Random Slope Models

Hedonism example

Our questions in the last session

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Group lines

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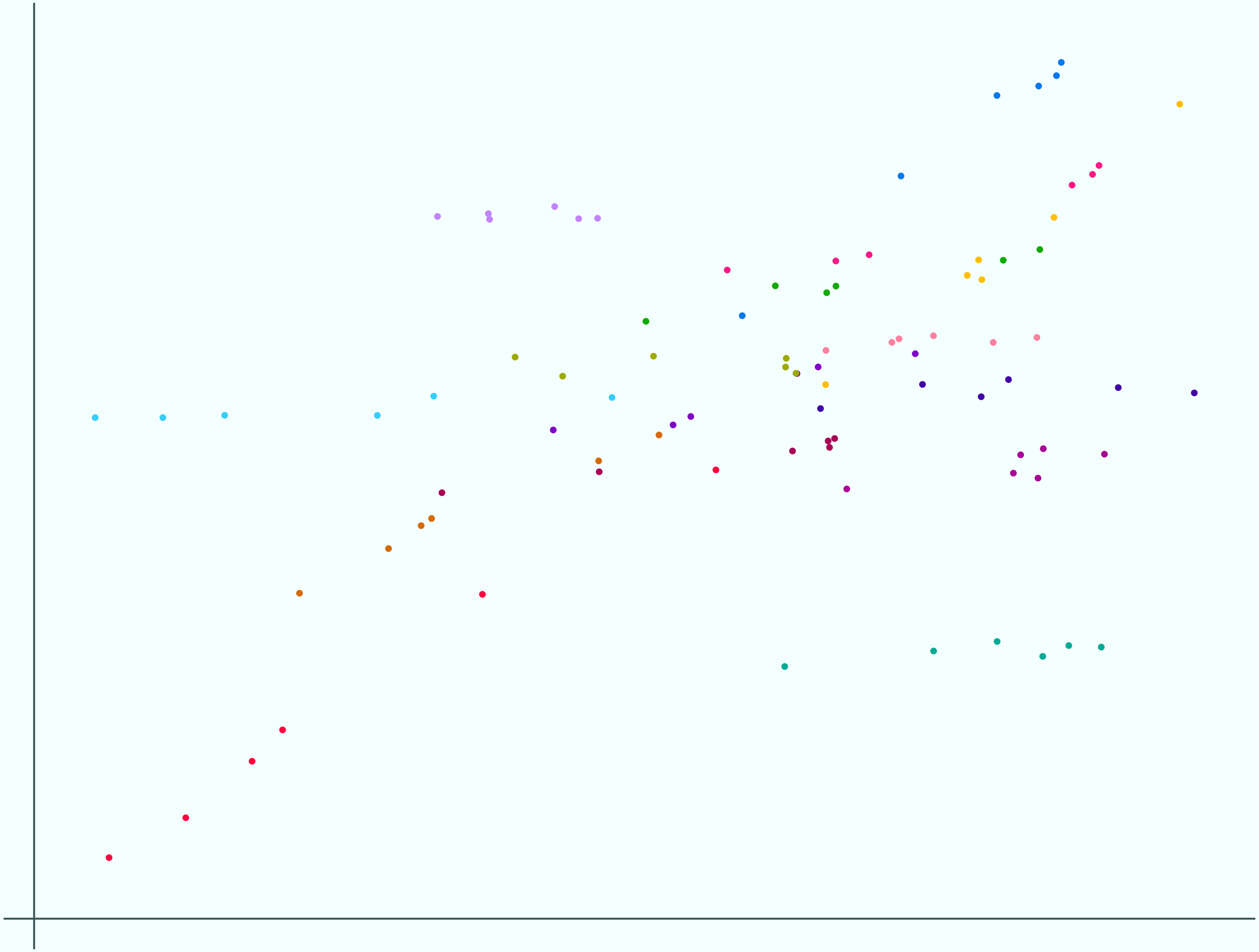
  - Recall that for the variance components model all lines were flat i.e. they had slope 0
- So in every group, the relationship between the explanatory variable and the response is the same
- This is one of the assumptions of the random intercept model
- However, sometimes the effect of the explanatory variable may differ from group to group and this may be of interest

A possible situation



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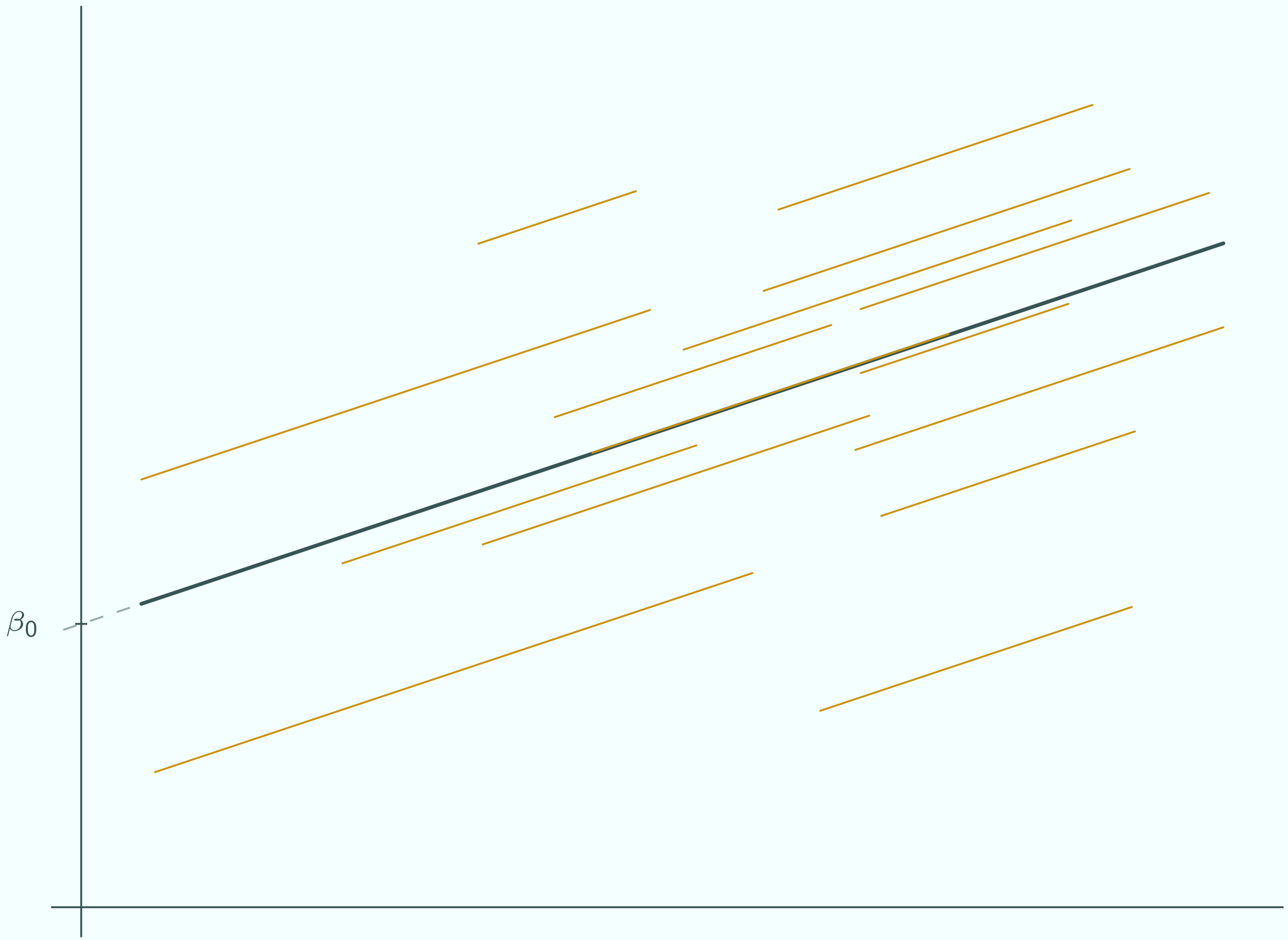

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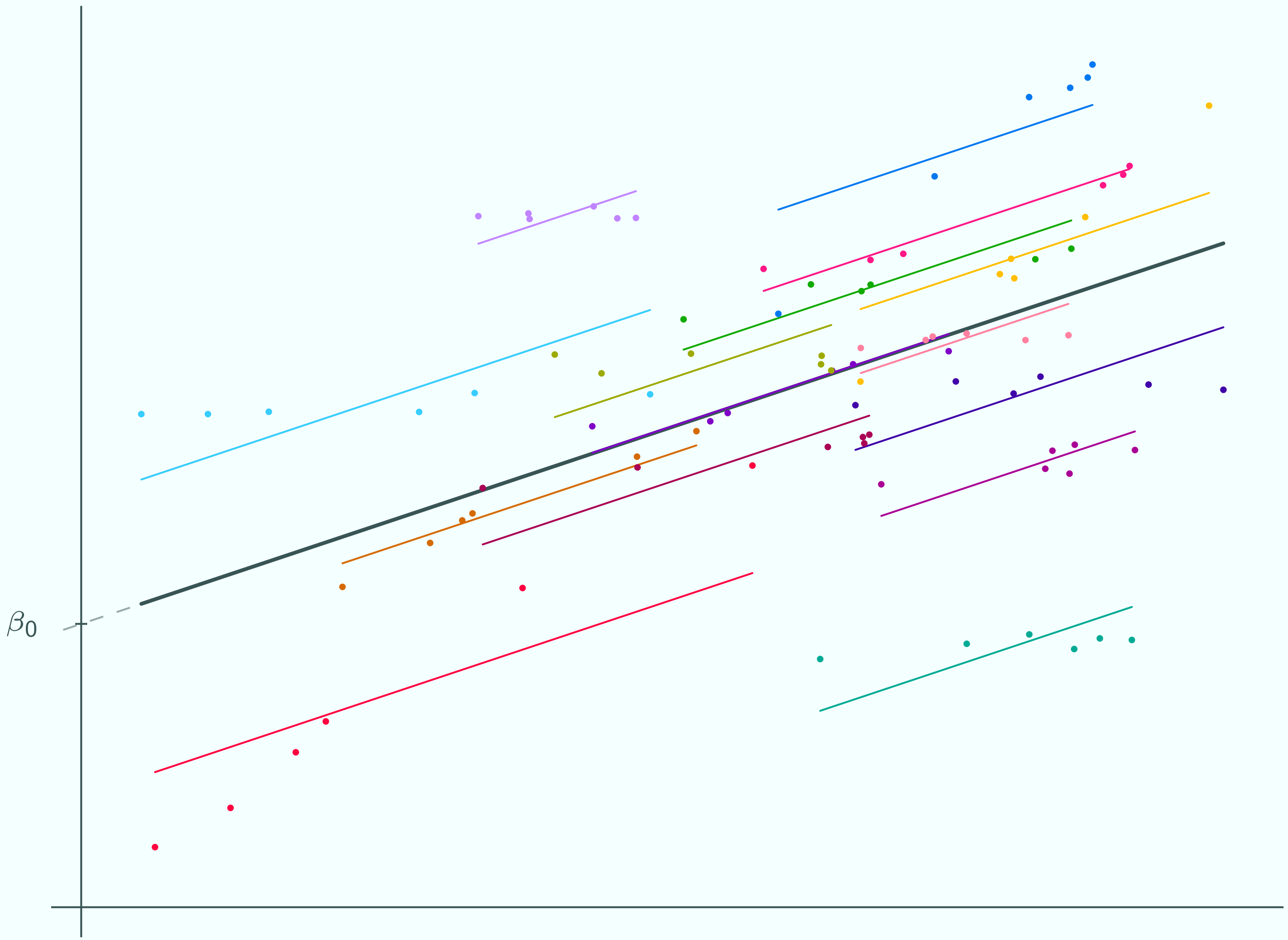

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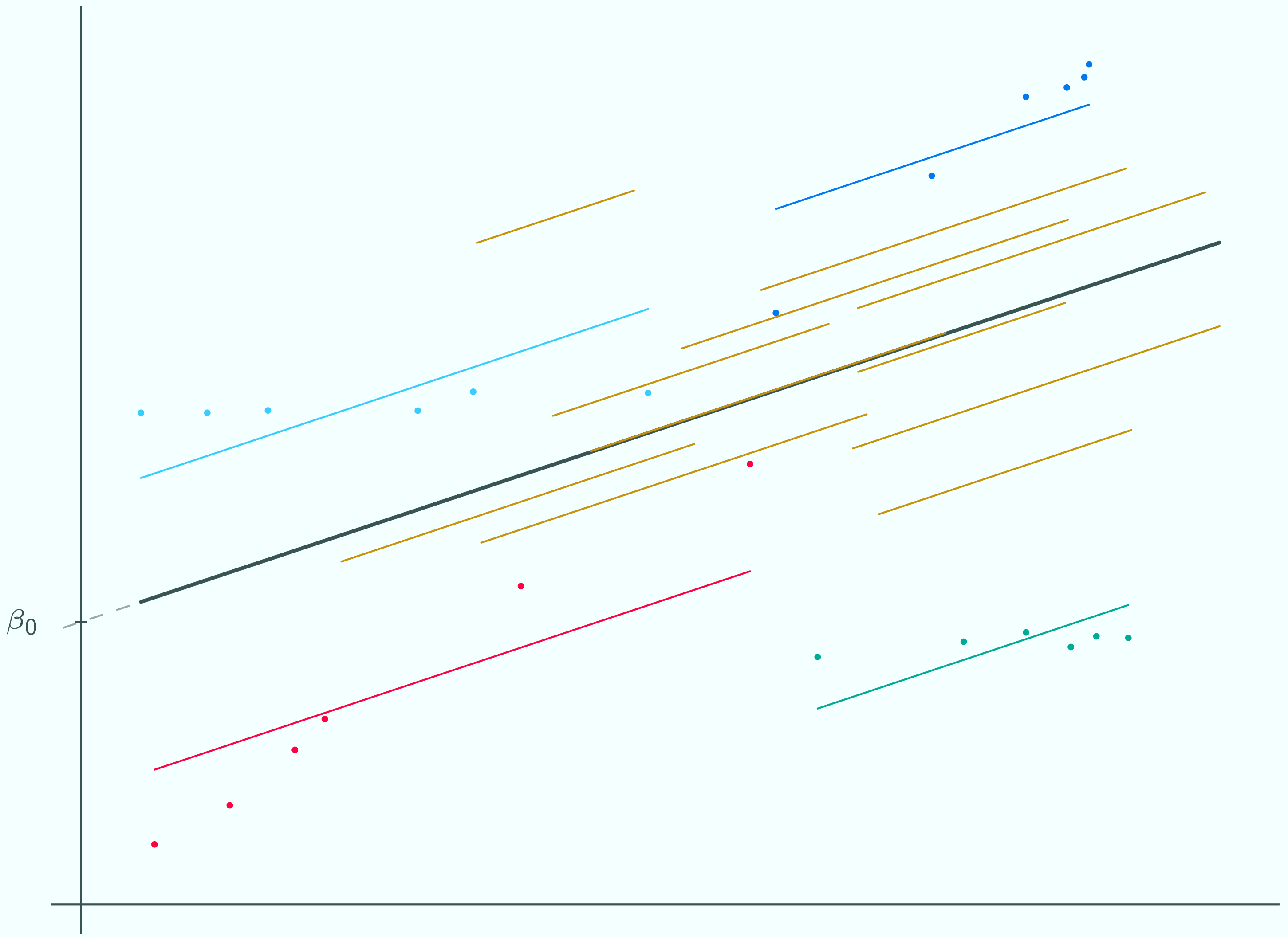

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Real examples

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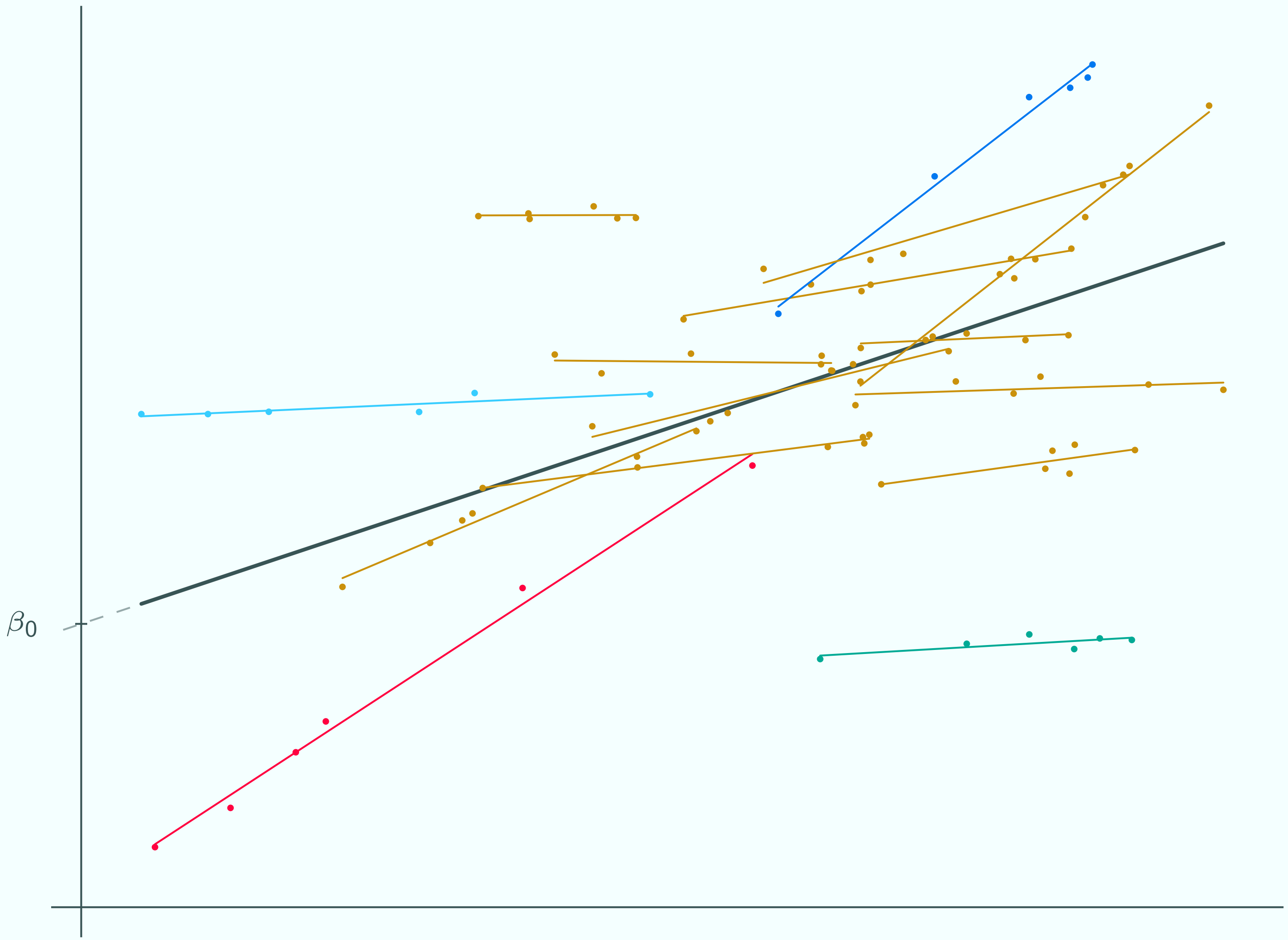
  - For some datasets there is only enough power to fit a random intercepts model in any case

Solution: Random Slopes Model



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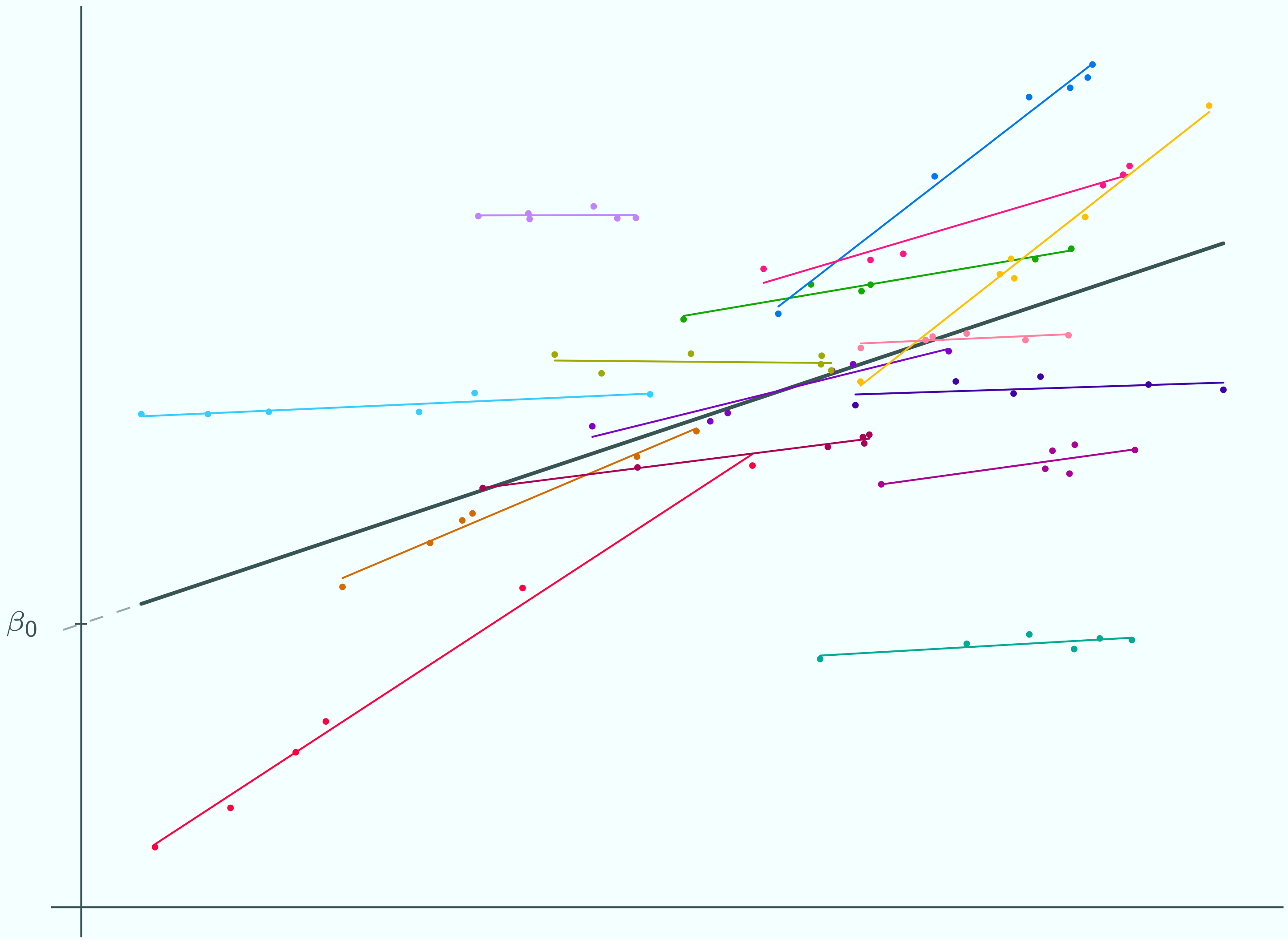

