Kendall & Stuart, 1961, p. 569; Fisher, 1938, §49·2).

The maximization equations are

$$Z_{12} x_2 - 2\lambda_1 Z_{11} x_1 - \mu_1 S_1 Q = 0, \quad Z_{12}^T x_1 - 2\lambda_2 Z_{22} x_2 - \mu_2 S_2 R = 0,$$

leading to

$$(Z_{11}^- Z_{12} Z_{22}^- Z_{12}^T - \lambda^2 I) x_1 = 0$$

in terms of generalized inverses of the singular matrices Z_{11} and Z_{22}. This can be solved by standard techniques which allow for the singularities; see Appendix. The largest root is $\lambda = 1$, giving the trivial solution, and the required scores are the elements of the vectors corresponding to the next largest root. Further sets of uncorrelated scores can be derived from the vectors corresponding to subsequent roots.

In some applications of the above method, the two sets of attributes are in fact the same set measured at two different occasions, and it may then be of interest to impose the extra constraint that the two scoring systems shall be identical. We now have to maximize $E = x^T Z_{12} x$ subject to

$$x^T (Z_{11} + Z_{22}) x = 1, \quad x^T (S_1 + S_2) Q = 0,$$

which leads to

$$\{(Z_{12} + Z_{12}^T) - \lambda (Z_{11} + Z_{22})\} x = 0.$$

The same procedure can clearly be applied to the more usual canonical correlation situation, in which the Z matrices contain sums of squares and products of continuous variables.

4·3. *Linear constraints*

We can alternatively impose the $h_1 + h_2 + 2$ linear constraints, for all i ($k = 1, 2$),

$$\Sigma_i w_{ki} x_{kip_{ki}} = 1, \quad x_{ki1} = 0.$$

In matrix form these are

$$x_1^T r_1 = 1, \quad x_2^T r_2 = 1, \quad x_1^T Q_1 = 0, \quad x_2^T Q_2 = 0, \tag{4·3}$$

where r is defined in (2·5) and Q_k is of size $n_k \times h_k$ with unity in the ith column corresponding to the first stage of the ith attribute and zeros elsewhere. Minimization leads to

$$2Z_{11} x_1 - 2Z_{12} x_2 - \lambda_1 r_1 - Q_1 \mu_1 = 0, \quad -2Z_{12}^T x_1 + 2Z_{22} x_2 - \lambda_2 r_2 - Q_2 \mu_2 = 0,$$

and these with (4·3) provide $n_1 + n_2 + h_1 + h_2 + 2$ equations in as many unknowns. Further sets of uncorrelated scores can be determined as before.

If we impose the extra condition that the two systems of scores should be identical, we are led to

$$x^T Q = 0, \quad x^T r = 1, \quad 2(Z_{11} + Z_{22} - Z_{12} - Z_{12}^T) x - \lambda r - Q\mu = 0.$$

5. Example

We use the data from § 3 with responses to the same questions by the mothers of the same children at the age of 7. The number of responses was 10 180 and the estimated scores using quadratic constraints are given in Table 3, along with the optimal scores for the two ages separately and those for the paired data constrained to be the same at both ages. The separate systems at the two ages are very similar, and it is perhaps not too surprising that the optimal linked scores, although based on a different kind of disagreement, are not very different from them.

Table 3. *Scaled 'canonical' scores for behaviour attributes*

Attribute	Category	11-year score	7-year score	Equal 11-year and 7-year scores
1	Never	0 (0)	0 (0)	0
	Sometimes	18·0 (19·7)	16·9 (18·0)	14·3
	Frequently	44·9 (39·4)	40·9 (31·0)	35·4
2	Never	0 (0)	0 (0)	0
	Sometimes	13·3 (9·2)	13·7 (11·8)	15·4
	Frequently	26·3 (27·6)	30·3 (31·2)	32·6
3	Never	0 (0)	0 (0)	0
	Sometimes	9·0 (9·7)	10·1 (12·5)	10·9
	Frequently	28·7 (33·0)	28·8 (37·8)	32·0

Separate age scores given in brackets

Correlation between 7-year and 11-year individual scores: (*a*) unequal scores, 0·441; (*b*) equal scores, 0·410; (*c*) separate systems at each age, 0·433.

6. Relating more than two sets of attributes

6·1. *General considerations*

We can extend the methods of § 4 to the situation in which each subject has more than two sets of attributes. If there are p sets, we introduce a further set of weights and define the disagreement for the mth subject to be

$$d_m = \sum_{k=1}^{p} c_k(\bar{z}_{km} - \tilde{z}_m)^2$$

with

$$\tilde{z}_m = \sum_{k=1}^{p} c_k \bar{z}_{km}, \quad \Sigma_k c_k = 1.$$

The total disagreement is

$$D = \Sigma_m d_m = \Sigma_k c_k(1-c_k) x_k^{\mathrm{T}} Z_{kk} x_k - \sum_{k \neq l} c_k c_l x_k^{\mathrm{T}} Z_{kl} x_l,$$

where x_k, of length n_k, is the vector of scores for the kth set of attributes and the Z matrices are as defined in § 4. We need constraints to avoid the trivial solution with all the x_{kij} equal.