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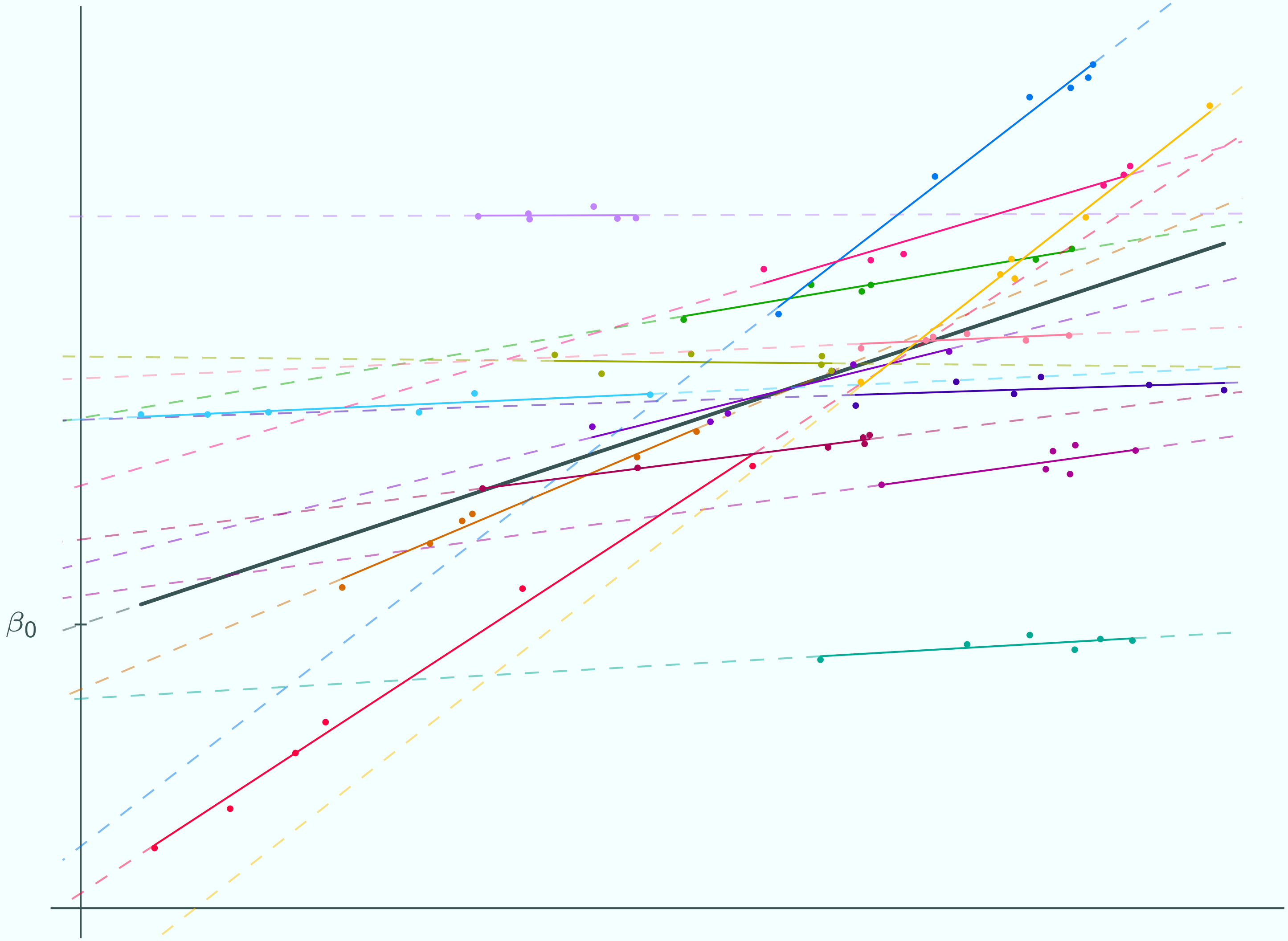
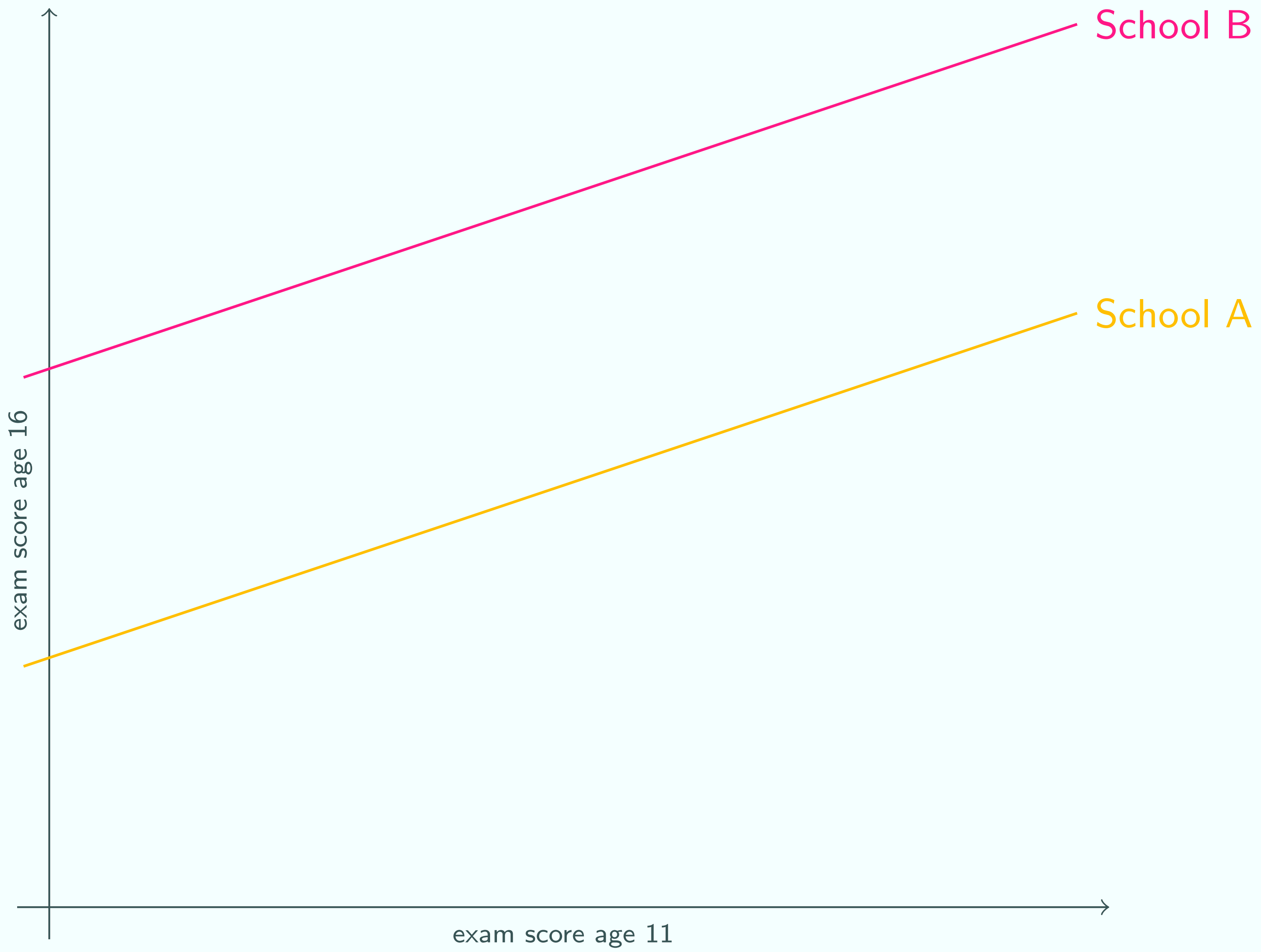


Using dummy variables



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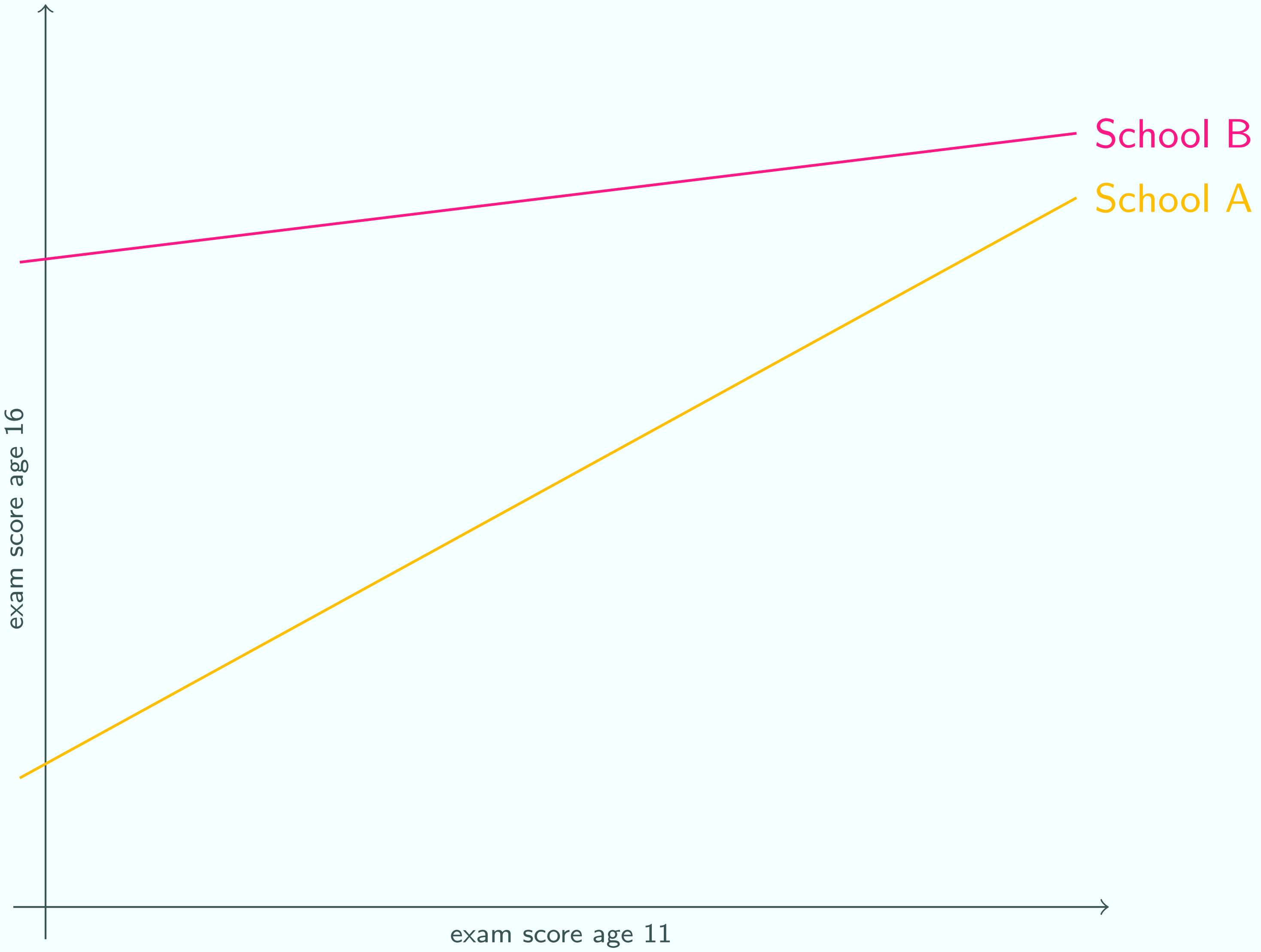

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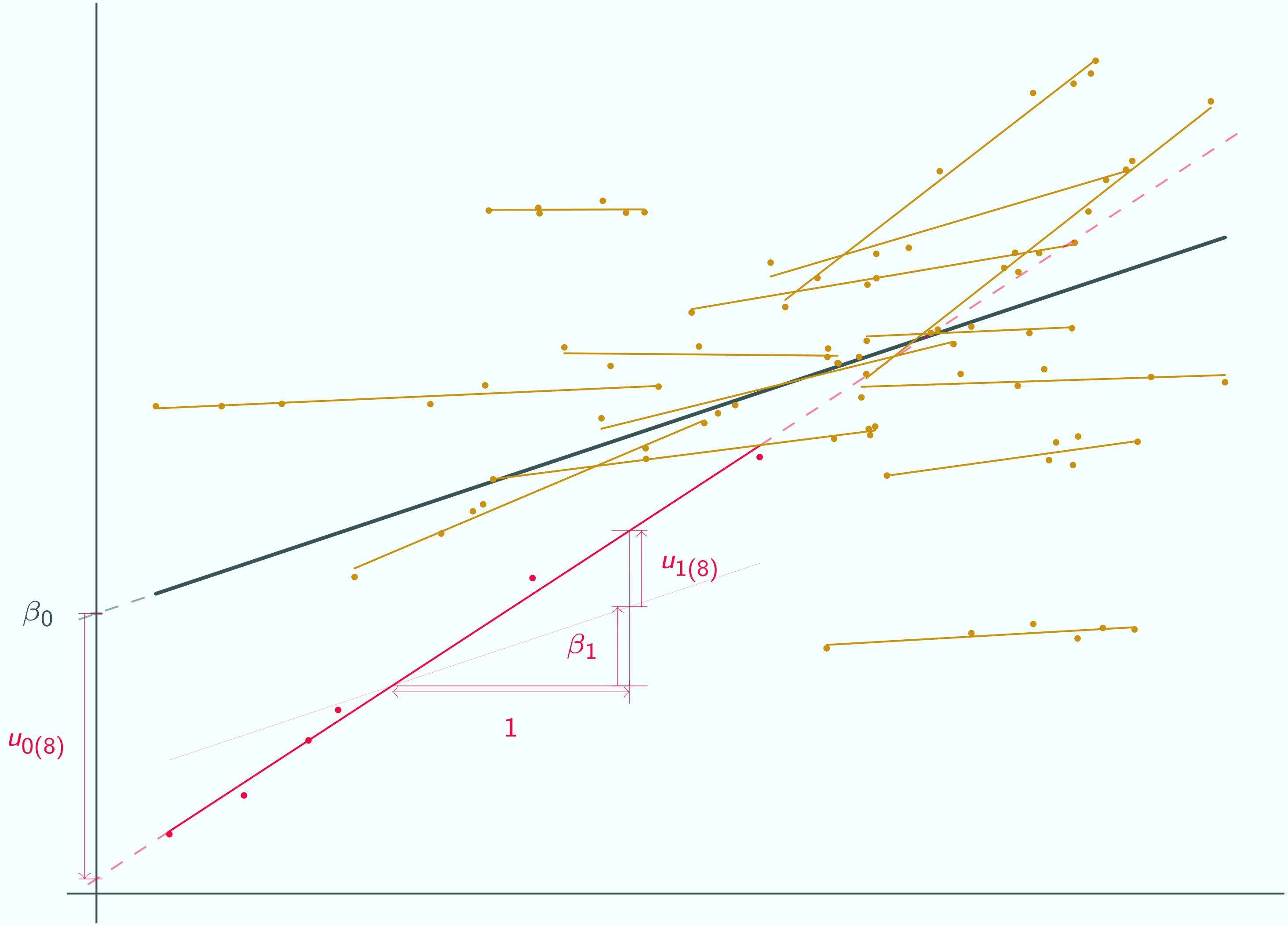

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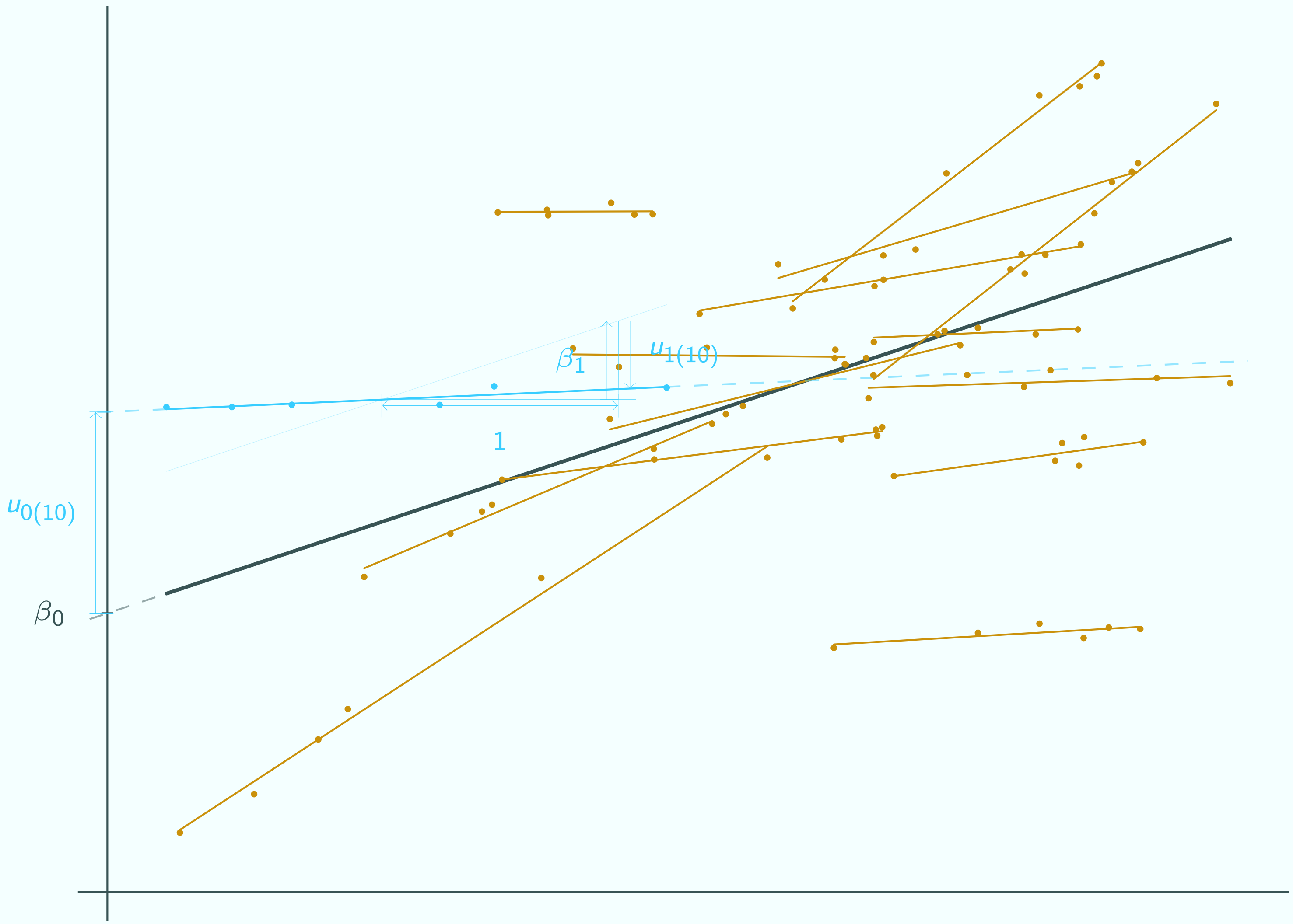

Interpreting the parameters

$\beta_0, \beta_1, \sigma_e^2$

- β_0 and σ_e^2 can be interpreted as for the random intercepts model

$\sigma_{u0}^2, \sigma_{u1}^2, \sigma_{u01}$

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- β_0 and σ_e^2 can be interpreted as for the random intercepts model
- β_1 is the slope of the average line: the average increase (across all groups) in y for a 1 unit change in x_1

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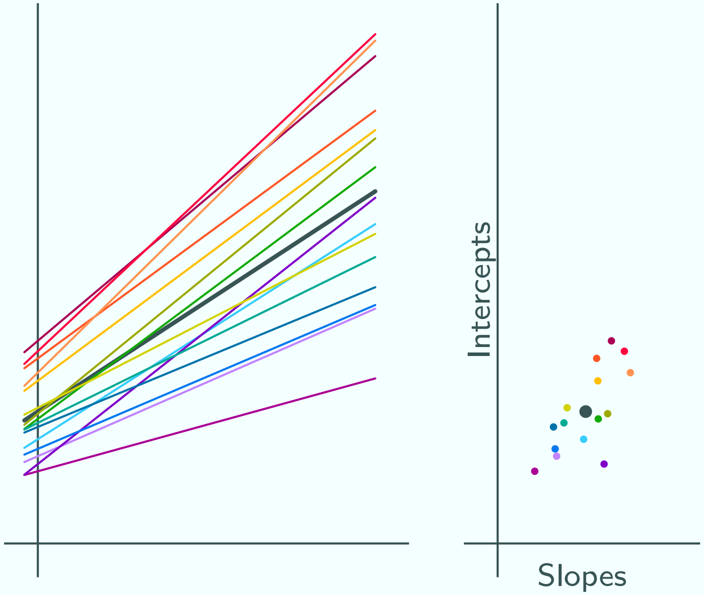
  - σ_{u01} is the covariance between intercepts and slopes
- BUT the estimates of σ_{u1}^2 and σ_{u0}^2 are not very meaningful in themselves.
- We will explain why that is after having a look at what the 'covariance between intercepts and slopes' means

Covariance between intercepts and slopes

Single level model

Random int. model



- The lines fan out
- Lines with larger slopes u_{1j} have larger intercepts u_{0j}
- σ_{u01} is positive

Random slopes model

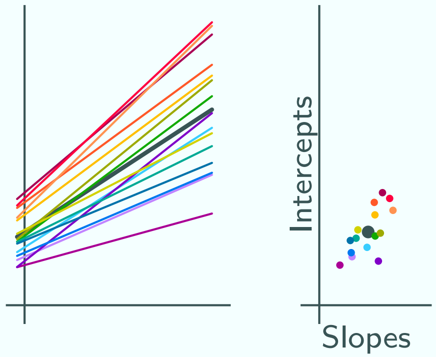
Covariance between intercepts and slopes

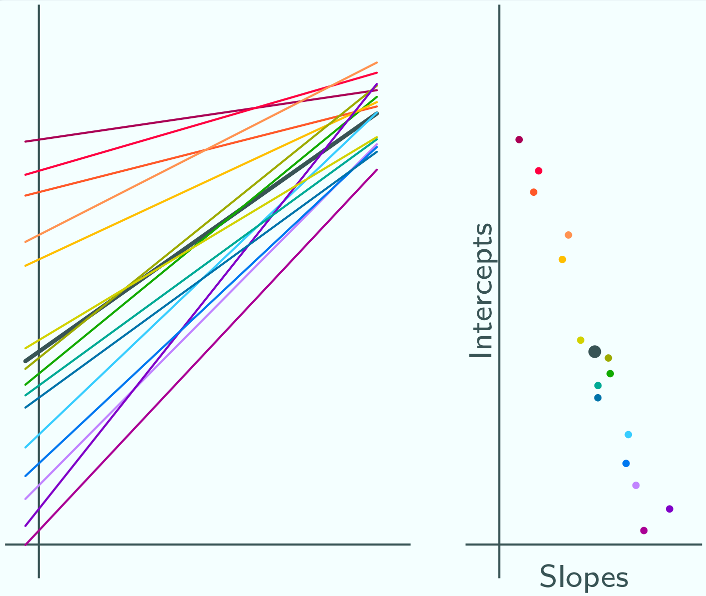
Single level model

Random int. model

Random slopes model

(a)
σ_{u01}
positive





- The lines fan in
- Lines with larger slopes u_{1j} have smaller intercepts u_{0j}
- σ_{u01} is negative

Covariance between intercepts and slopes

Single level model

Random int. model

Random slopes model

(a)
σ_{u01}
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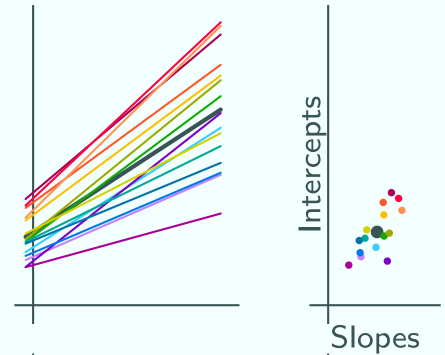
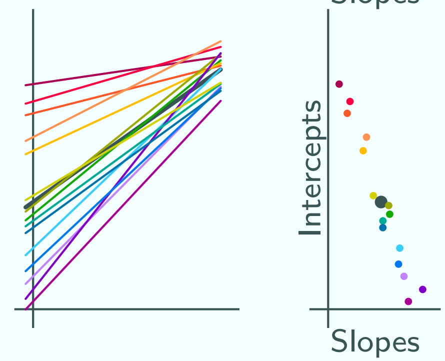
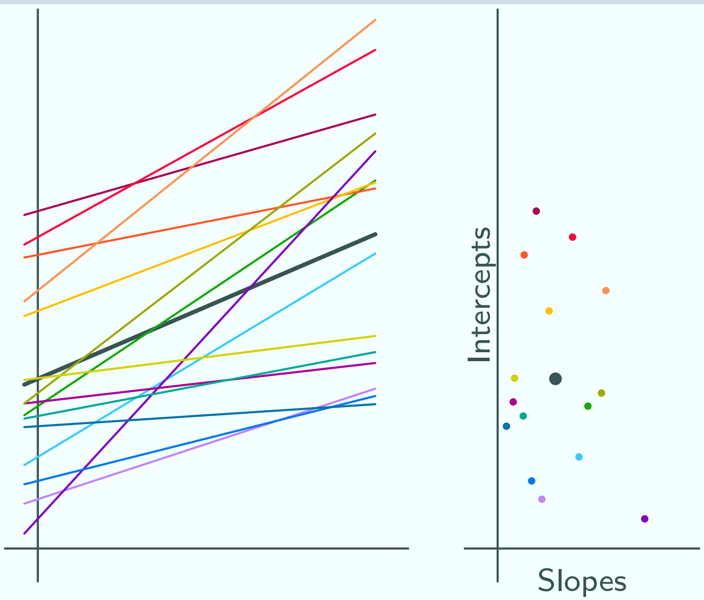


(b)
σ_{u01}
negative



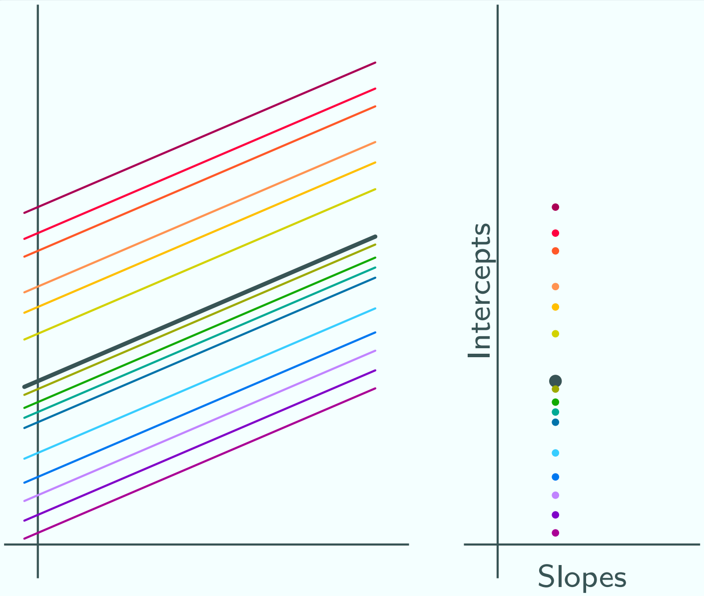


- The lines show no pattern
- There is no relation between slopes u_{1j} and intercepts u_{0j}
- $\sigma_{u01} = 0$

Covariance between intercepts and slopes

Single level model

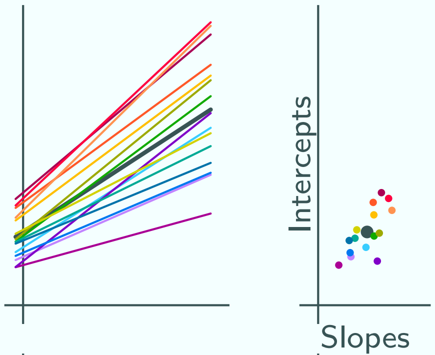
Random int. model



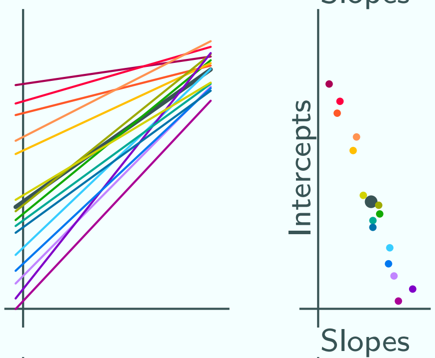
- All lines have the same slope, regardless of their intercept
- There is no variation in slope
- It does not make sense to estimate σ_{u01}

Random slopes model

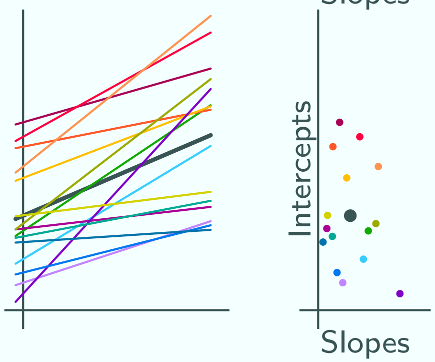
(a) σ_{u01} positive



(b) σ_{u01} negative



(c) $\sigma_{u01} = 0$



Covariance between intercepts and slopes

Single level model

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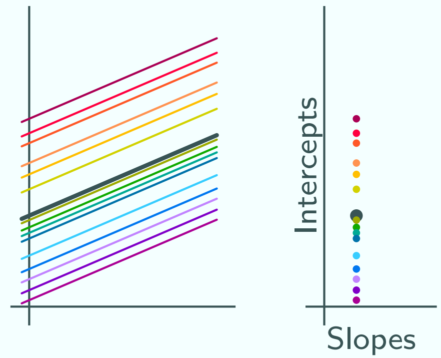
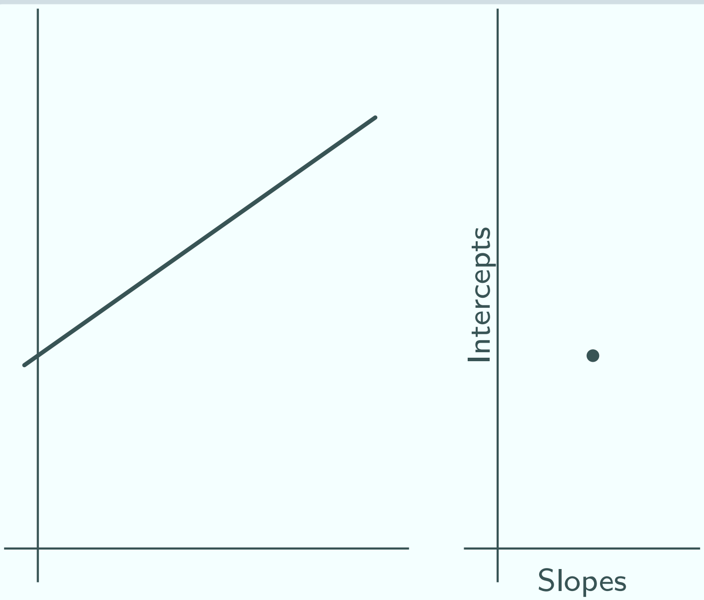




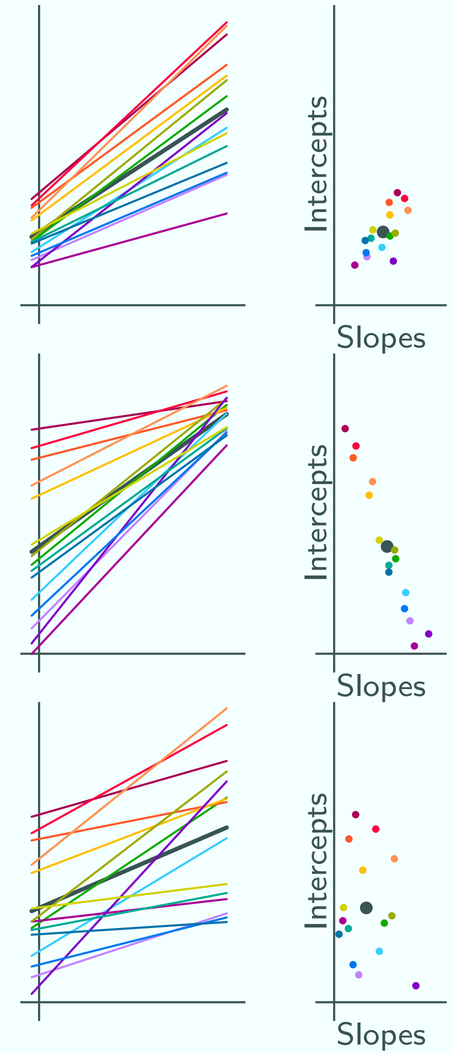
- There is only one line
- It does not make sense to estimate σ_{u01}

Random slopes model

(a) σ_{u01} positive

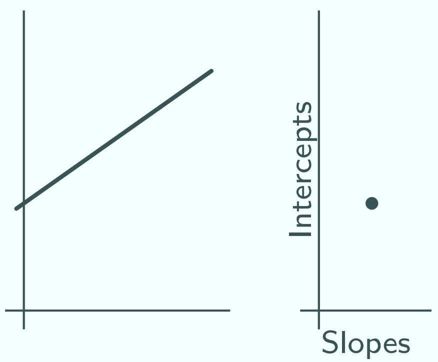
(b) σ_{u01} negative

(c) $\sigma_{u01} = 0$

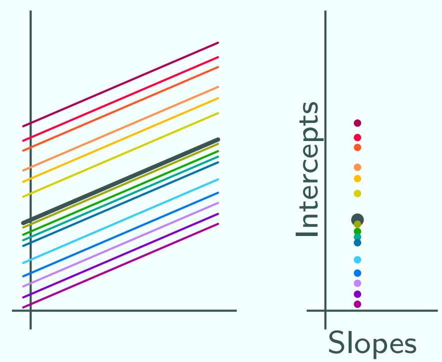


Covariance between intercepts and slopes

Single level model



Random int. model

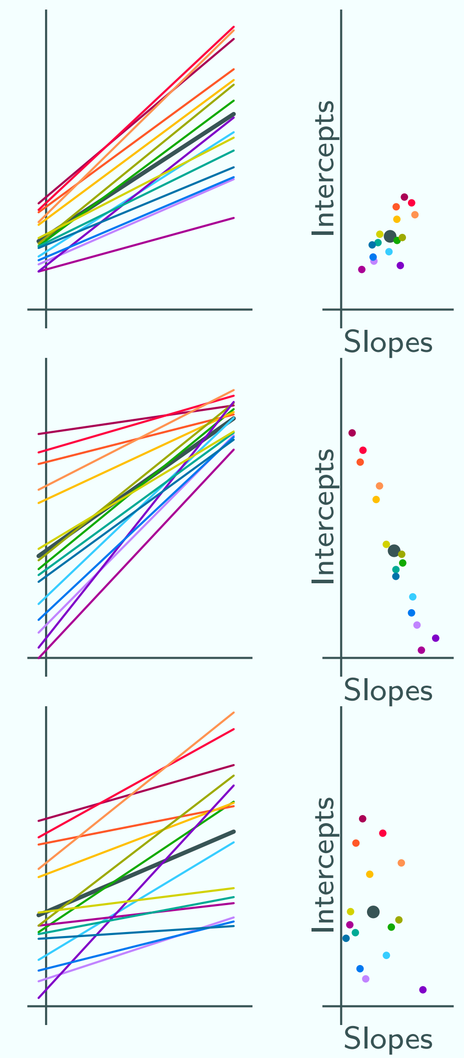


Random slopes model

(a) σ_{u01} positive

(b) σ_{u01} nega-tive

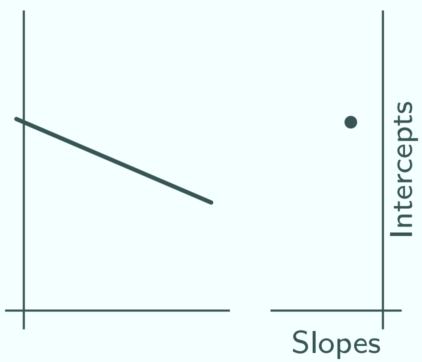
(c) $\sigma_{u01} = 0$



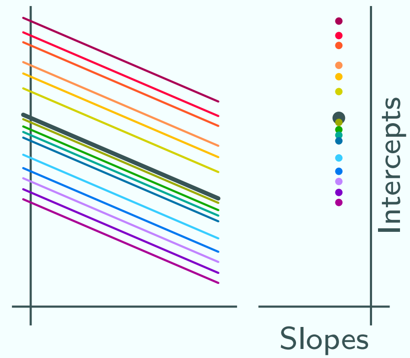
- For single level or random intercept models, σ_{u01} is not defined (there is no variation in slopes)
- For random slope models,
  - σ_{u01} positive means a pattern of fanning out
  - σ_{u01} negative means a pattern of fanning in
  - $\sigma_{u01} = 0$ means no pattern

Covariance between intercepts and slopes

Single level model



Random int. model



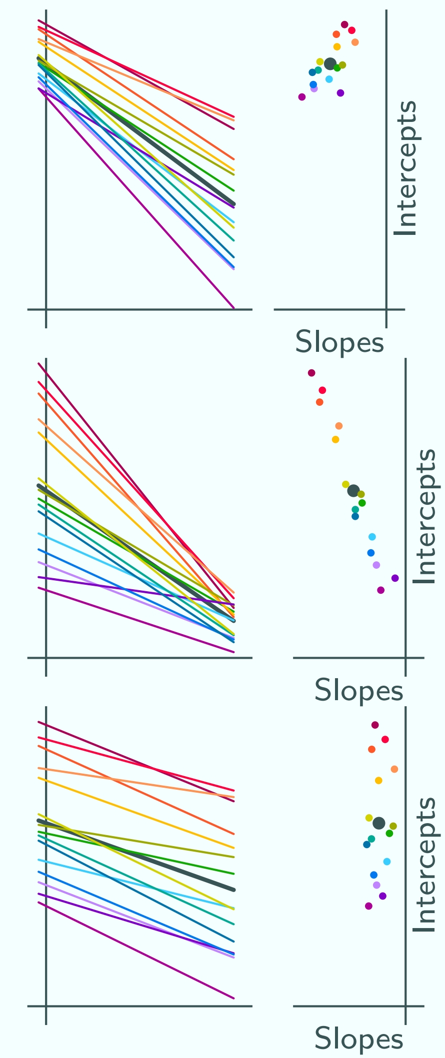
- If β_1 is negative we still have the same relation between the value of σ_{u01} and the pattern
- For random slope models,
  - σ_{u01} positive means a pattern of fanning out
  - σ_{u01} negative means a pattern of fanning in
  - $\sigma_{u01} = 0$ means no pattern

Random slopes model

(a) σ_{u01} positive

(b) σ_{u01} negative

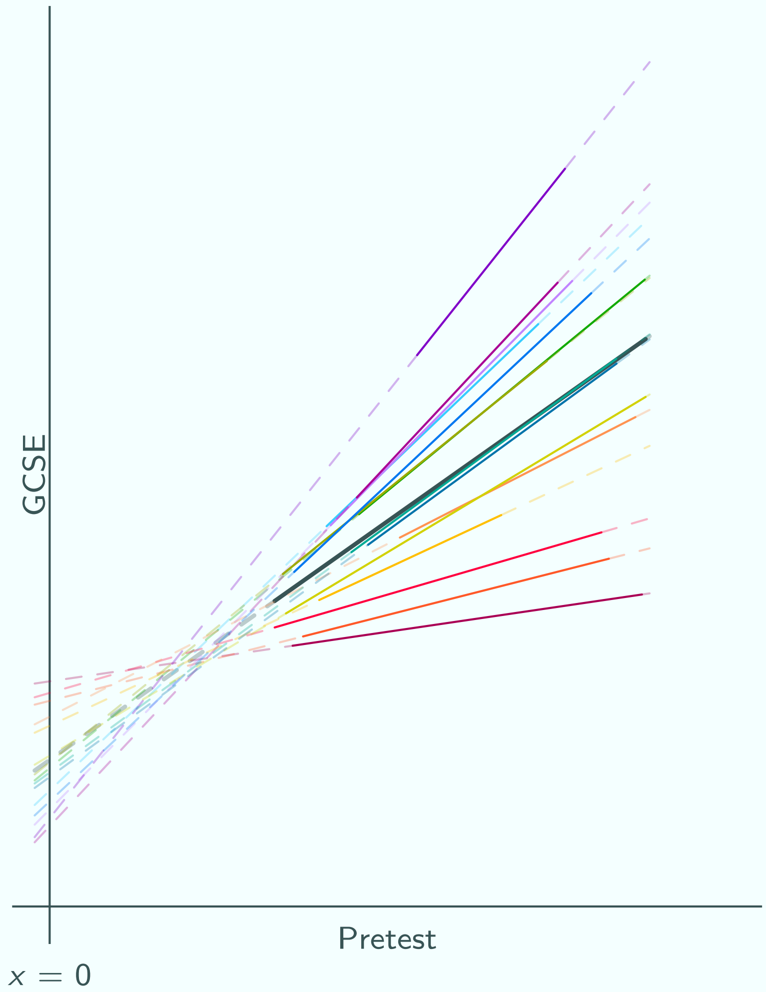
(c) $\sigma_{u01} = 0$



σ_{u01} and the scale of x

Example

- We fit a random slopes model:

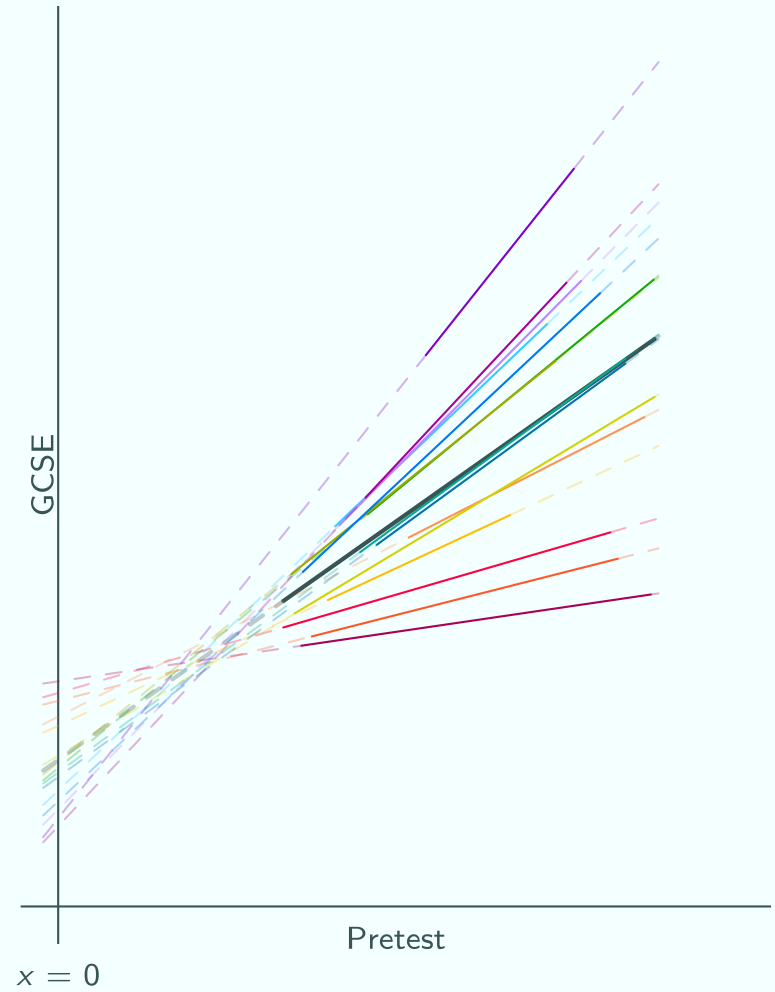


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

Example

- We fit a random slopes model:
  - response: **GCSE**

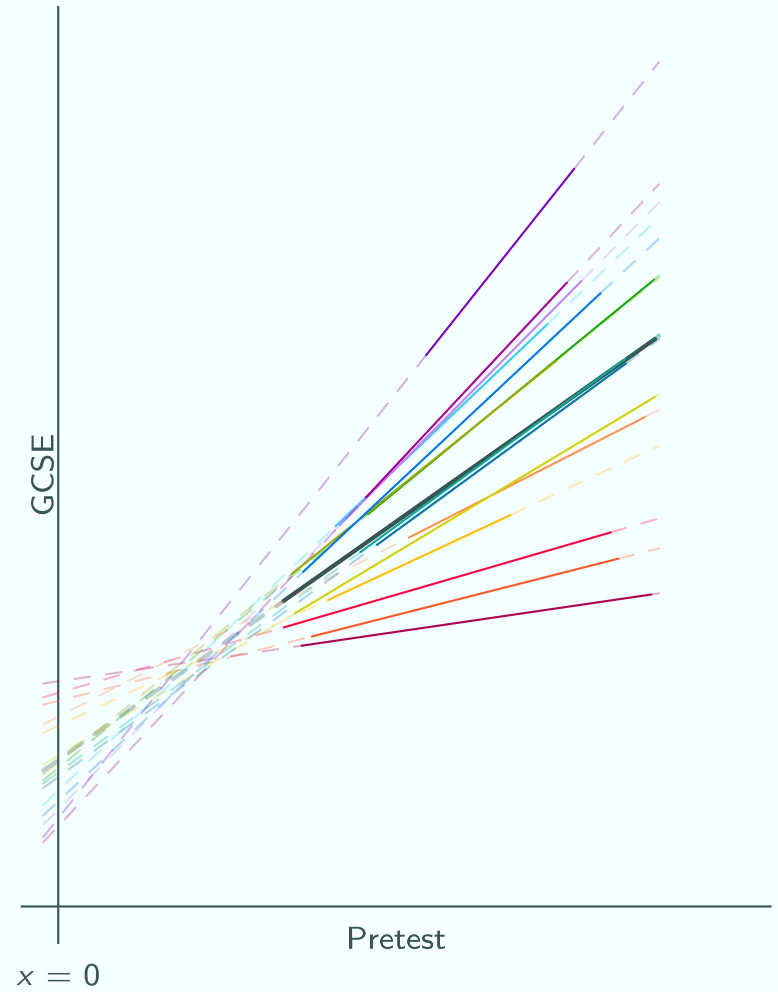


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

Example

- We fit a random slopes model:
  - response: **GCSE**
  - explanatory variable: previous exam score (%)

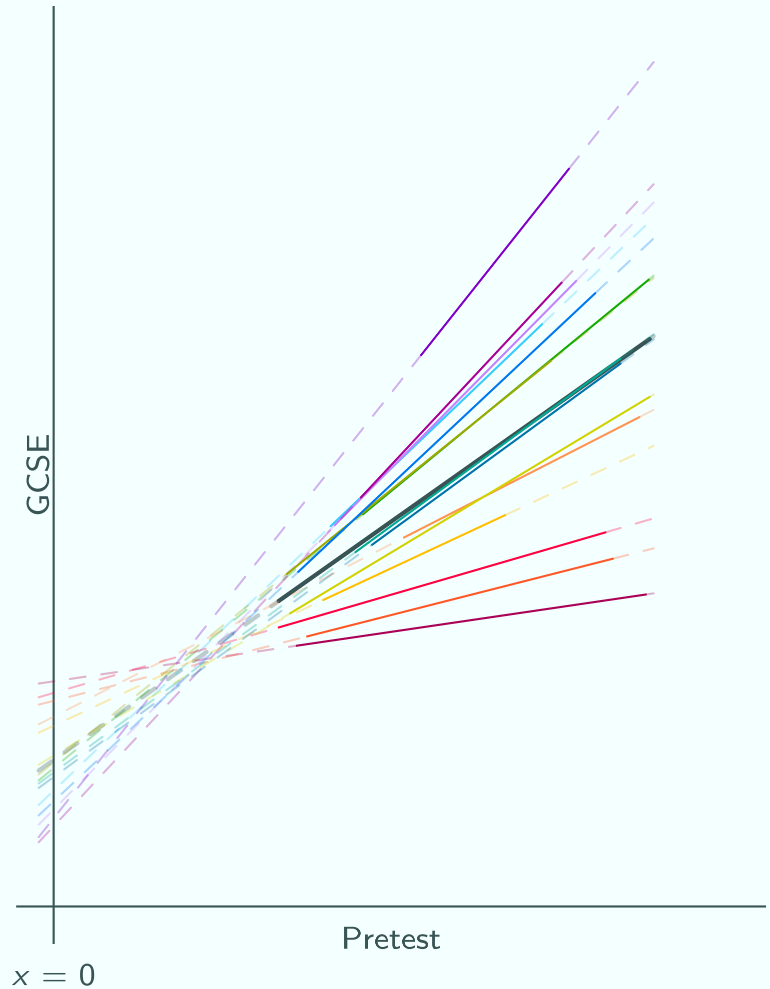


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

Example

- We fit a random slopes model:
  - response: **GCSE**
  - explanatory variable: previous exam score (%)
  - units: students within schools

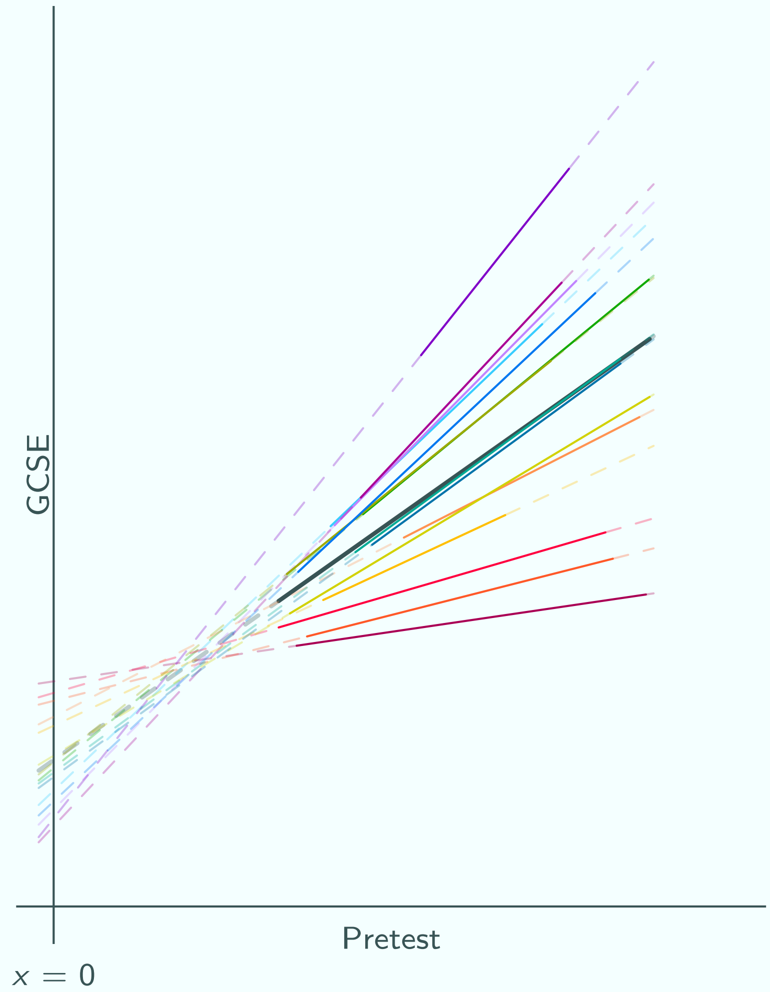


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

Example

- We fit a random slopes model:
  - response: **GCSE**
  - explanatory variable: previous exam score (%)
  - units: students within schools
- If we only look at the value of σ_{u01} we will think the pattern is of fanning in