6·2. *Quadratic constraints*

We may impose

$$c_k(1-c_k)\,\Sigma_m(\bar{z}_{km}-\tilde{z}_k)^2 = 1,$$

where, for all i and k,

$$\tilde{z}_k = \Sigma_m \bar{z}_{km}/N, \quad \Sigma_j N_{kij} x_{kij} = 0.$$

In matrix form these are

$$c_k(1-c_k)x_k^{\mathrm{T}} Z_{kk} x_k = 1, \quad x_k^{\mathrm{T}} S_k R_k = 0,$$

with S_k, R_k defined in the same way as S_1, S_2 and R in (4·2). We minimize the disagreement by maximizing

$$E = \sum_{k+l} c_k c_l x_k^{\mathrm{T}} Z_{kl} x_l,$$

which is the weighted sum of all the correlations between pairs of sets. This is analogous to the SUMCOR method described by Kettenring (1971) for continuous variables. The result is p sets of equations

$$\sum_{l+k} c_k c_l Z_{kl} x_l - \lambda c_k(1-c_k) Z_{kk} x_k = 0 \quad (k = 1, \ldots, p),$$

which can be solved by a straightforward extension of the method described in §4. The largest latent root $\lambda = 1$ corresponds to the trivial solution, the optimal scores correspond to the second largest root and further sets of scores can be derived from the vectors corresponding to subsequent roots.

If we make all the scoring systems the same, by imposing

$$x_k = x \quad (k = 1, \ldots, p), \tag{6·1}$$

we need to maximize

$$E = \sum_{k+l} c_k c_l x^{\mathrm{T}} Z_{kl} x$$

subject to

$$\Sigma_k c_k(1-c_k) x^{\mathrm{T}} Z_{kk} x = 1, \quad \Sigma_k c_k x^{\mathrm{T}} S_k R_k = 0,$$

which leads to the equations

$$\{\sum_{k+l} c_k c_l Z_{kl} - \lambda \Sigma_k c_k(1-c_k) Z_k\} x = 0.$$

6·3. *Linear constraints*

We impose

$$x_k^{\mathrm{T}} r_k = 1, \quad x_k^{\mathrm{T}} Q_k = 0 \quad (k = 1, \ldots, p), \tag{6·2}$$

where r_k and Q_k are defined as in (4·3). We then obtain the p sets of equations

$$2c_k(1-c_k) Z_{kk} x_k - 2\sum_{k+l} c_k c_l Z_{kl} x_l - \lambda_k r_k - Q_k \mu_k = 0 \quad (k = 1, \ldots, p),$$

which gives with (6·2) $\Sigma_k(h_k + n_k + 1)$ equations in as many unknowns. If we impose the extra condition (6·1), we obtain the equations

$$2\{\Sigma_k c_k(1-c_k) Z_{kk} - \sum_{k+l} c_k c_l Z_{kl}\} x - \lambda r - Q\mu = 0.$$

As before, further uncorrelated sets of scores can be determined.

7. Sampling considerations

We have expressed our results in terms of matrices of numbers of subjects in a standardizing group who exhibit various combinations of stages in different attributes. If we divide by the total number of subjects they become the proportions in the standardizing group and may in certain circumstances be regarded as estimates, even maximum likelihood estimates, of probabilities in an appropriate population.

In the application to physical maturity, the reference population is one with all ages equally represented, though it is necessary to truncate this at some suitable ages to avoid the inclusion of large numbers of wholly immature or fully mature subjects. It is difficult in practice to obtain a random sample from such a population, and some form of age-stratification will usually be needed, the probabilities being estimated separately in the different strata and combined to give estimates of those in the desired standard population.

8. Discussion

We have shown that the simple notion of minimizing disagreement between parallel assessments of the same quantity unifies a number of current methods of scale construction and leads straightforwardly to extensions covering new problems. Our work is closely related to that of Guttman (1941) and previous workers, notably Thurstone (1929); an extensive bibliography is given by Torgerson (1958). Much recent work on scaling has stemmed from papers by Shephard (1962) and Gower (1966); these are based on the concept of distance betweeen subjects and seem to us to be only loosely connected with our problems.