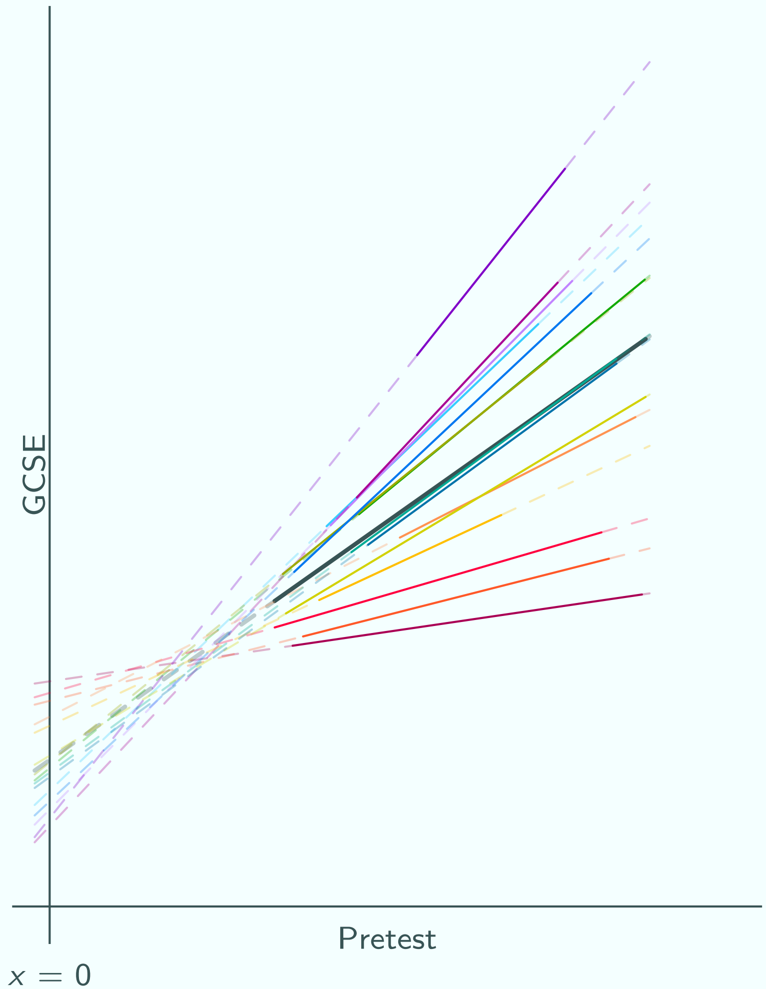


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

Example

- We fit a random slopes model:
  - response: **GCSE**
  - explanatory variable: previous exam score (%)
  - units: students within schools
- If we only look at the value of σ_{u01} we will think the pattern is of fanning in
- Actually over the range of our data the pattern is of fanning out

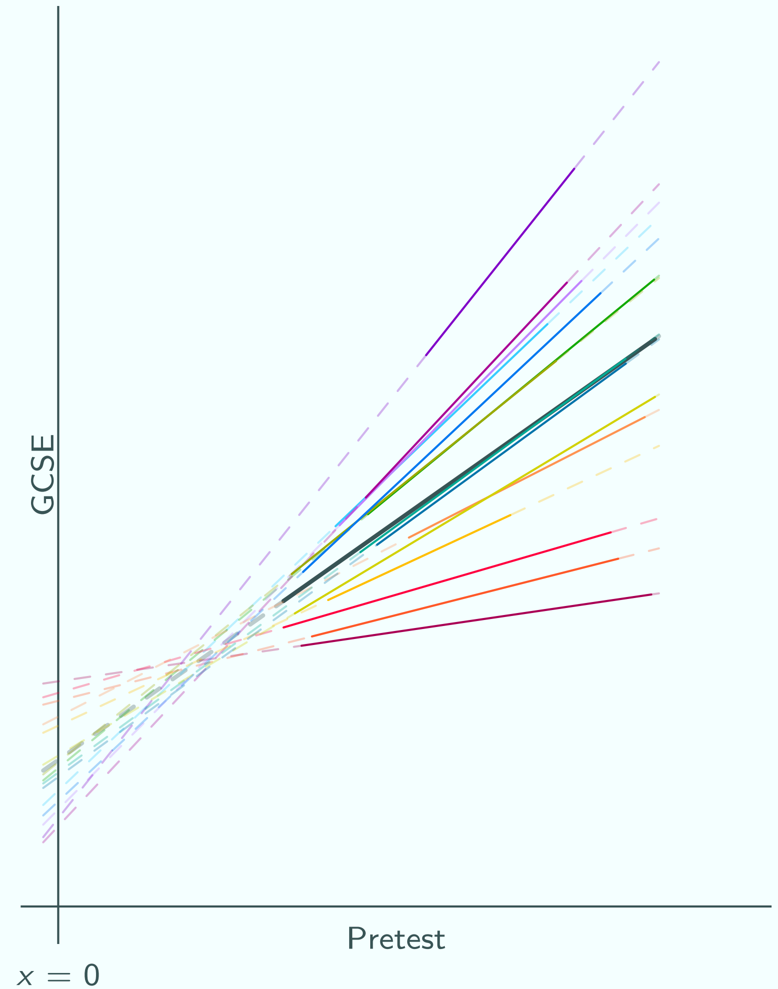


$\sigma_{u01} < 0$

σ_{u01} and the scale of x

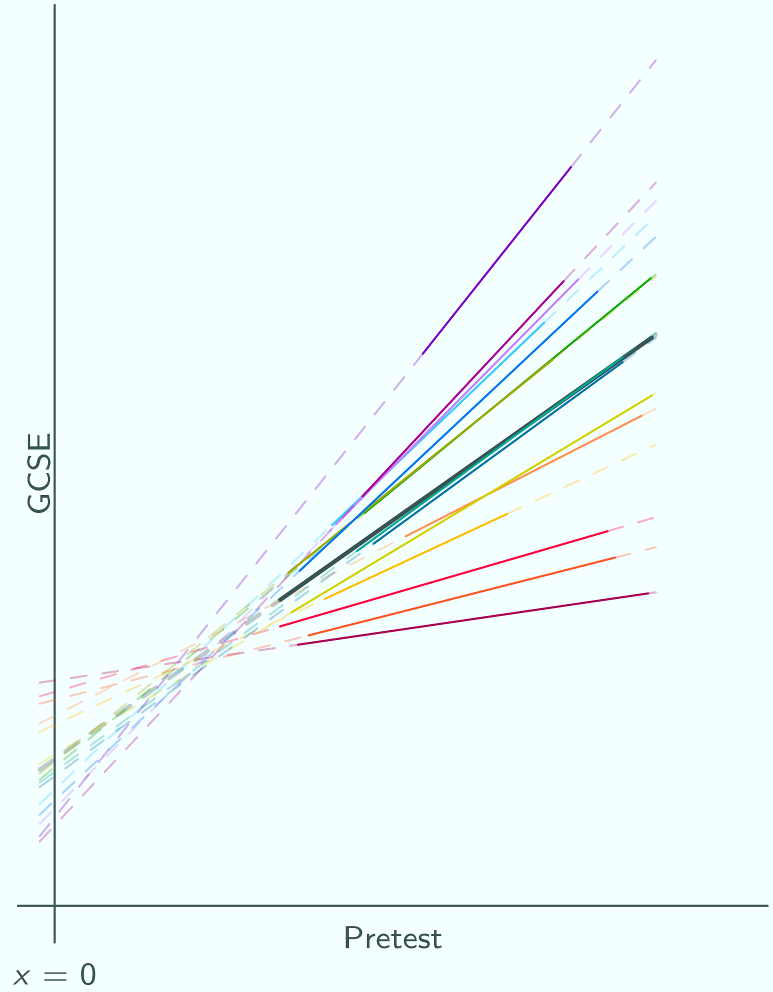
Example

- We fit a random slopes model:
  - response: **GCSE**
  - explanatory variable: previous exam score (%)
  - units: students within schools
- If we only look at the value of σ_{u01} we will think the pattern is of fanning in
- Actually over the range of our data the pattern is of fanning out
- We can see this if we look at the graph



$\sigma_{u01} < 0$

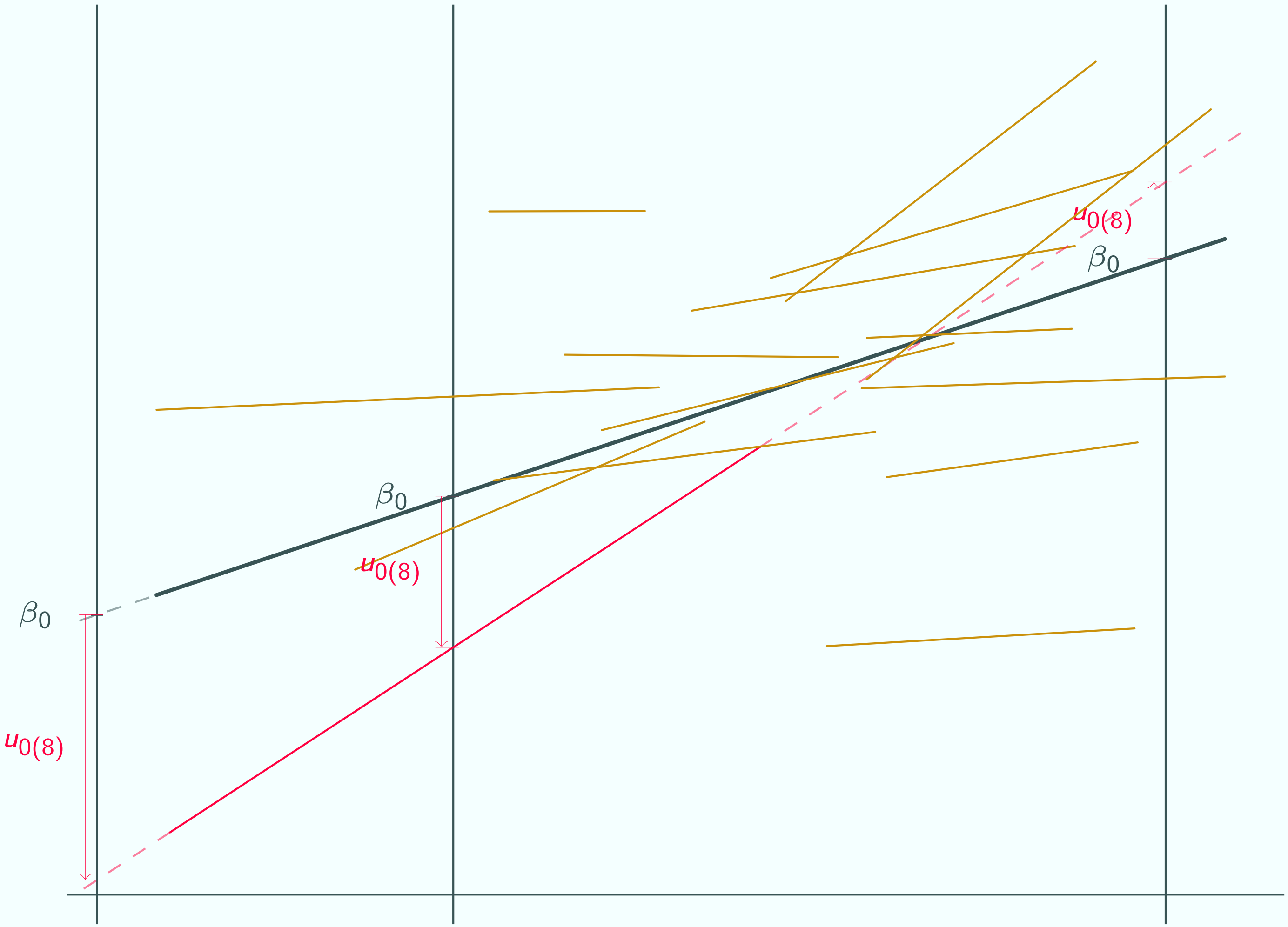
σ_{u01} and the scale of x



$\sigma_{u01} < 0$

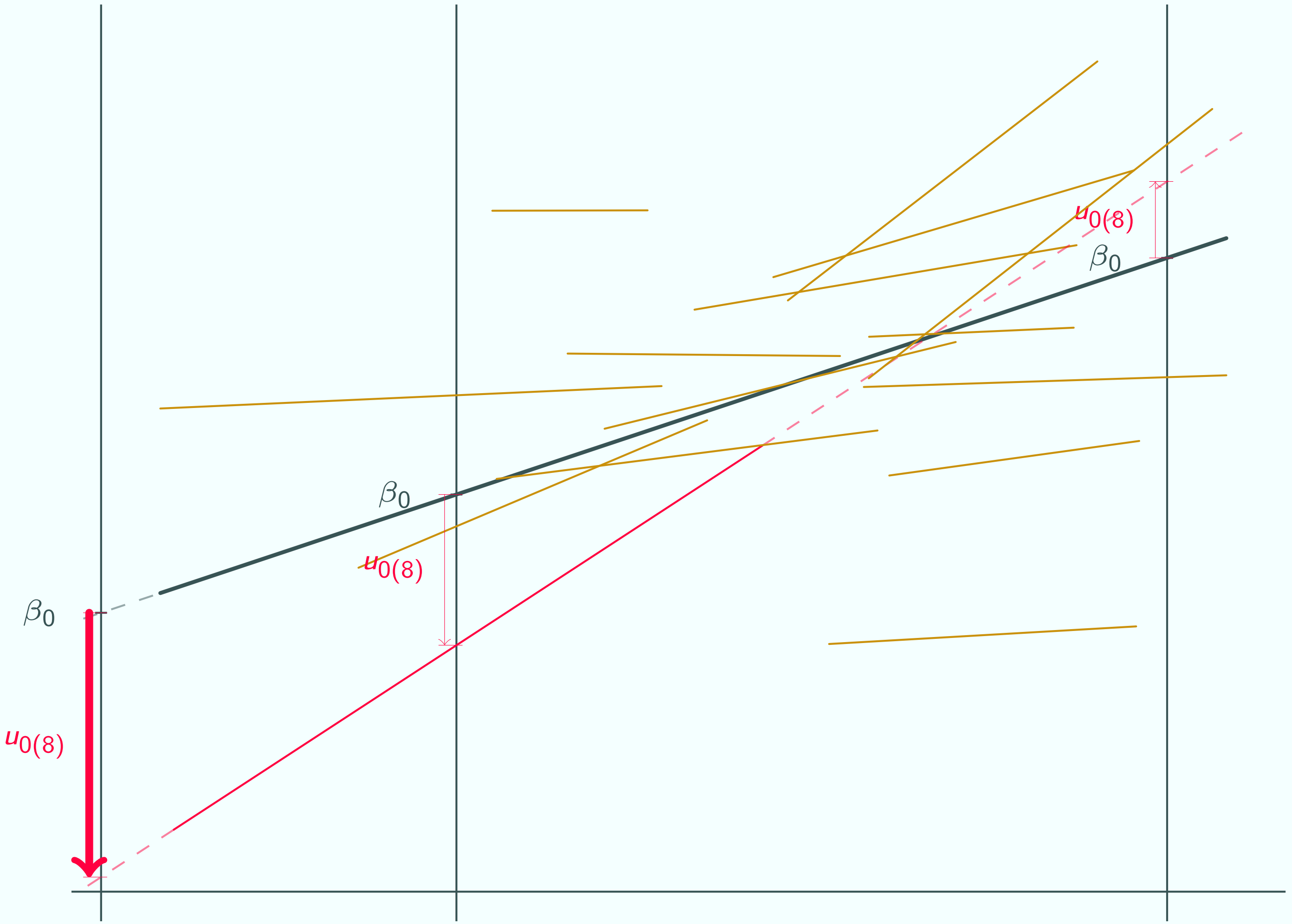
u_{0j} and the scale of x

Random slope model



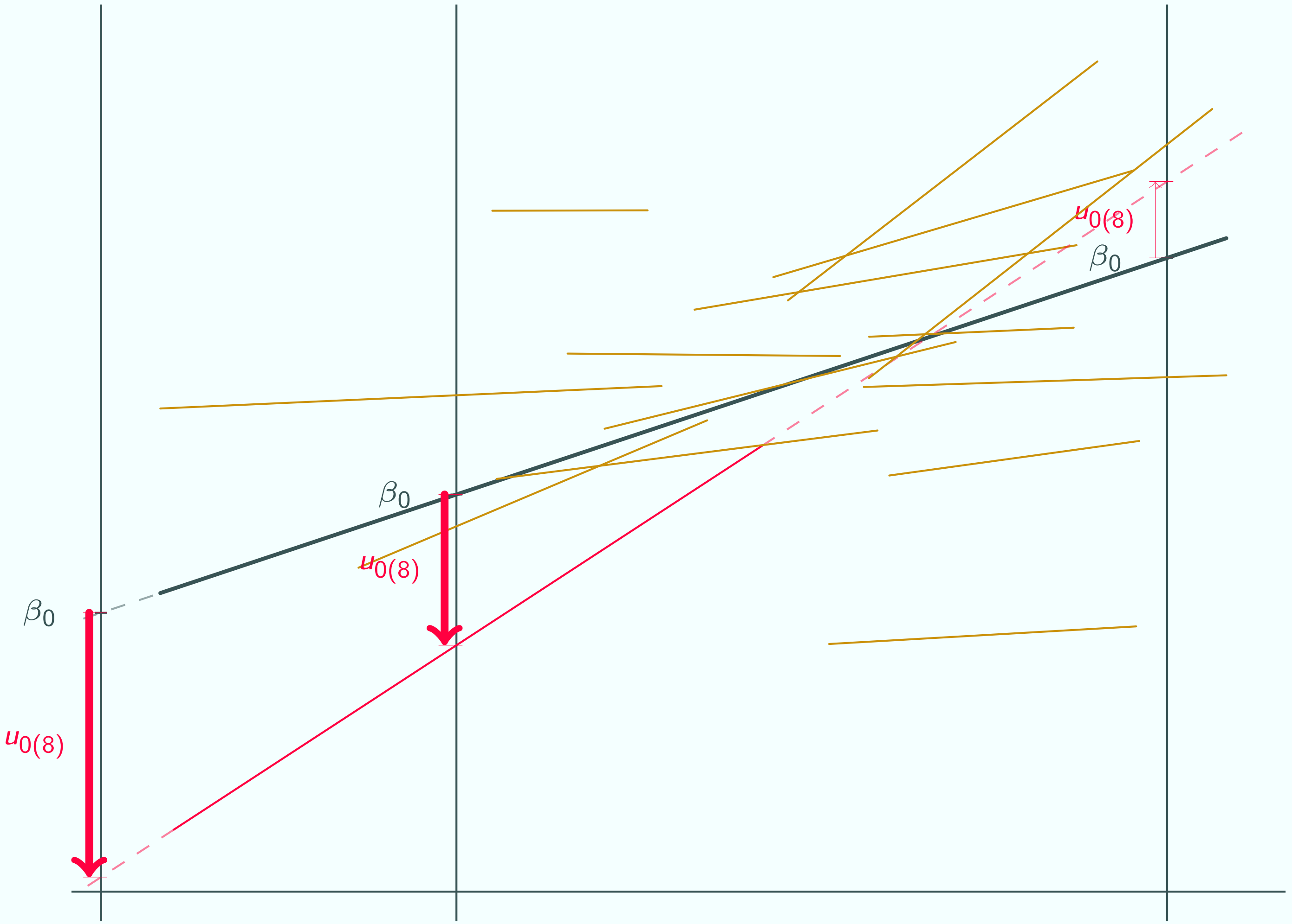
u_{0j} and the scale of x

Random slope model



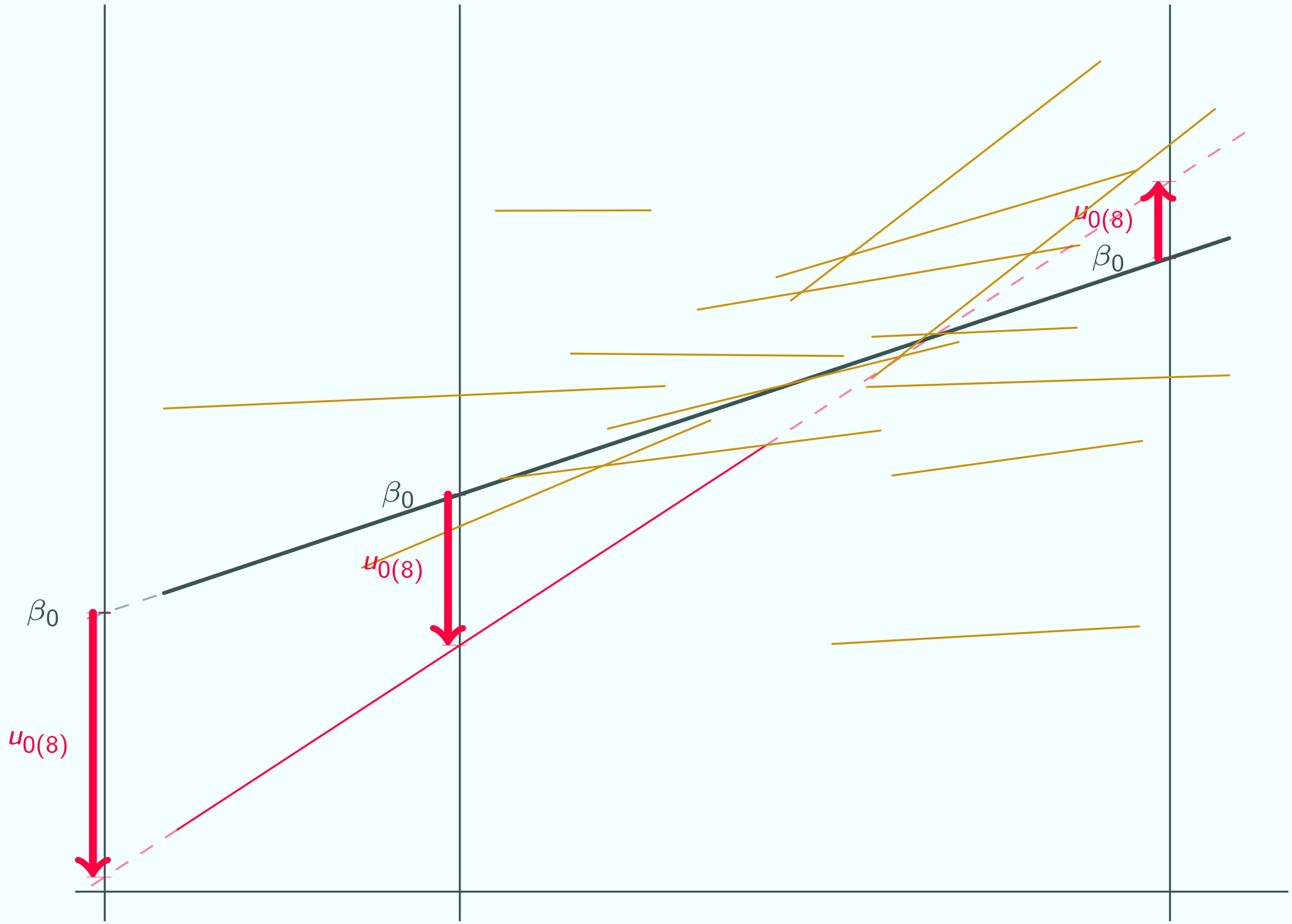
u_{0j} and the scale of x

Random slope model



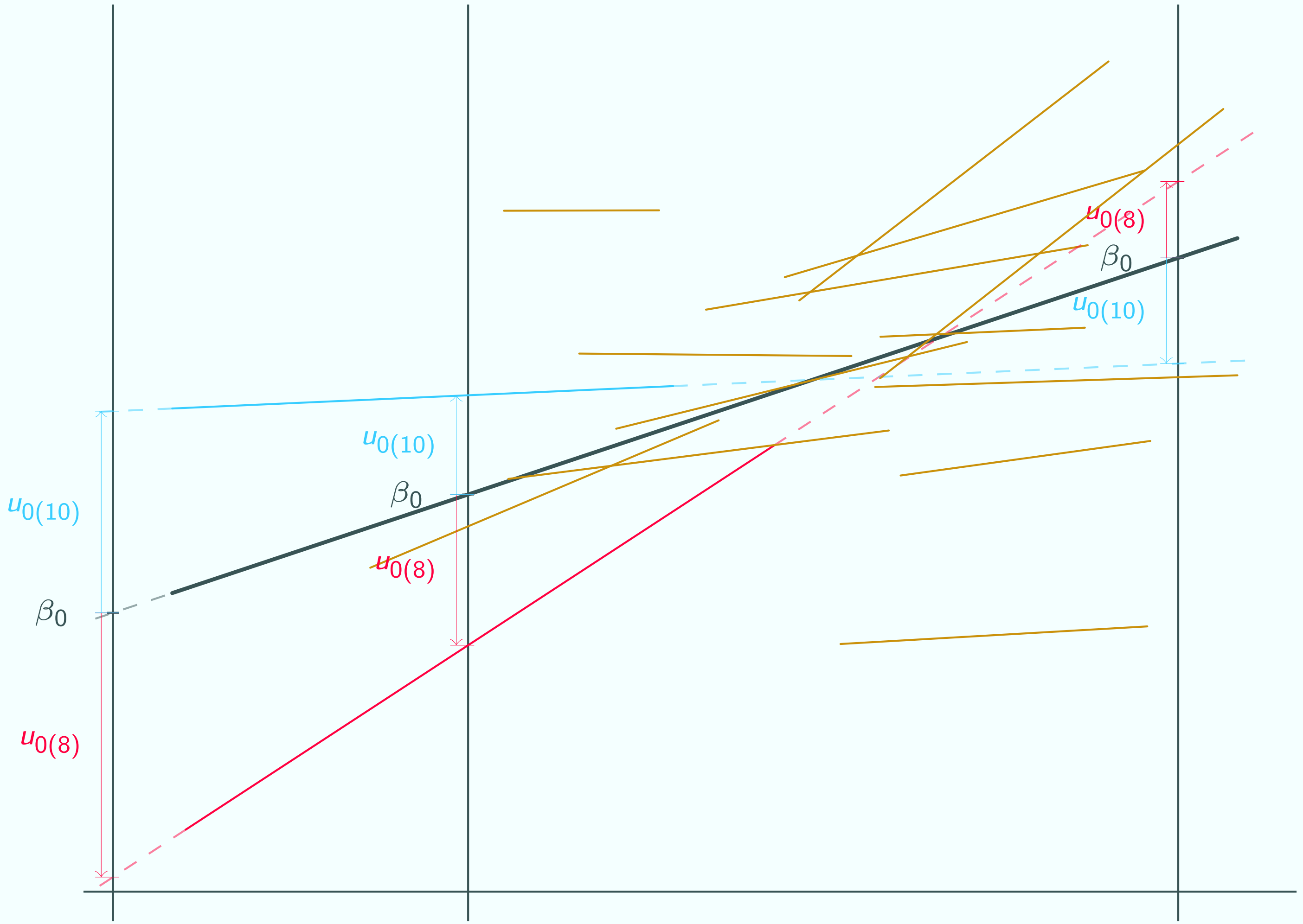
u_{0j} and the scale of x

Random slope model



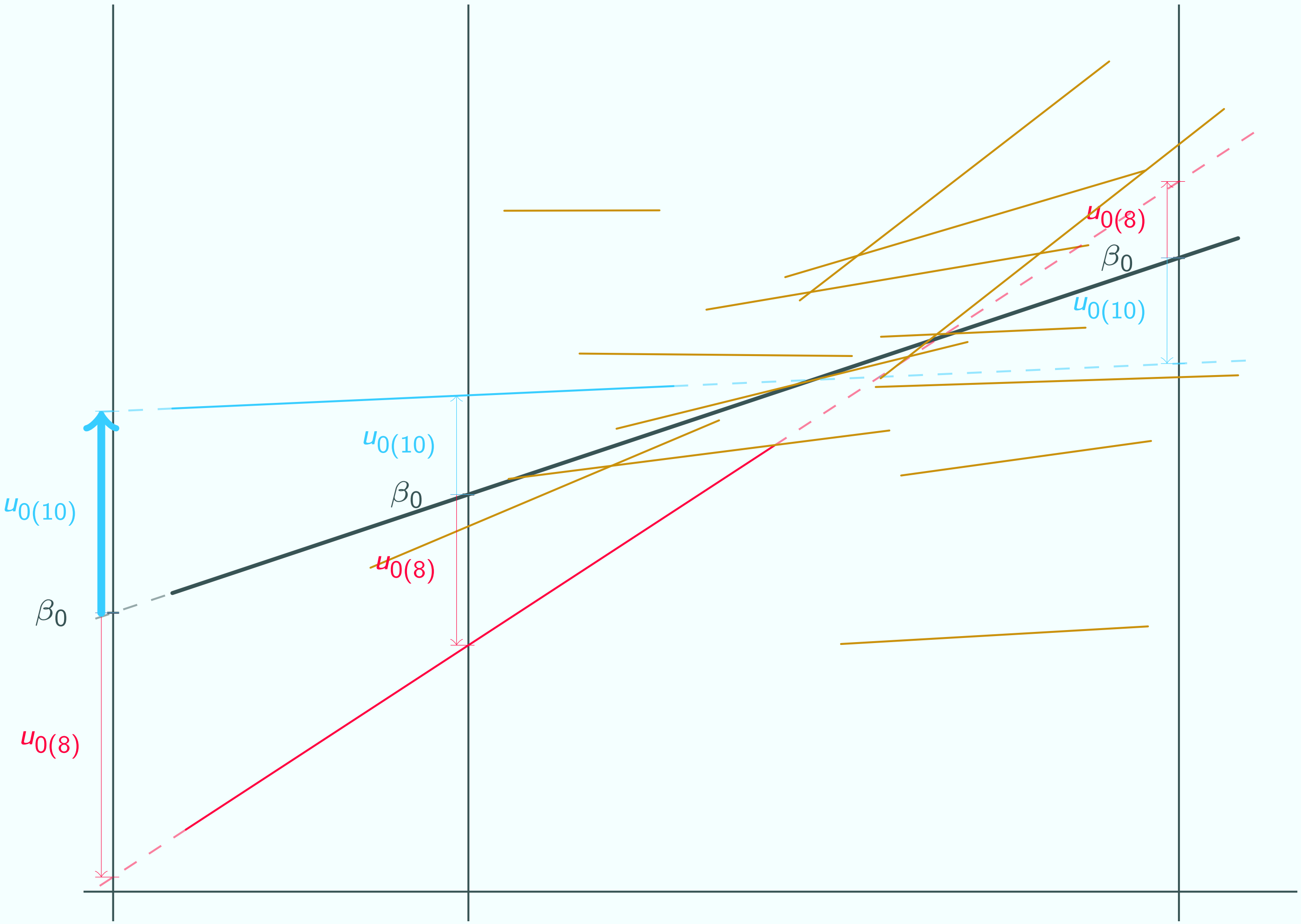
u_{0j} and the scale of x

Random slope model



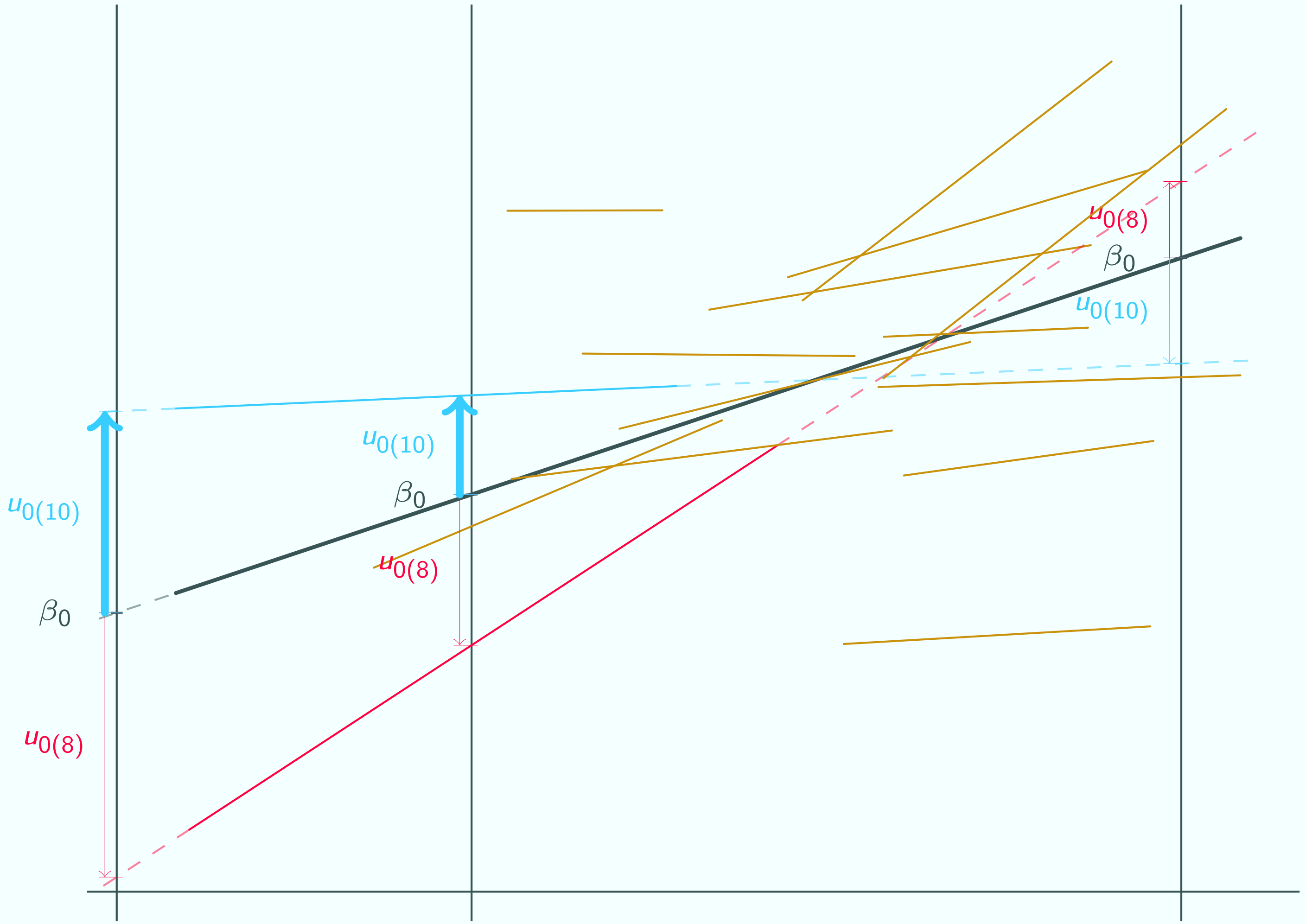
u_{0j} and the scale of x

Random slope model



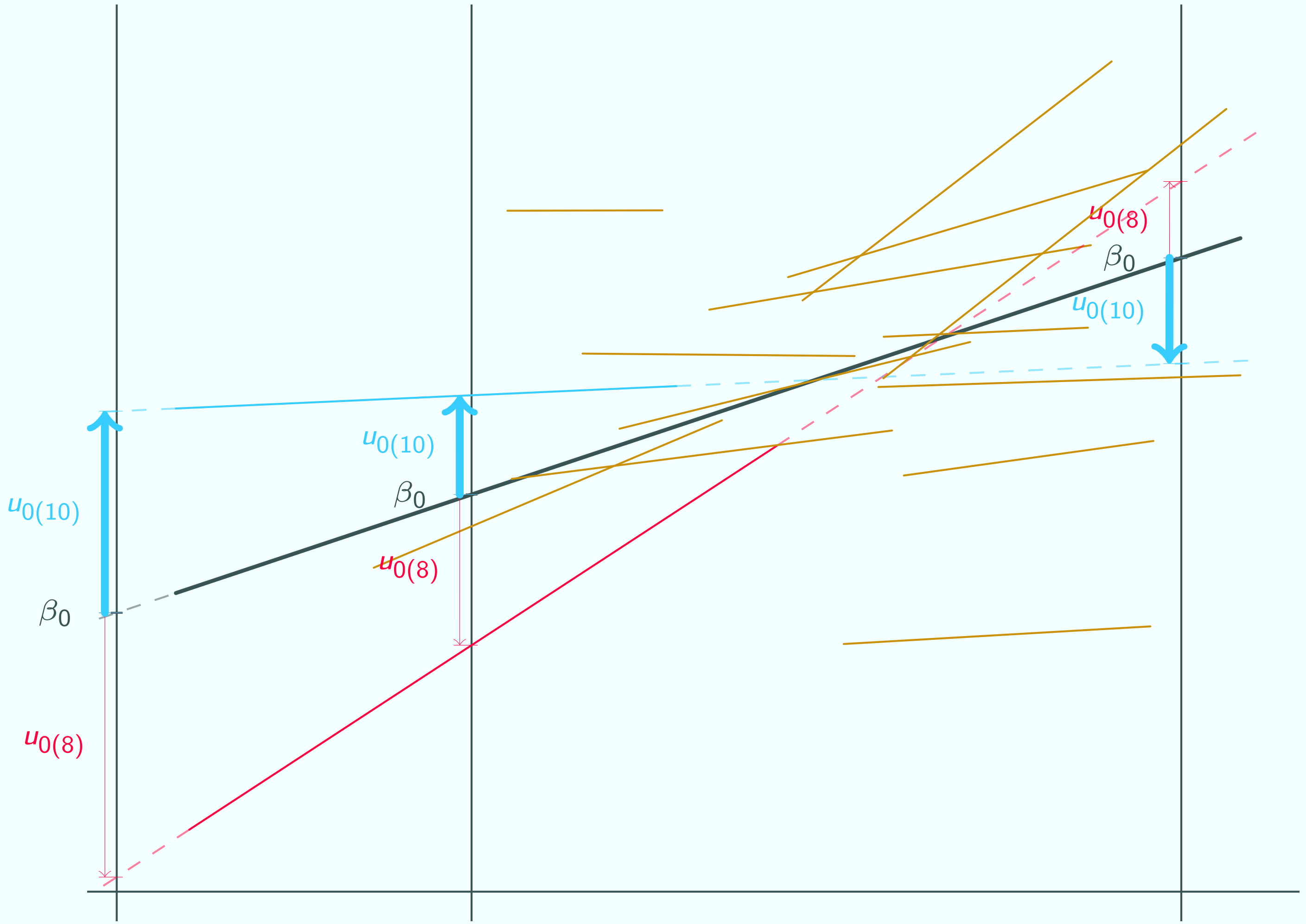
u_{0j} and the scale of x

Random slope model



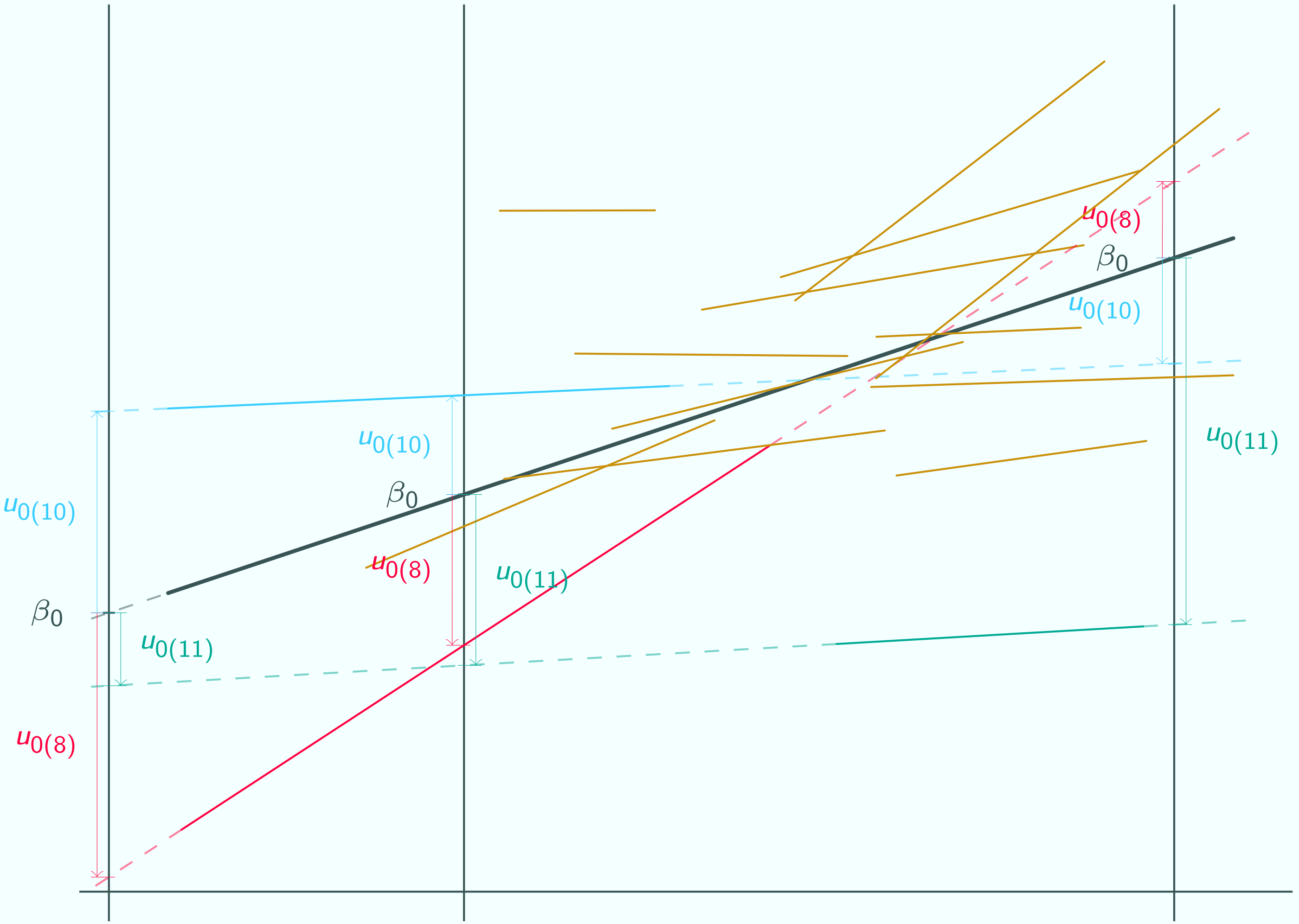
u_{0j} and the scale of x