The scoring methods for relating several sets of attributes may be compared with those for continuous variables given by Kettenring (1971). Our proposals could readily be extended to this situation and would appear to lead to much simpler calculations. Most applications to categorical attribute data seem to have been restricted to the two-set case, usually with just one attribute in each set, although Lingoes (1973, Chapter 19) considers the p-set case with one attribute in each set. Srikantan (1970) investigates the sampling distributions of functions of the latent roots in the two-set case with one attribute in each set. The methods for relating several sets may be of practical interest in longitudinal studies; in intelligence testing, for example, it is necessary to use different attributes on each occasion, but a combined measure of some underlying factor assumed to be present at each occasion is required. Also in longitudinal studies we may wish to give less weight to those ages which are, for example, closer together.

The approach used in this paper can readily be adapted to special systems. One such case has been considered by Hill (1974) among others under the name of 'first order correspondence analysis', and is essentially intermediate between the system defined in § 2·2 and that defined in § 6·2, with just one attribute per set. It imposes a single quadratic constraint analogous to (2·1) and a constraint for each attribute analogous to (4·2). Thus it may be regarded either as additionally constraining the system (2·1) by requiring each attribute to have the same zero mean, or as allowing each attribute in § 6·1 to operate on a different scale, subject only to an overall quadratic constraint.

It would be of interest to investigate systems in which the data themselves determine the weights, which could then be applied to single stages as well as to whole attributes. In an application to the assessment of physical maturity, the attributes are the bones of the hand and wrist and the categories are the recognizable stages of development through which the bones pass. If a particular stage normally lasts for a long time, observing it clearly gives less information than the observation of a stage which is more transitory and so more characteristic of a particular level of maturity; presumably therefore the more transitory stage should receive a higher weight. Such a system would provide a total weight which differed from subject to subject, indicating that certain combinations of attribute categories gave a more precise indication of the overall score than others. However, this approach leads to very large scale nonlinear minimization problems and we have not pursued it further.

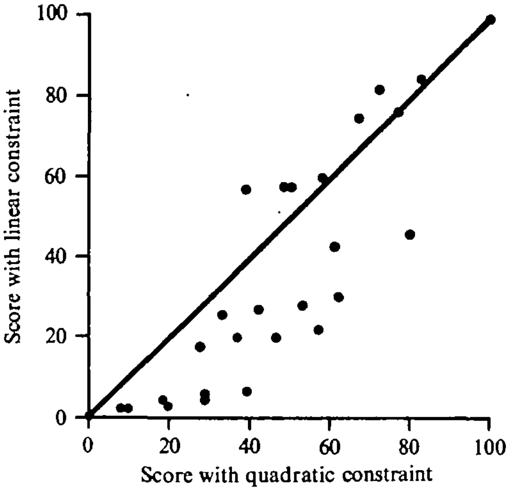

Fig. 1. Antisocial behaviour attributes. Comparison of scores with two different constraints.

The imposition of constraints has played a major role in the development of our methods. Quadratic constraints of the kind we have used are universally used in multivariate statistics, and characterize the two basic techniques of principal components and canonical correlation analysis. We have found it interesting that our linear constraints lead to algebra that is as straightforward and computation that is somewhat simpler. More importantly, however, our use of linear constraints is motivated by the nature of the attribute categories themselves. Since these have a defined ordering within each attribute, they allow us to specify the two end-points of the scale. The lowest point occurs when a subject responds with the lowest category of each attribute, and the highest point occurs when a subject responds with the highest category of each attribute. It seems natural to use this information, therefore, to specify the constraints in the way we have done. An important point emerges from the example in Table 2, where we show that the two types of constraint produce noticeably different scoring systems. Figure 1 shows the resulting scores on the two systems from the 27 possible combinations, linearly scaled so that the three responses 'never' and the three responses 'frequently' score 0 and 100, respectively; Kendall's τ for these 27 pairs of values is only 0·55. This seems to show that a particular scoring system can be almost as much determined by the constraints imposed as by the data on which it is based.

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APPENDIX

We wish to solve

$$(Z_x^- Z_{xy} Z_y^- Z_{xy}^T - \lambda^2 I)x = 0. \tag{A 1}$$

The matrices Z_x and Z_y have nullity $h_x - 1$ and $h_y - 1$, respectively, and the vectors formed by summing the rows for each attribute are equal. Hence a simple way of constructing the generalized inverses Z_x^- and Z_y^- is to form the reduced matrix obtained by omitting the first category of each attribute except the last, inverting the resulting nonsingular symmetric matrix and then inserting rows and columns of zeros corresponding to the first categories of all attributes except the last.

The equations (A 1) then give the particular solution with the scores for the omitted categories equal to zero. When the corresponding rows and columns of the matrices in (A 1) are omitted we have the reduced form $(A - \lambda^2 I)u = 0$, which can be solved by standard methods to give the vector u containing the nonzero elements of x.

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