Random slope model



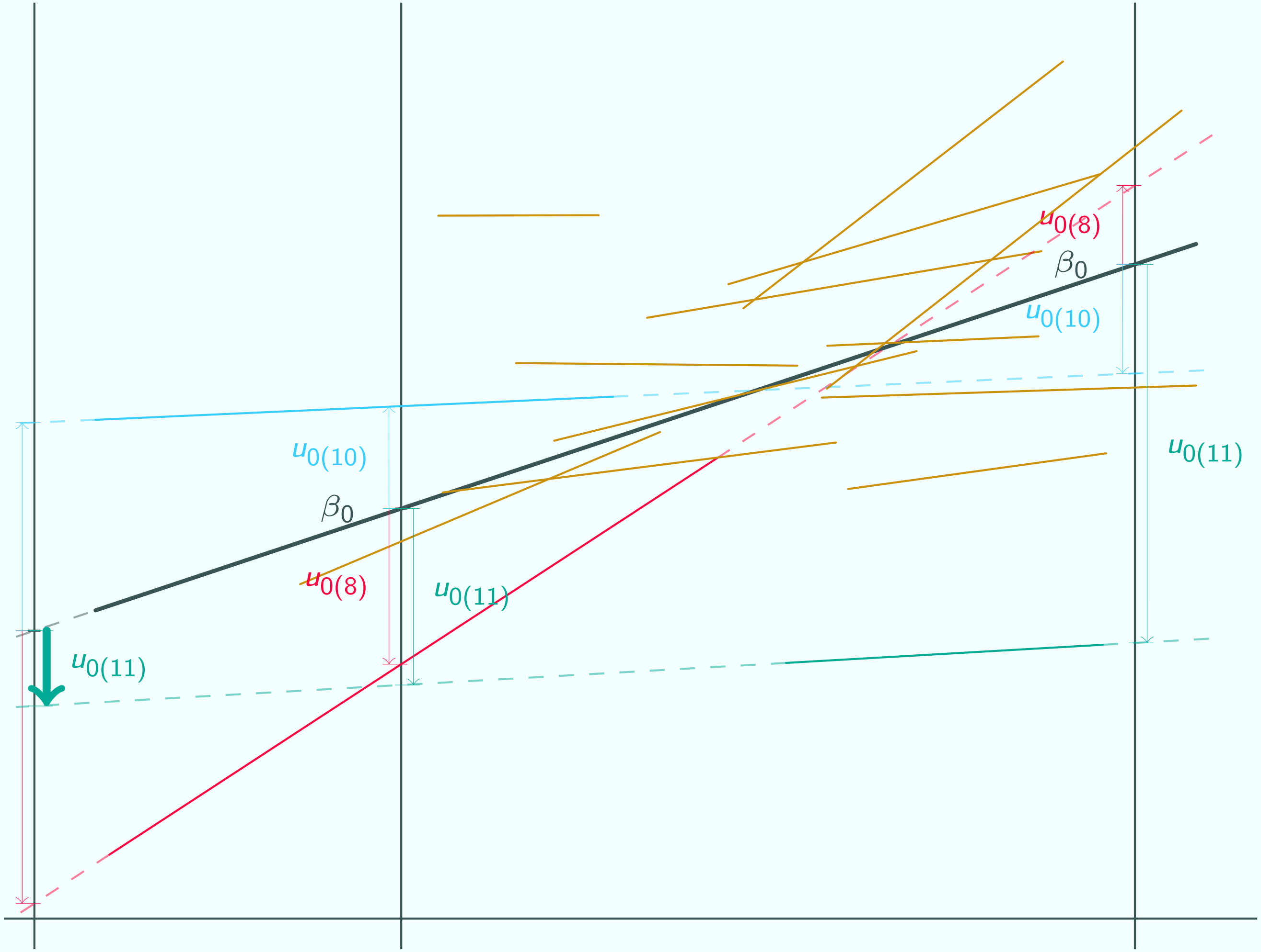
u_{0j} and the scale of x

Random slope model



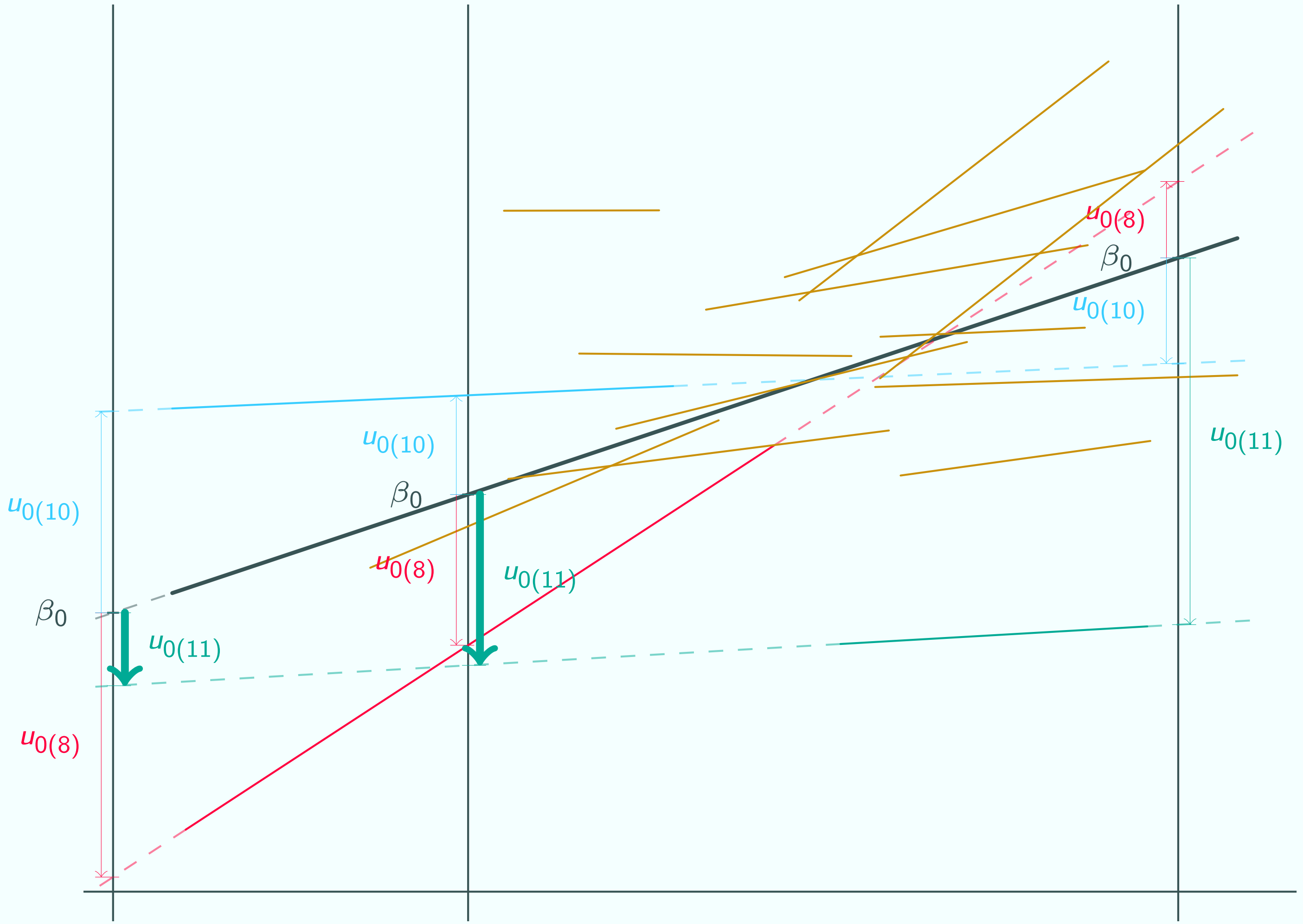
u_{0j} and the scale of x

Random slope model



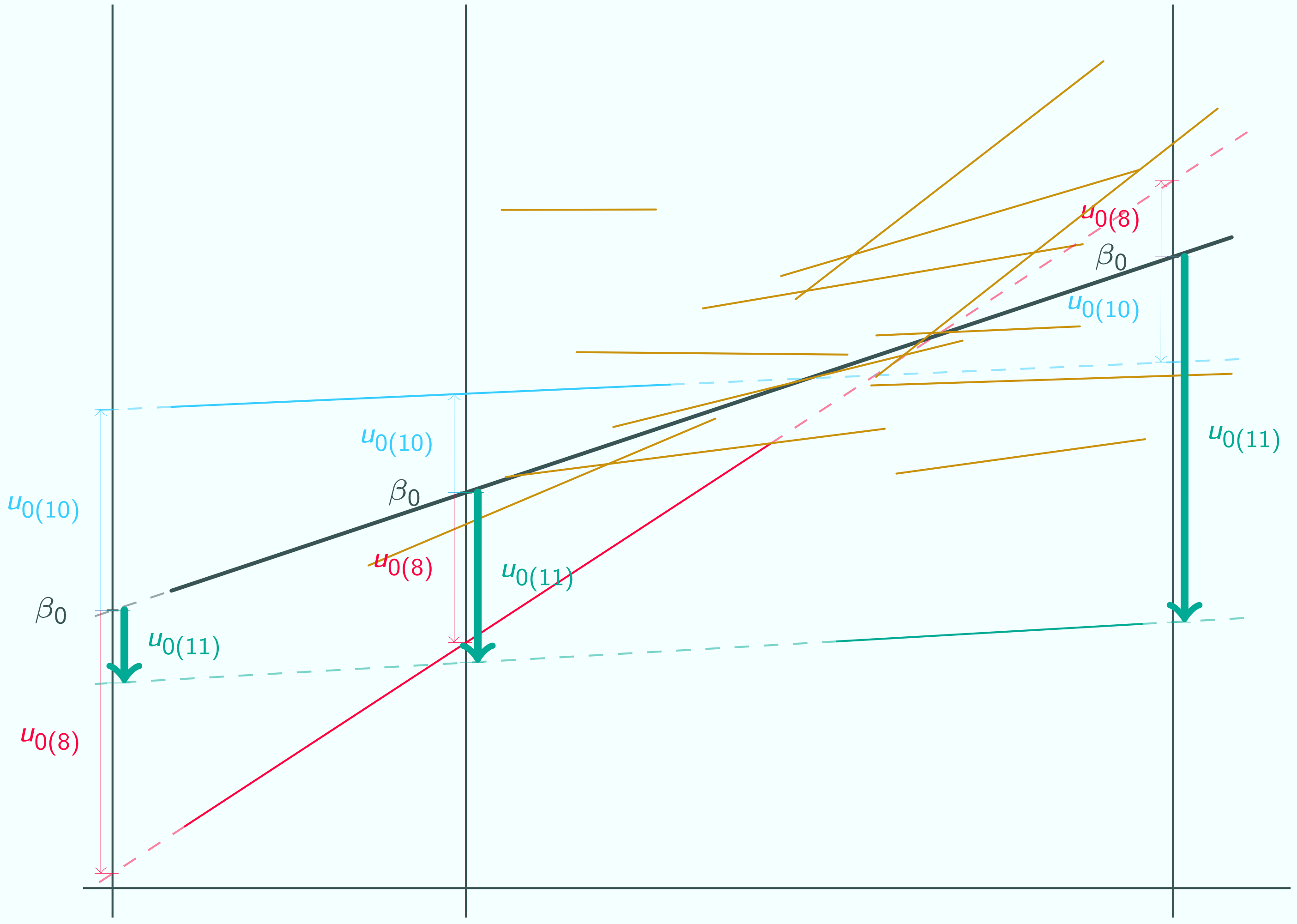
u_{0j} and the scale of x

Random slope model



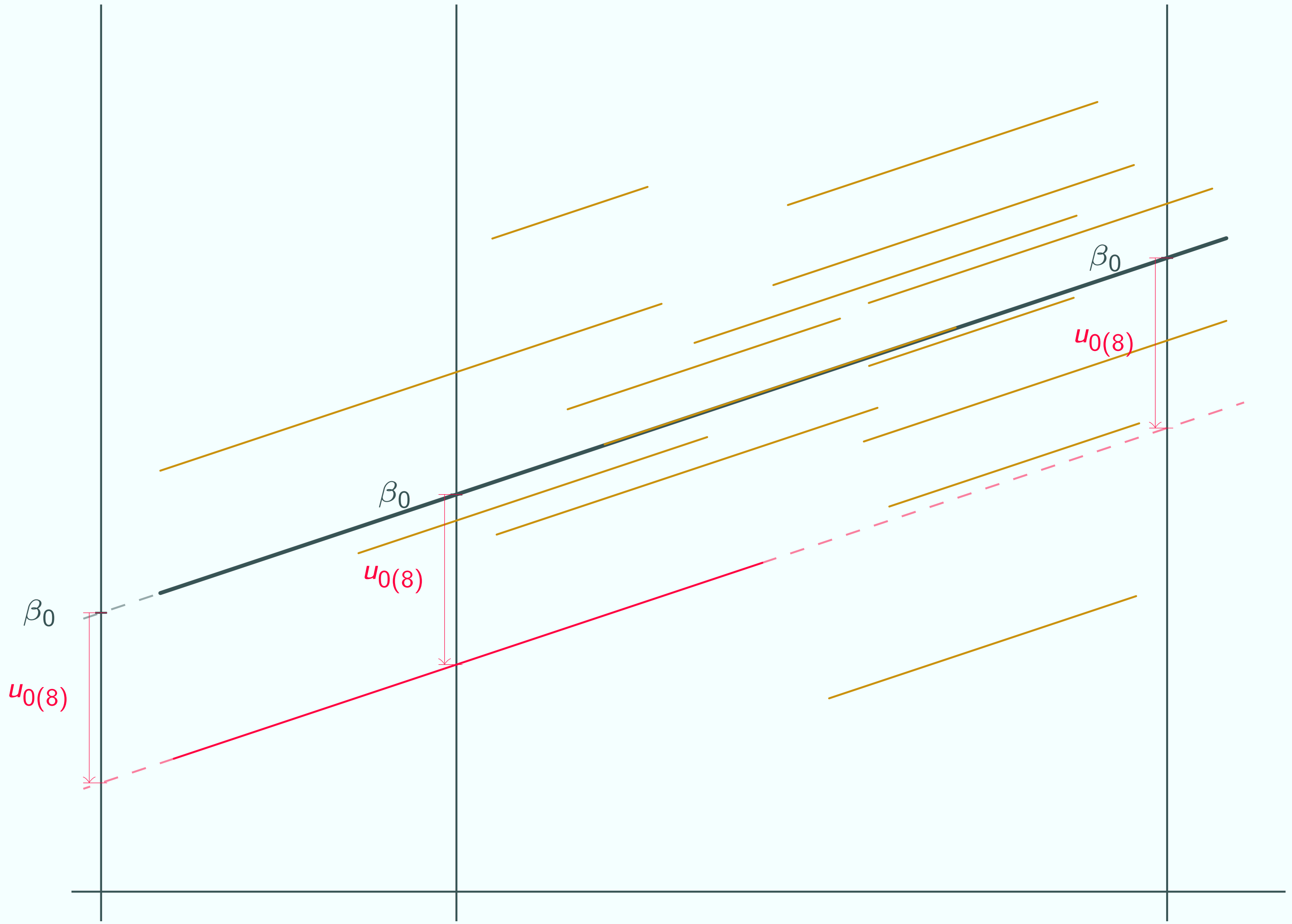
u_{0j} and the scale of x

Random slope model



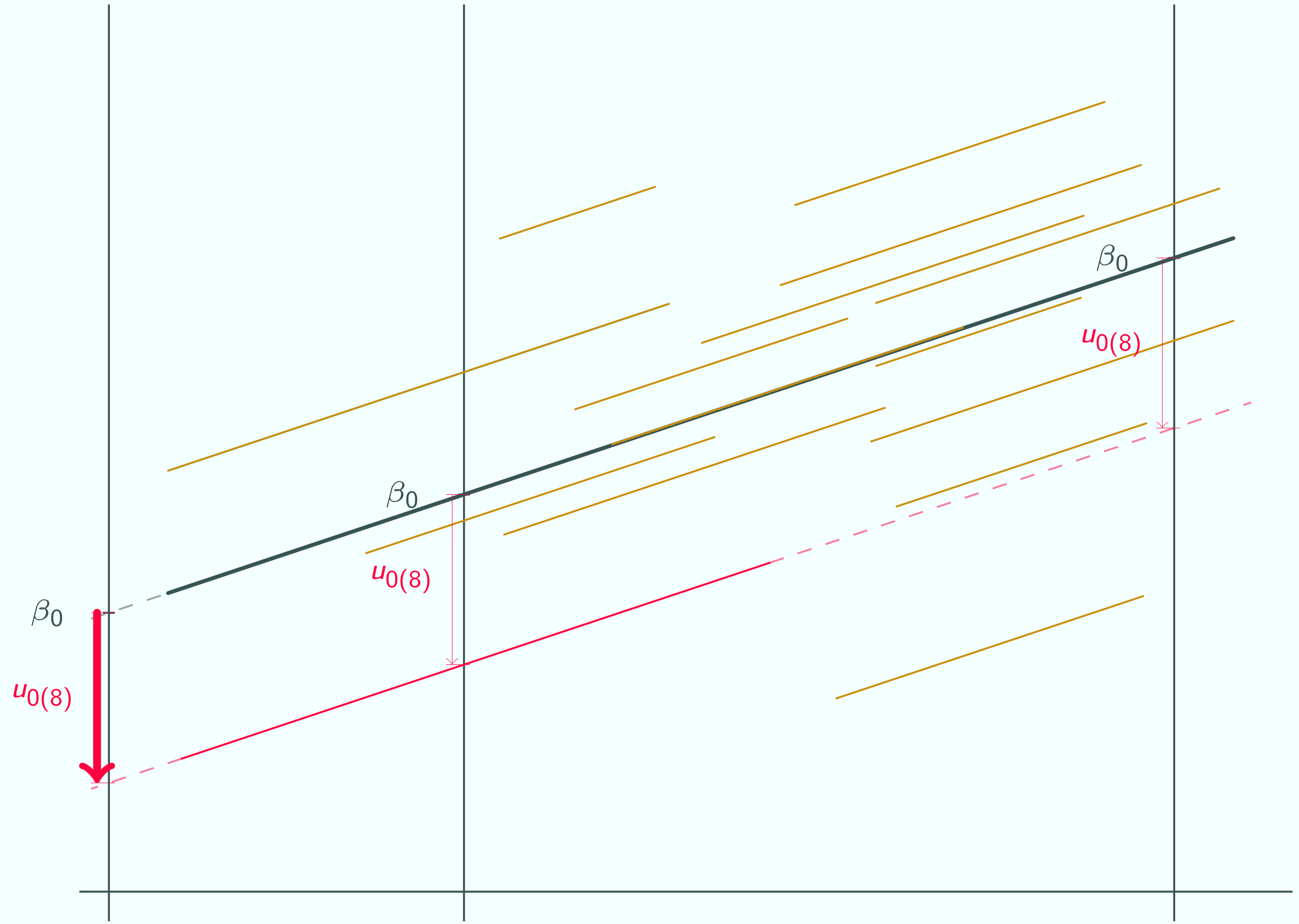
u_{0j} and the scale of x

Random intercept model



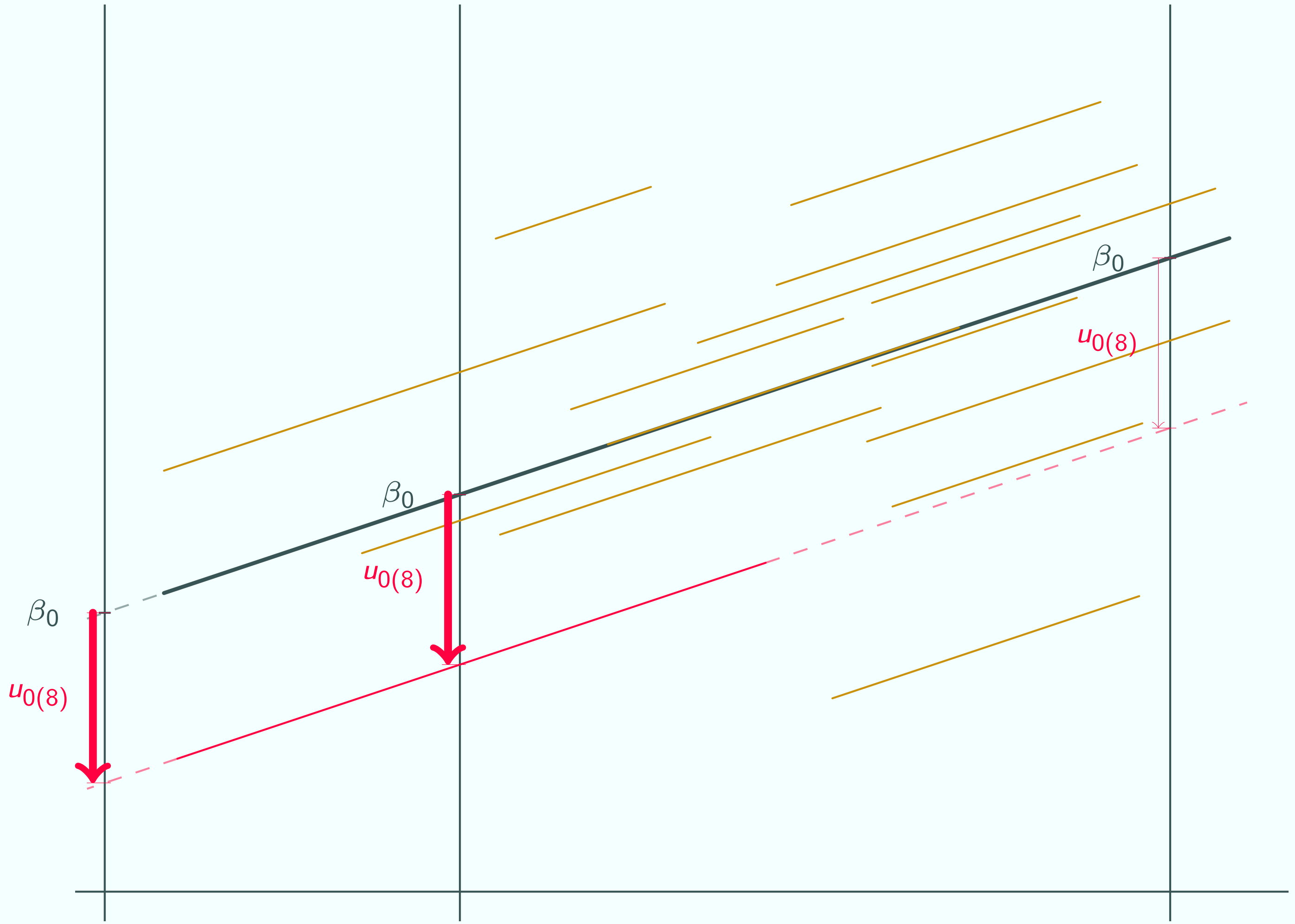
u_{0j} and the scale of x

Random intercept model



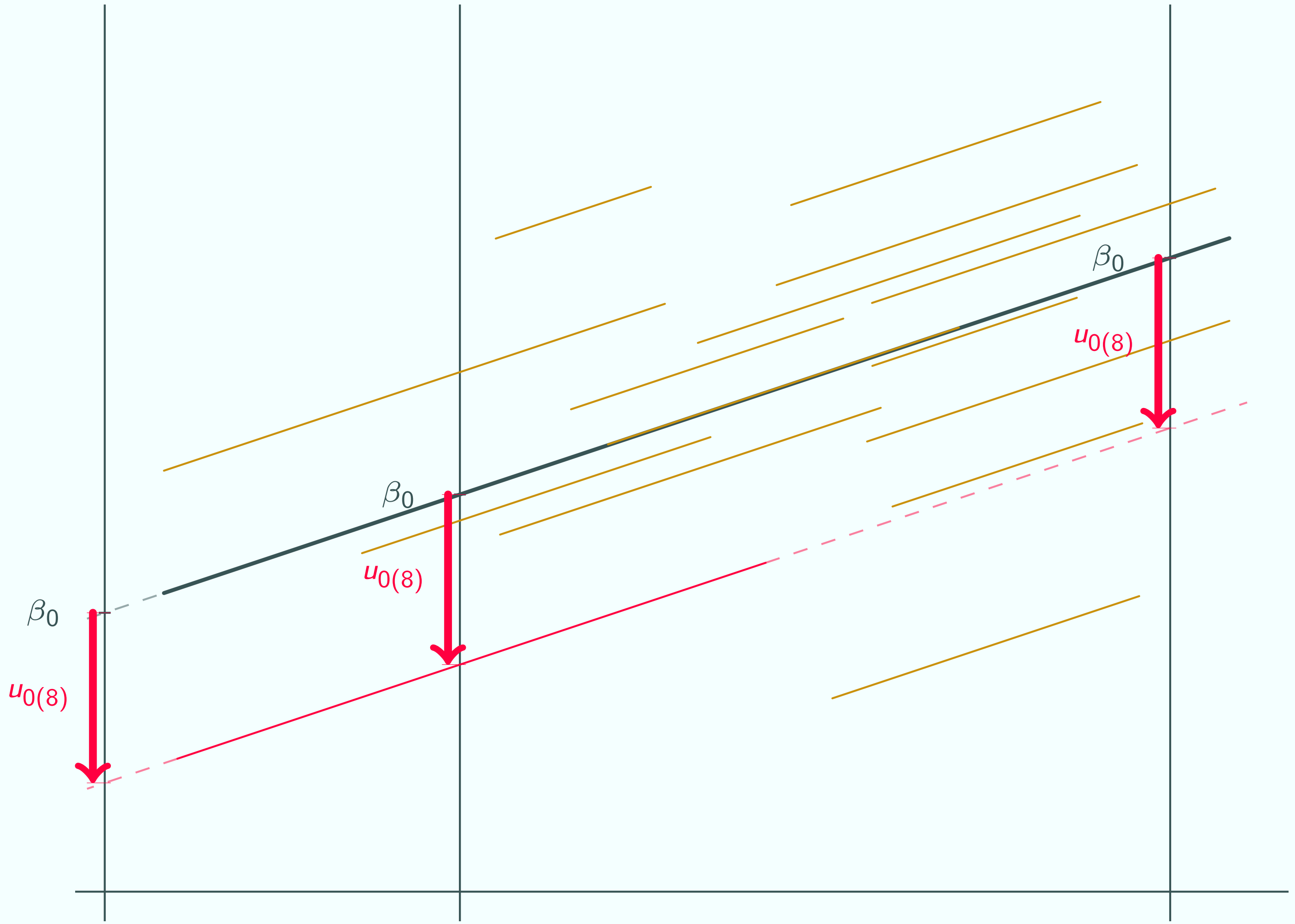
u_{0j} and the scale of x

Random intercept model



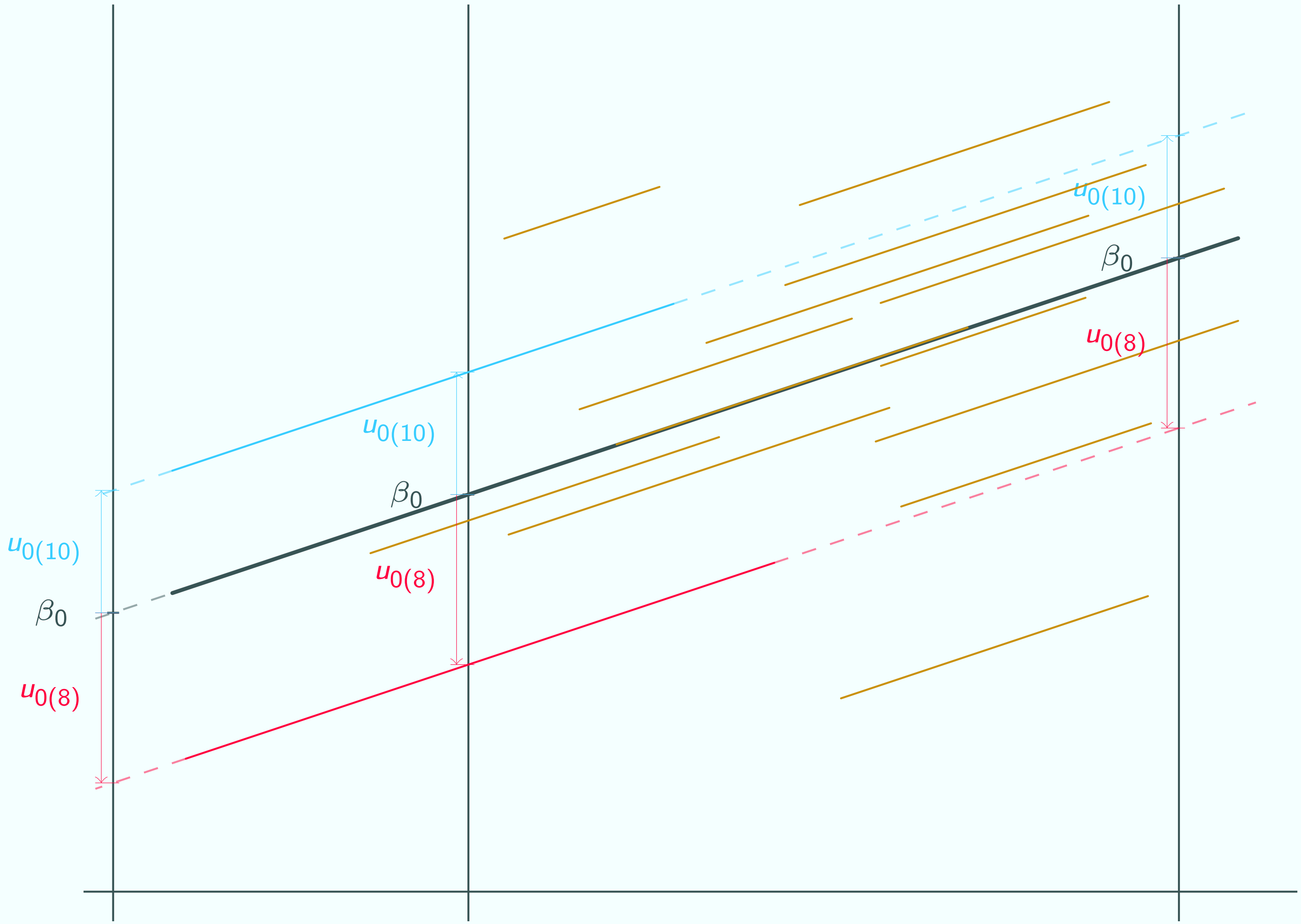
u_{0j} and the scale of x

Random intercept model



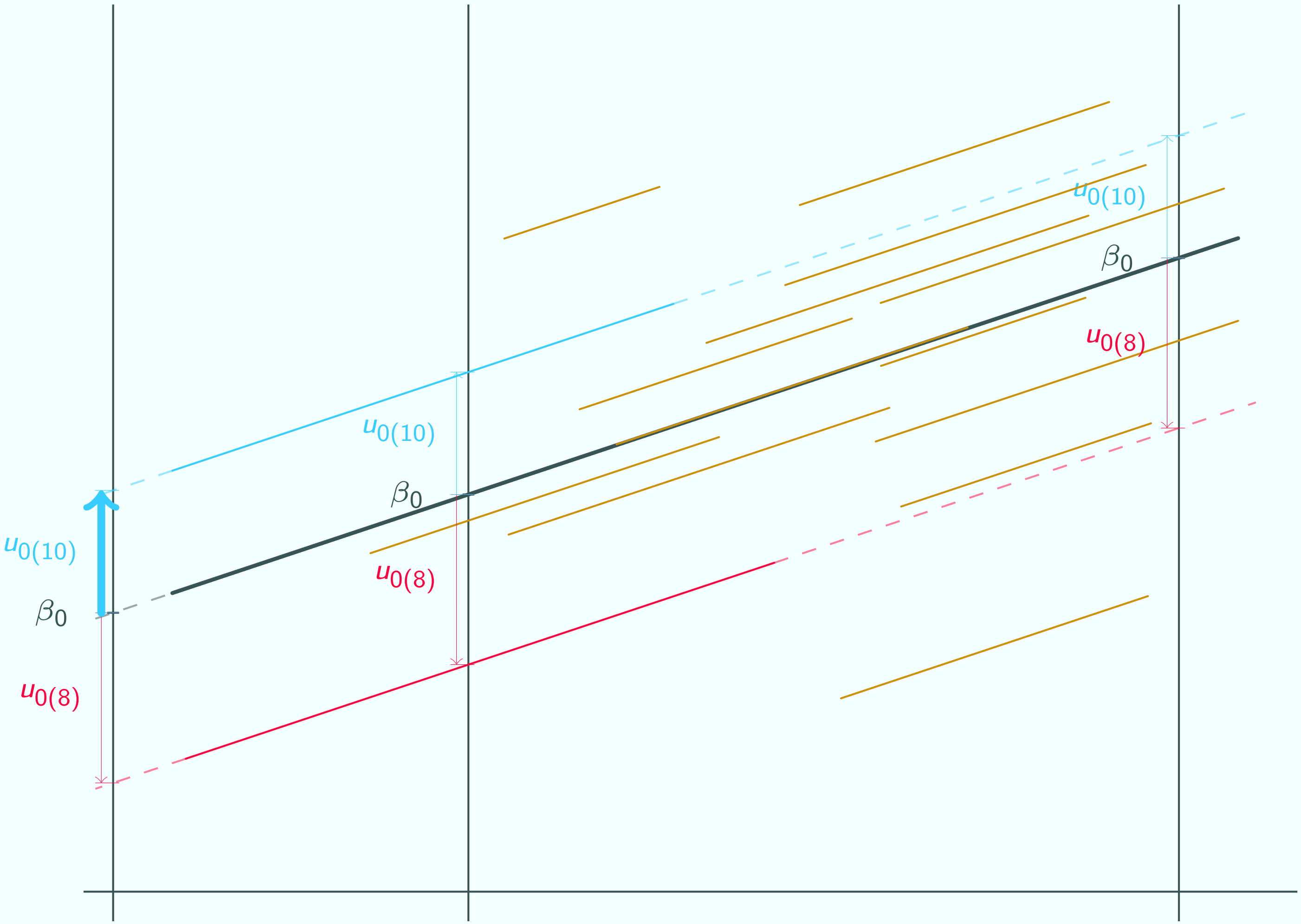
u_{0j} and the scale of x

Random intercept model



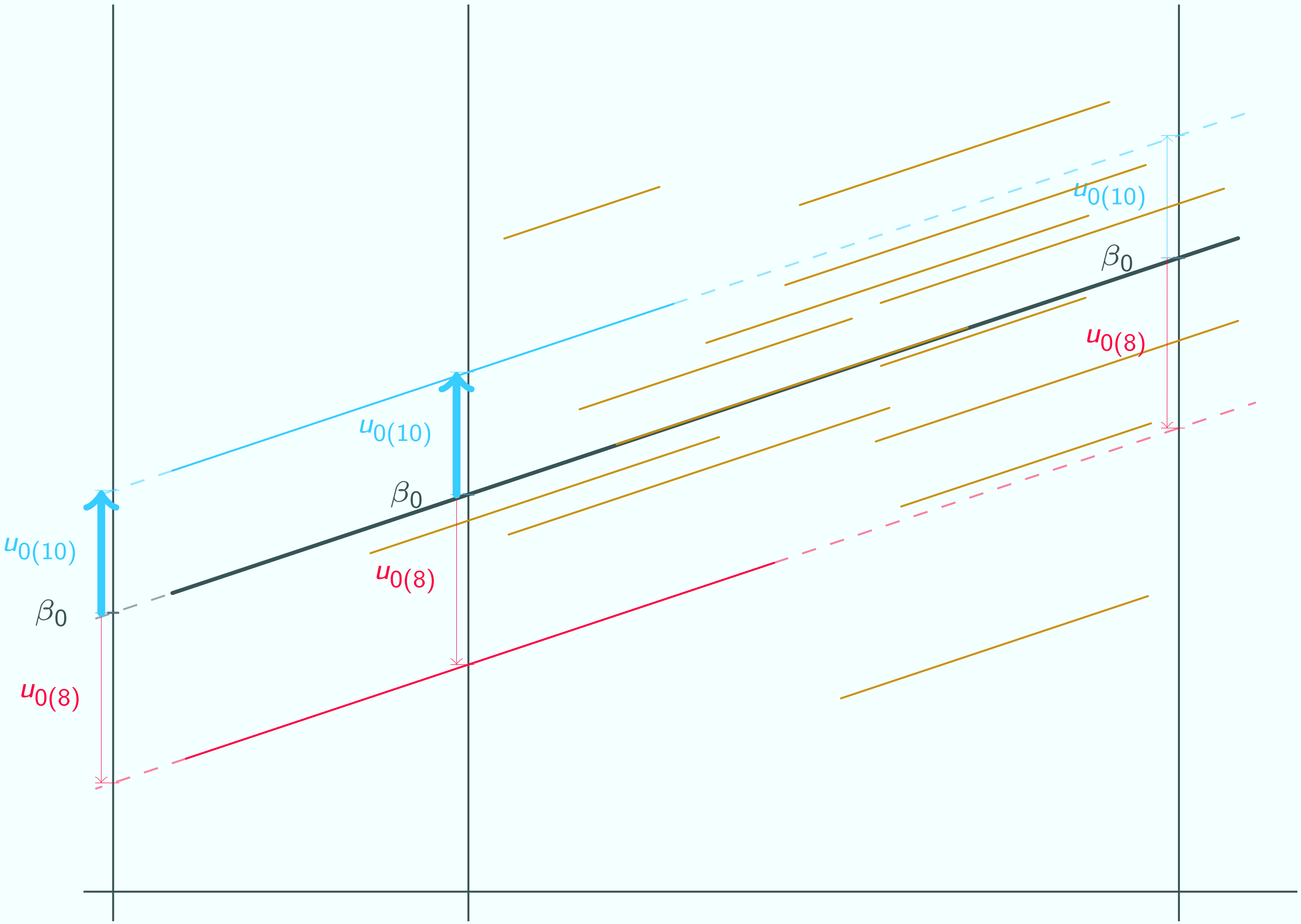
u_{0j} and the scale of x

Random intercept model



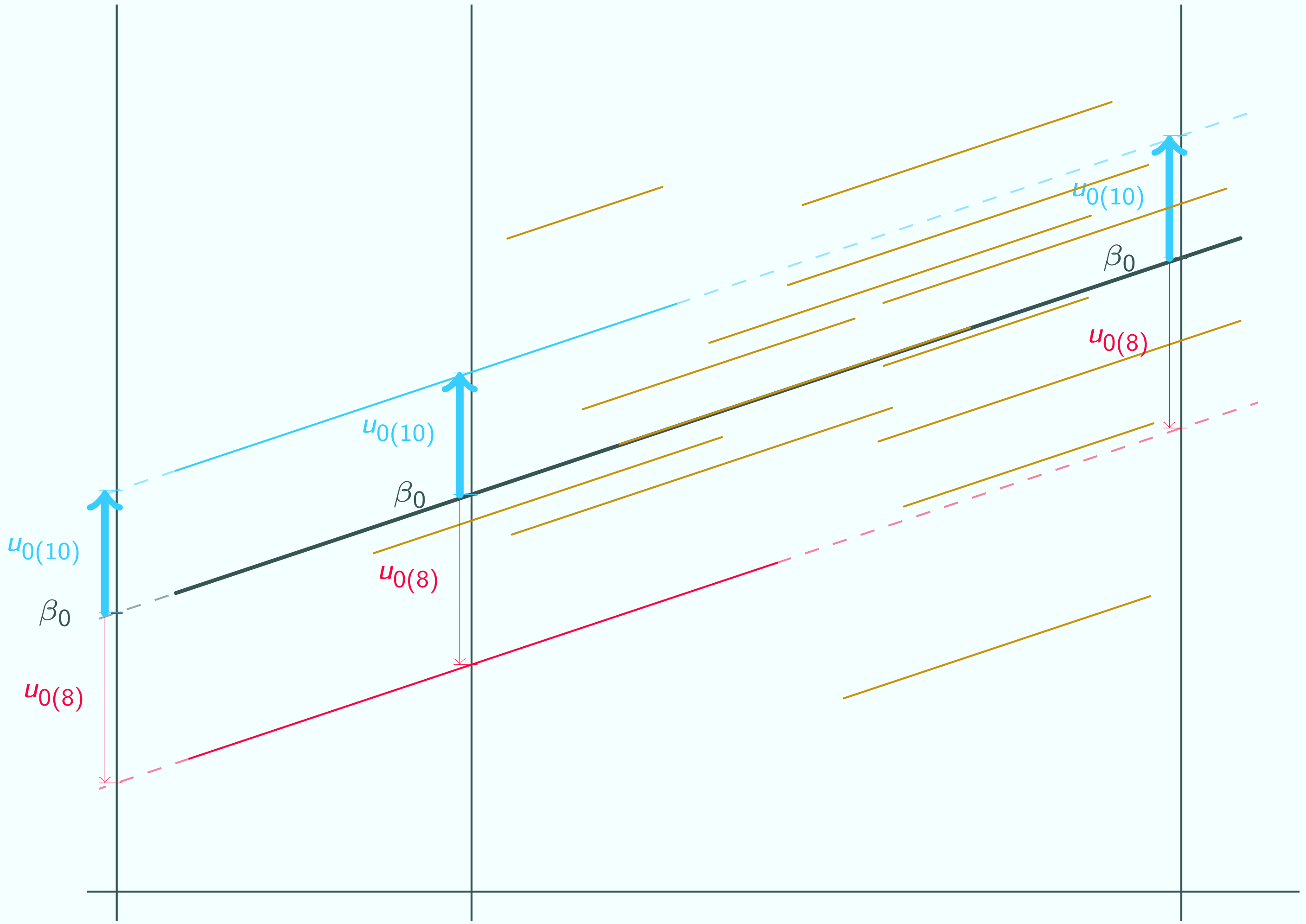
u_{0j} and the scale of x

Random intercept model



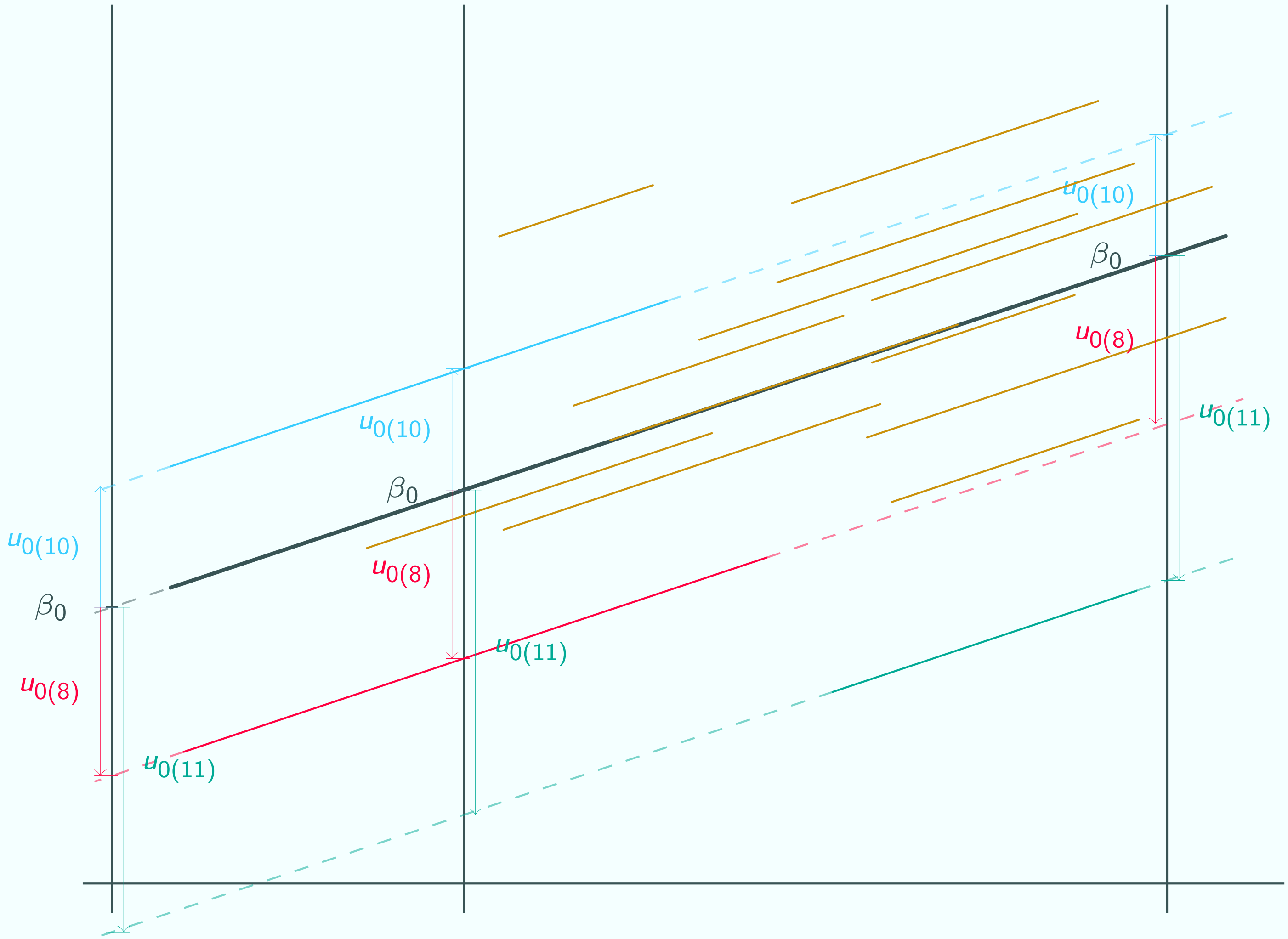
u_{0j} and the scale of x

Random intercept model



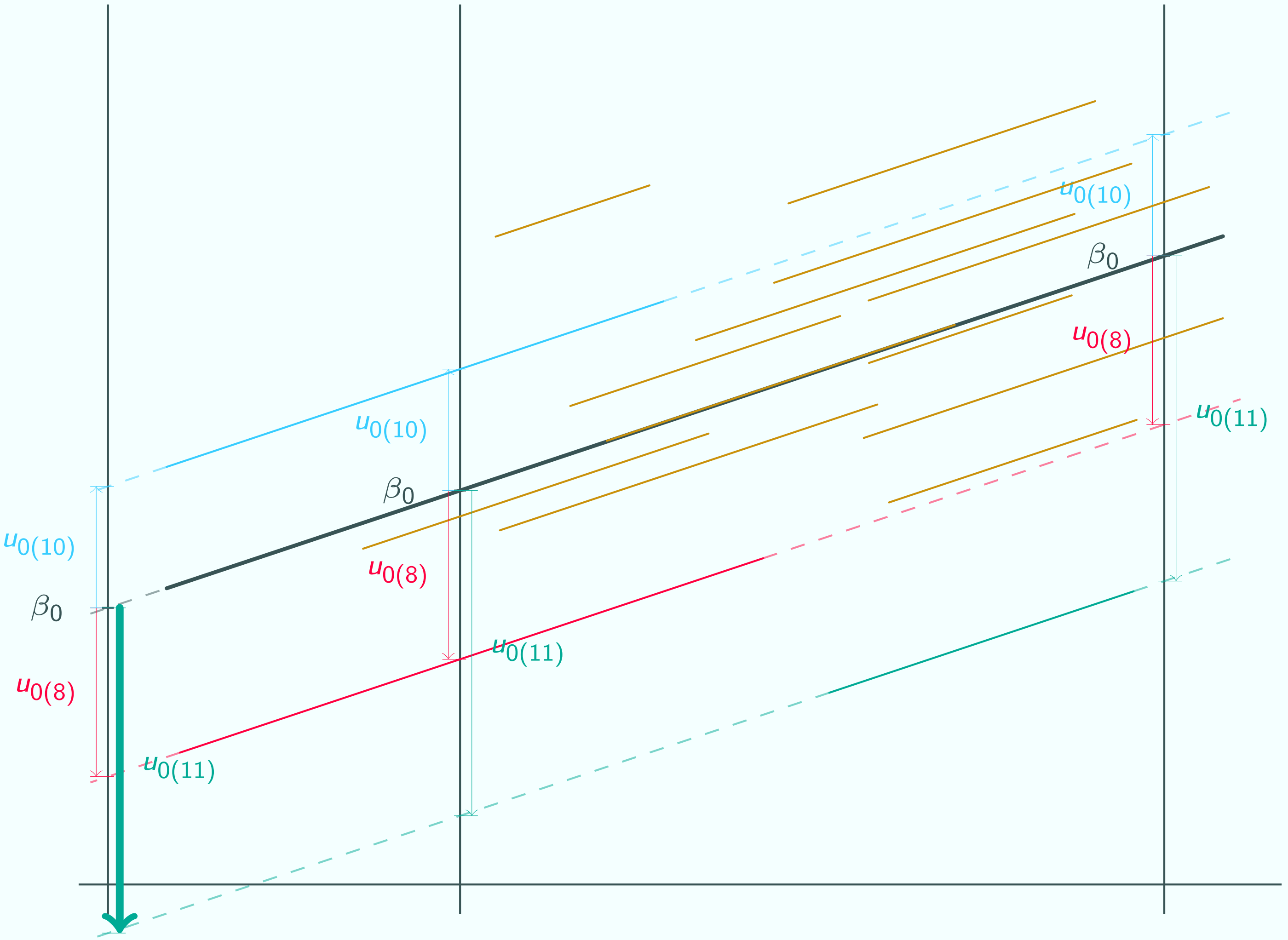
u_{0j} and the scale of x

Random intercept model



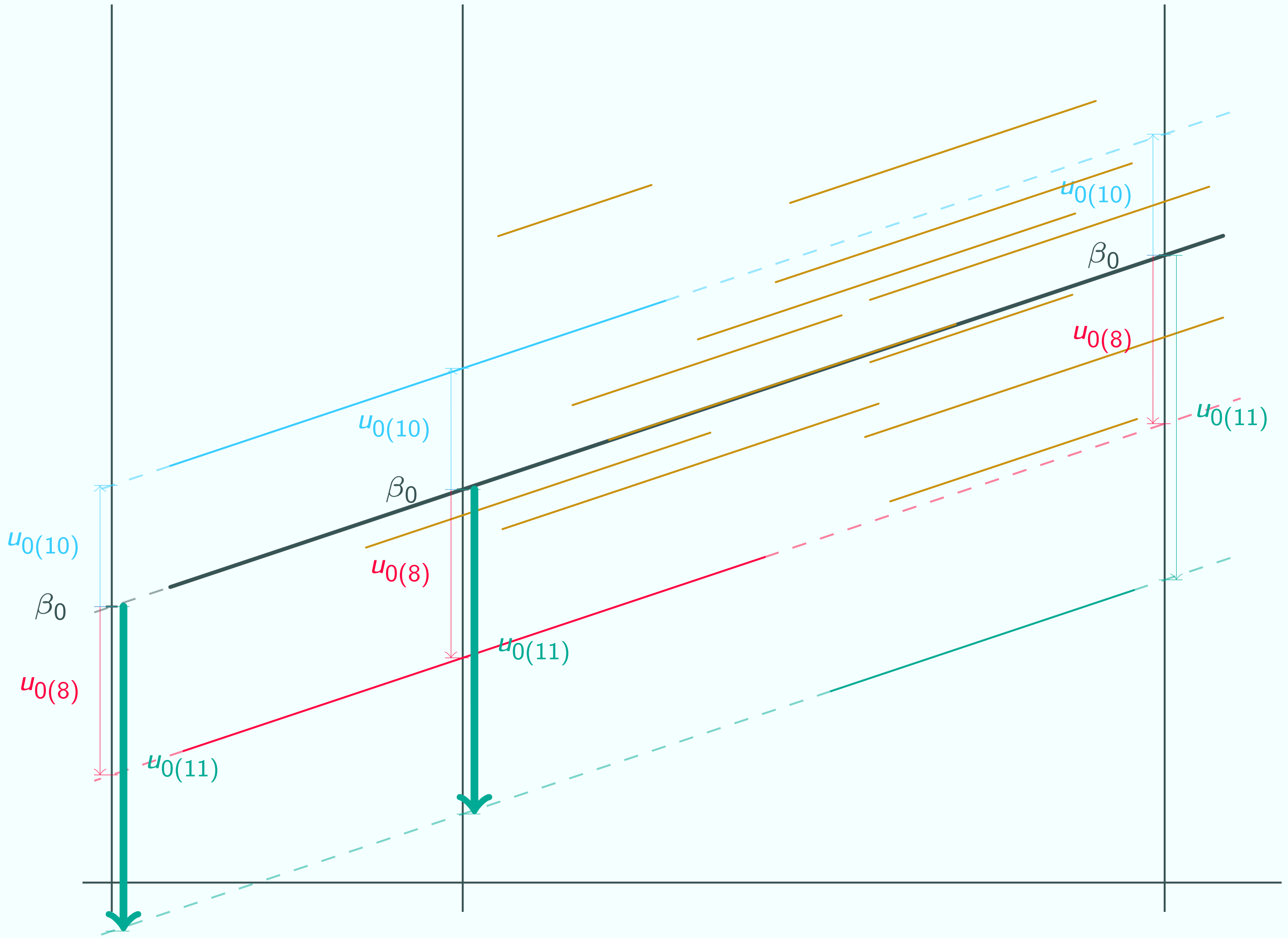
u_{0j} and the scale of x

Random intercept model



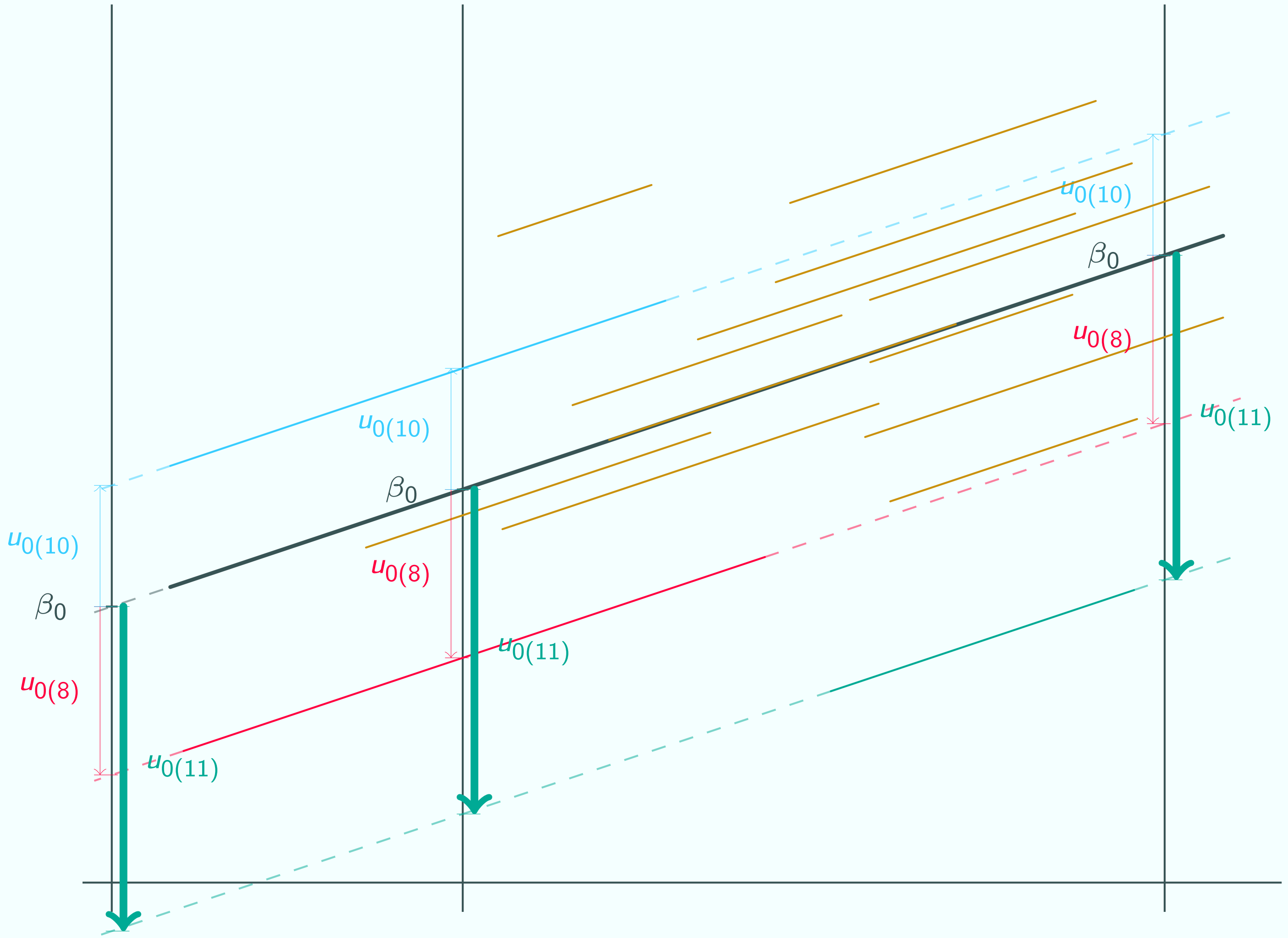
u_{0j} and the scale of x

Random intercept model



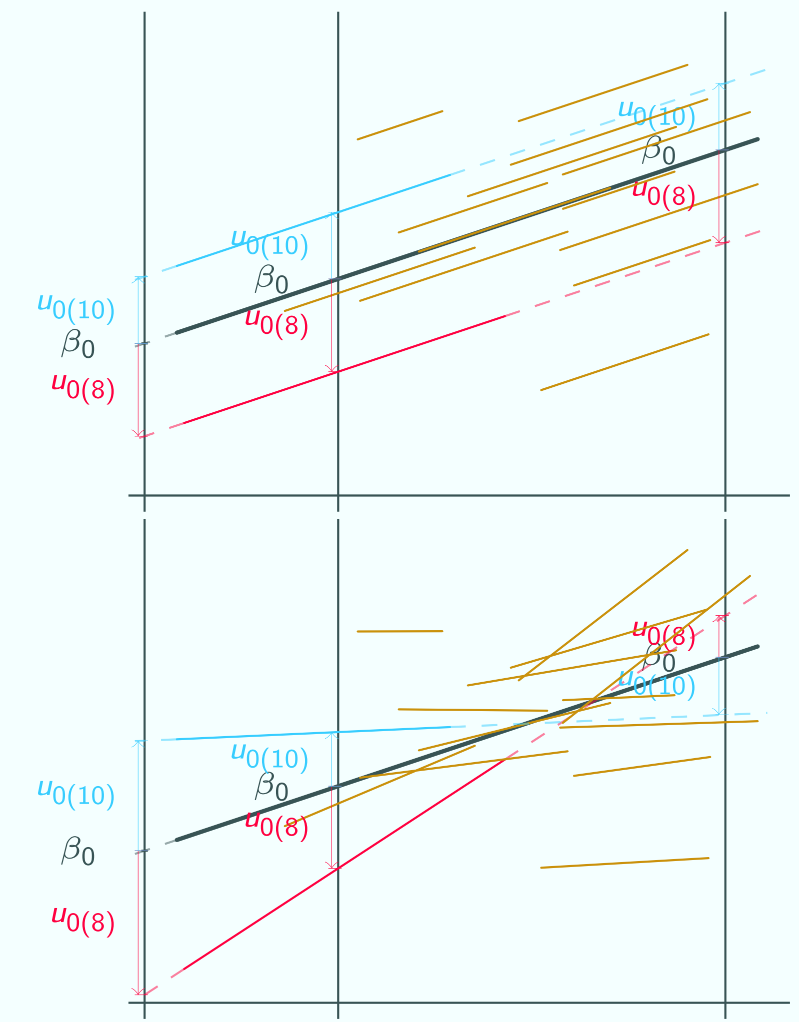
u_{0j} and the scale of x

Random intercept model



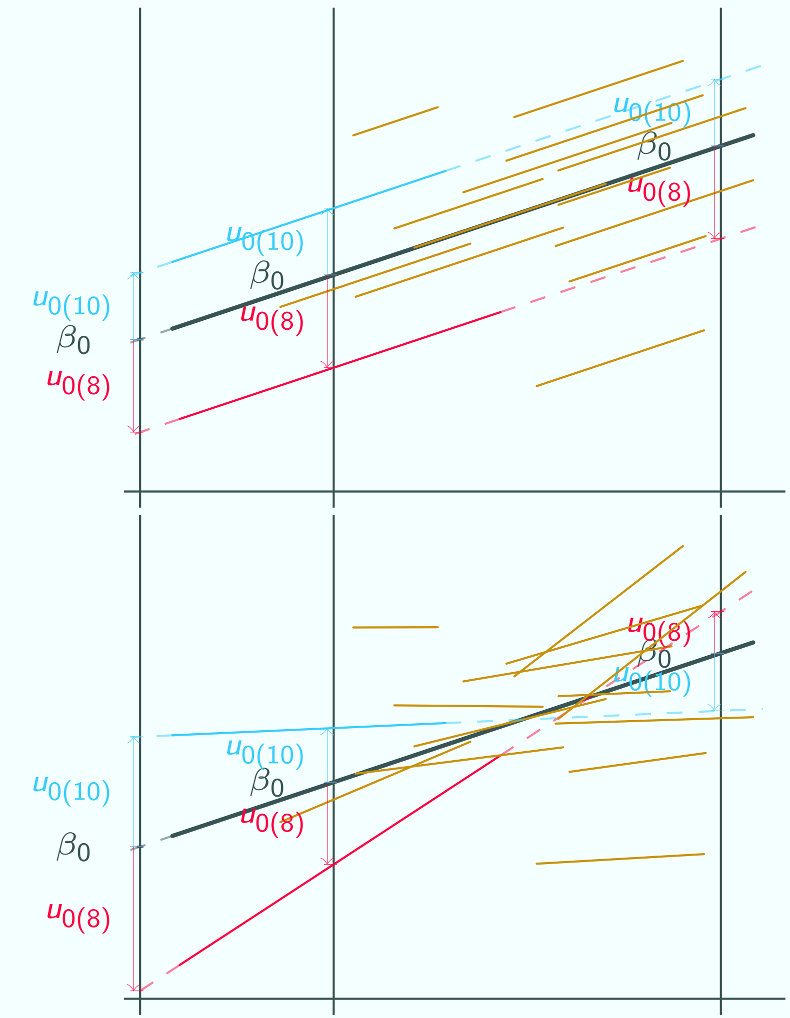
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}



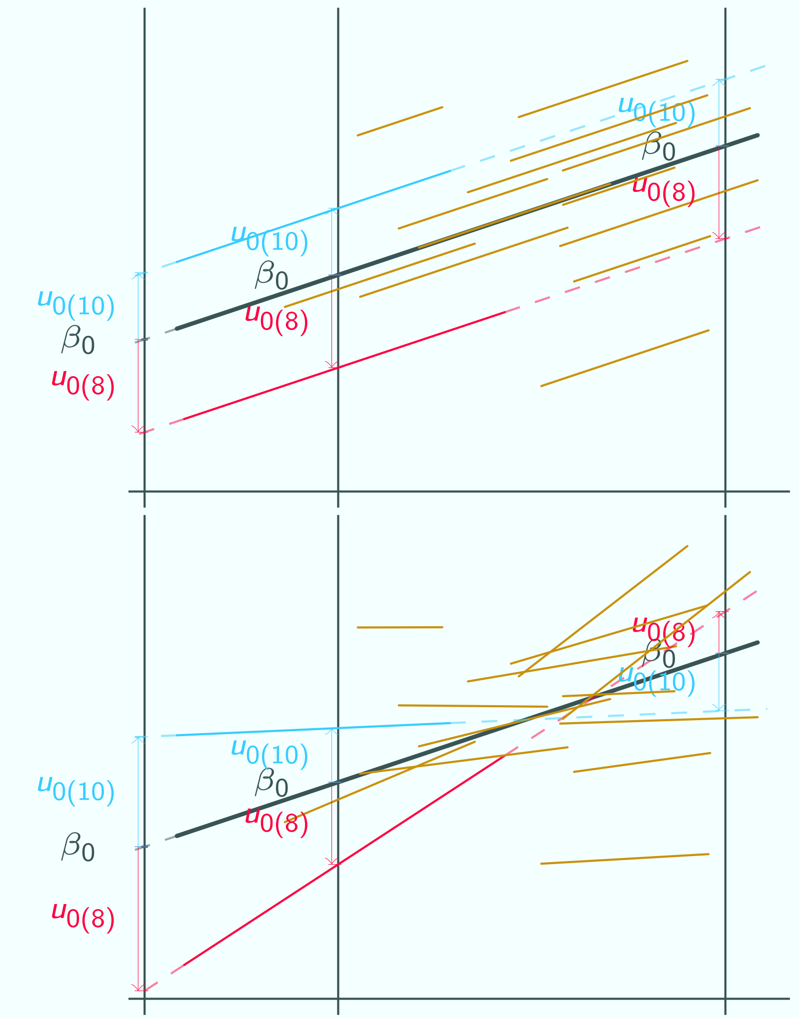
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}



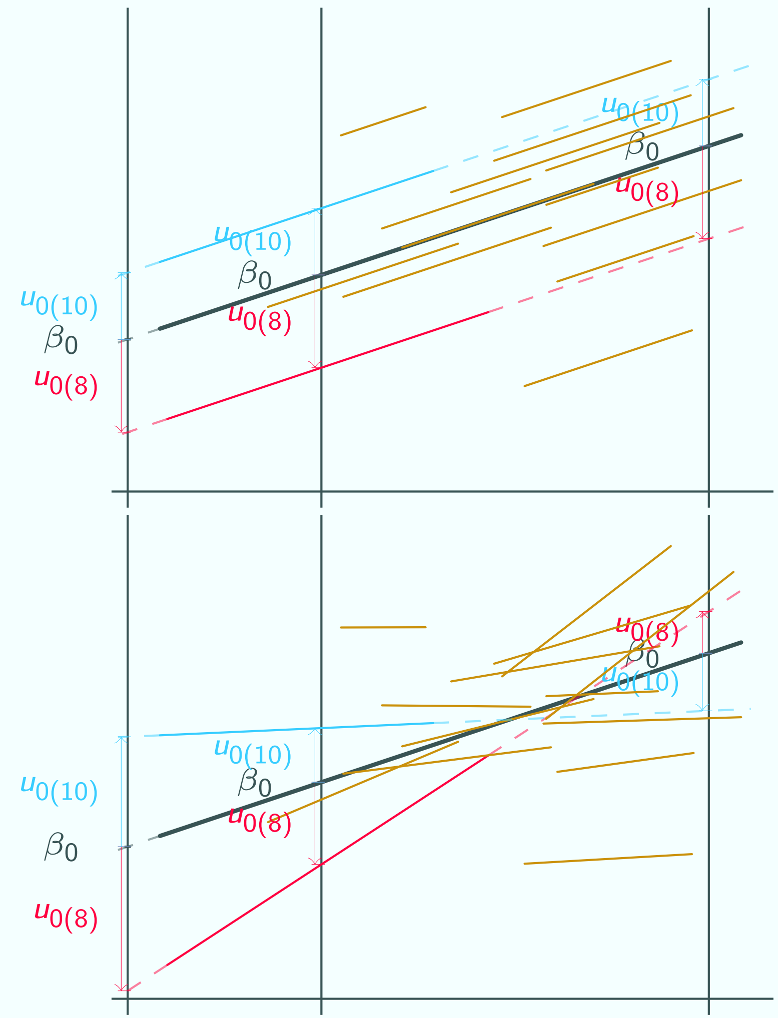
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}
- The variance σ^2_{u0} will also be affected



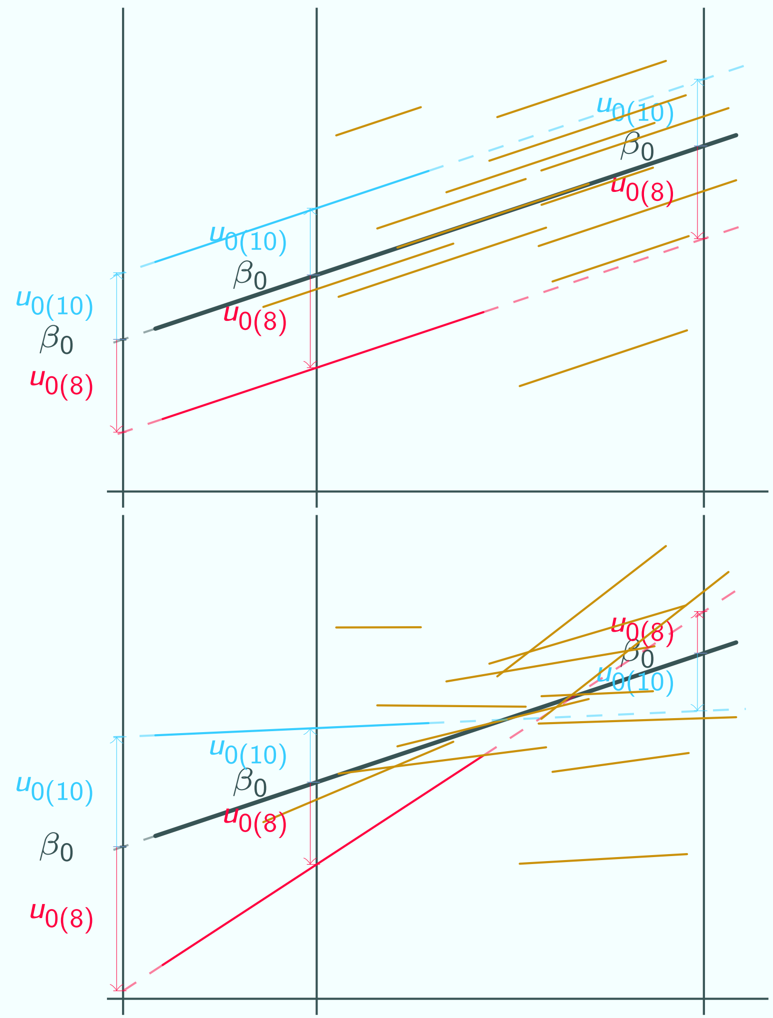
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}
- The variance σ^2_{u0} will also be affected
- as will the covariance σ_{u01}



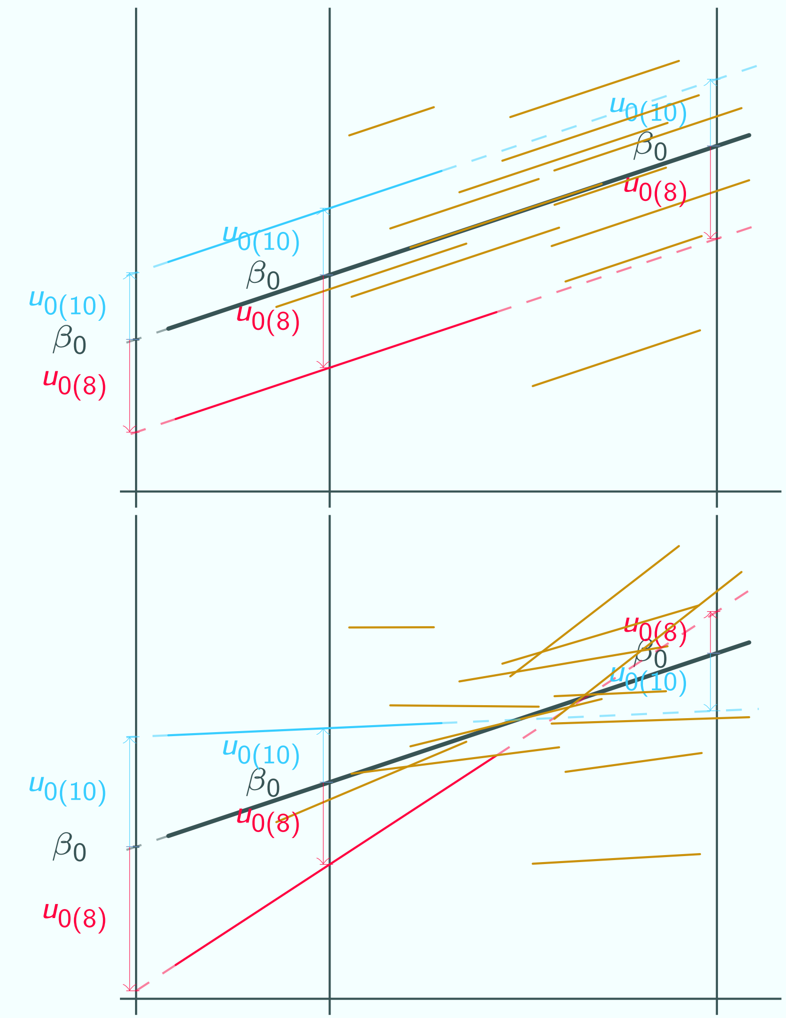
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}
- The variance σ^2_{u0} will also be affected
- as will the covariance σ_{u01}
- This is why we have to interpret σ^2_{u1}, σ^2_{u0} and σ_{u01}



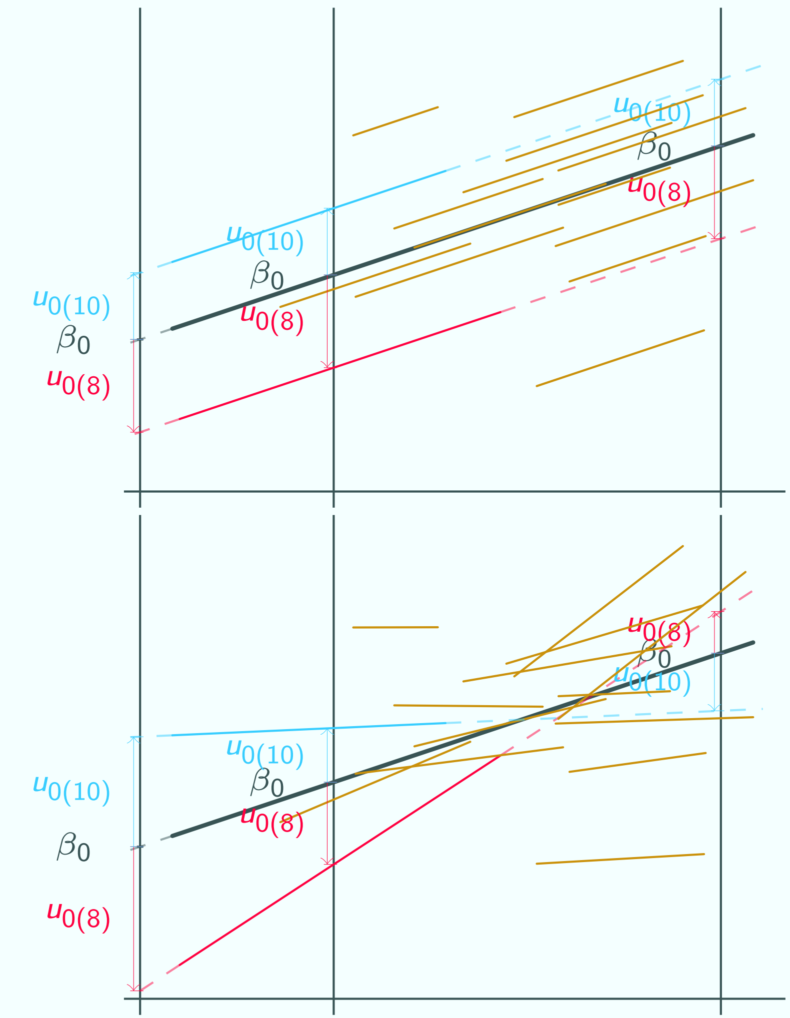
u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}
- The variance σ^2_{u0} will also be affected
- as will the covariance σ_{u01}
- This is why we have to interpret σ^2_{u1}, σ^2_{u0} and σ_{u01}
    - together



u_{0j} and the scale of x

- For a random intercepts model, where $x = 0$ occurs makes no difference to the value of u_{0j}
- For a random slopes model, it makes no difference to the value of u_{1j}, but it does make a difference to the value of u_{0j}
- The variance σ^2_{u0} will also be affected
- as will the covariance σ_{u01}
- This is why we have to interpret σ^2_{u1}, σ^2_{u0} and σ_{u01}
  - together
  - and in light of where we have put $x = 0$



Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (①)

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (1)
- and without $u_{1j}x_{1ij}$ (0)

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (1)
- and without $u_{1j}x_{1ij}$ (0)

- In other words we are comparing the random slope model to a random intercept model

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (1)
- and without $u_{1j}x_{1ij}$ (0)

- In other words we are comparing the random slope model to a random intercept model
- The test statistic is again $2(\log(\text{likelihood}(1)) - \log(\text{likelihood}(0)))$

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (①)
- and without $u_{1j}x_{1ij}$ (⓪)

- In other words we are comparing the random slope model to a random intercept model
- The test statistic is again 2(log(likelihood(①)) − log(likelihood(⓪)))
- This time there are 2 degrees of freedom because there are 2 extra parameters in ① compared to ⓪

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (①)
- and without $u_{1j}x_{1ij}$ (⓪)

- In other words we are comparing the random slope model to a random intercept model
- The test statistic is again 2(log(likelihood(①)) − log(likelihood(⓪)))
- This time there are 2 degrees of freedom because there are 2 extra parameters in ① compared to ⓪
- So we compare the test statistic against the $\chi^2_{(2)}$ distribution

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (①)
- and without $u_{1j}x_{1ij}$ (⓪)

- In other words we are comparing the random slope model to a random intercept model
- The test statistic is again 2(log(likelihood(①)) − log(likelihood(⓪)))
- This time there are 2 degrees of freedom because there are 2 extra parameters in ① compared to ⓪
- So we compare the test statistic against the $\chi^2_{(2)}$ distribution
- The null hypothesis is that σ^2_{u1} and σ_{u01} are both 0

Hypothesis testing

Hypothesis testing is the same as for the random intercept model

Fixed part

β_k is significant at the 5% level if $|z_k| \geqslant 1.96$

Random part

We use a likelihood ratio test

- Fit the model with $u_{1j}x_{1ij}$ (①)
- and without $u_{1j}x_{1ij}$ (⓪)

- In other words we are comparing the random slope model to a random intercept model
- The test statistic is again $2(\log(\text{likelihood}(①)) - \log(\text{likelihood}(⓪)))$
- This time there are 2 degrees of freedom because there are 2 extra parameters in ① compared to ⓪
- So we compare the test statistic against the $\chi^2_{(2)}$ distribution
- The null hypothesis is that σ^2_{u1} and σ_{u01} are both 0 and hence that a random intercept model is more appropriate than a random slope model

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

1. Fit a model with a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value:

Exam scores example

1

Equations

$$\text{normexam}_{ij} = \beta_{0j} + \beta_{1j}\text{standlrt}_{ij} + e_{ij}$$

$$\beta_{0j} = -0.012(0.040) + u_{0j}$$

$$\beta_{1j} = 0.557(0.020) + u_{1j}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim \mathrm{N}(0,\ \Omega_u) \ : \ \Omega_u = \begin{bmatrix} 0.090(0.018) & \\ 0.018(0.007) & 0.015(0.004) \end{bmatrix}$$

$$e_{ij} \sim \mathrm{N}(0, \sigma_e^2) \quad \sigma_e^2 = 0.554(0.012)$$

$$-2{*}loglikelihood = 9316.870(4059 \text{ of } 4059 \text{ cases in use})$$

Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

2

Exam scores example

1

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2

Exam scores example

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Exam scores example

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Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

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2. Fit a model without a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value:

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Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

2

Equations

$$\text{normexam}_{ij} = \beta_{0j} + 0.563(0.012)\text{standlrt}_{ij} + e_{ij}$$

$$\beta_{0j} = 0.002(0.040) + u_{0j}$$

$$u_{0j} \sim \text{N}(0, \sigma_{u0}^2) \quad \sigma_{u0}^2 = 0.092(0.018)$$

$$e_{ij} \sim \text{N}(0, \sigma_e^2) \quad \sigma_e^2 = 0.566(0.013)$$

$-2{*}\mathit{loglikelihood} = 9357.242$(4059 of 4059 cases in use)

Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

Exam scores example

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Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

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Exam scores example

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Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

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1. Fit a model with a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value: 9316.870
2. Fit a model without a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value: 9357.242
3. Calculate the test statistic:

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

1. Fit a model with a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value: 9316.870
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3. Calculate the test statistic: $9357.242 - 9316.870 = 40.372$

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

1. Fit a model with a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value: 9316.870
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4. Compare to the χ^2 distribution with 2 degrees of freedom

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam scores at age 11 and 16?

Answer

1. Fit a model with a random slope on exam score age 11 and note the $-2 \times \log(\text{likelihood})$ value: 9316.870
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3. Calculate the test statistic: $9357.242 - 9316.870 = 40.372$
4. Compare to the χ^2 distribution with 2 degrees of freedom $p = 1.7113 \times 10^{-9}$
5. We conclude that there are differences between schools in the relationship between a pupil's exam scores at age 11 and 16

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

1. Fit a model with a random slope on gender and note the $-2 \times \log(\text{likelihood})$ value:

Exam scores example

1

$$\text{normexam}_{ij} = \beta_{0j} + \beta_{1j}\text{girl}_{ij} + e_{ij}$$

$$\beta_{0j} = -0.161(0.057) + u_{0j}$$

$$\beta_{1j} = 0.262(0.040) + u_{1j}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim \mathrm{N}(0,\ \Omega_u) \ :\ \Omega_u = \begin{bmatrix} 0.161(0.031) & \\ 0.000(0.000) & 0.000(0.000) \end{bmatrix}$$

$$e_{ij} \sim \mathrm{N}(0,\ \sigma_e^2) \quad \sigma_e^2 = 0.839(0.019)$$

$-2*loglikelihood$ = 10968.689(4059 of 4059 cases in use)

2

Exam scores example

1

Equations

$$\text{normexam}_{ij} = \beta_{0j} + \beta_{1j}\text{girl}_{ij} + e_{ij}$$

$$\beta_{0j} = -0.164(0.054) + u_{0j}$$

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$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim \text{N}(0,\ \Omega_u)\ :\ \Omega_u = \begin{bmatrix} 0.146(0.033) & \\ 0.017(0.014) & -0.009(0.008) \end{bmatrix}$$

$$e_{ij} \sim \text{N}(0, \sigma_e^2) \quad \sigma_e^2 = 0.841(0.019)$$

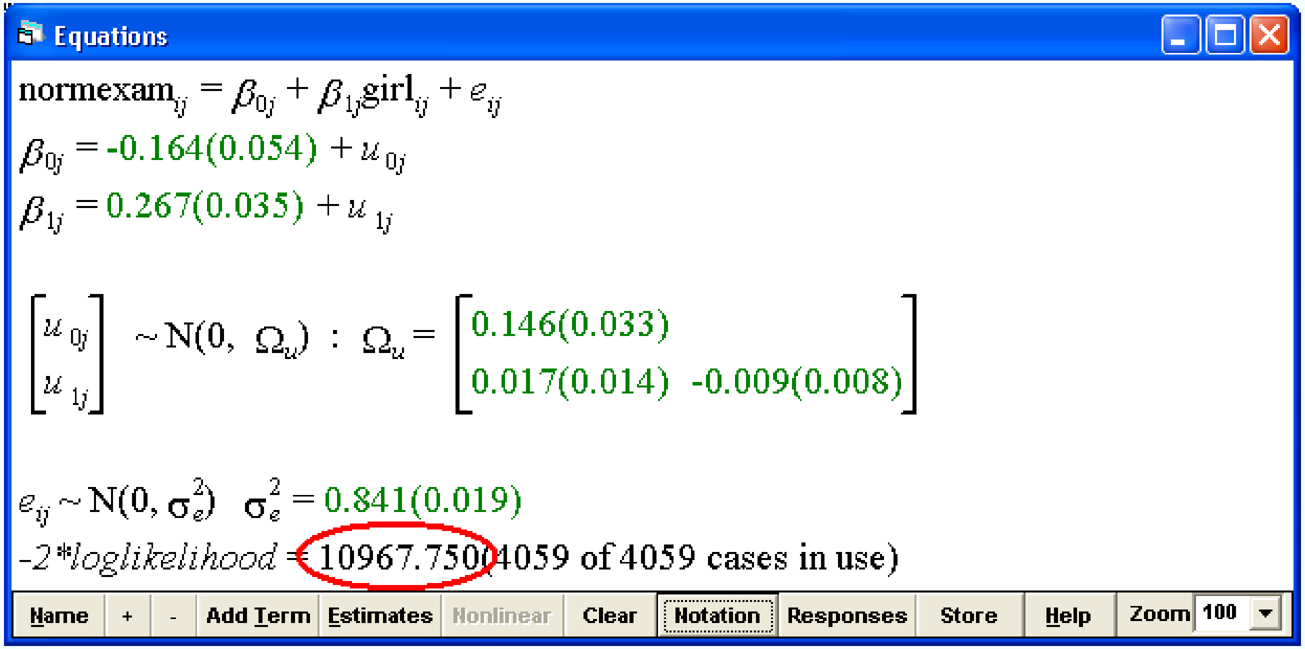
$-2*loglikelihood$ = 10967.750(4059 of 4059 cases in use)

Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

2

Exam scores example

1



2

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

1. Fit a model with a random slope on gender and note the $-2 \times \log(\text{likelihood})$ value: 10967.750

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

1. Fit a model with a random slope on gender and note the $-2 \times \log(\text{likelihood})$ value: 10967.750
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Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

2

Equations

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Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

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Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

1. Fit a model with a random slope on gender and note the $-2 \times \log(\text{likelihood})$ value: 10967.750
2. Fit a model without a random slope on gender and note the $-2 \times \log(\text{likelihood})$ value: 10968.689

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

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3. Calculate the test statistic:

Exam scores example

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3. Calculate the test statistic: $10968.689 - 10967.750 = 0.939$

Exam scores example

Question

Are there differences between schools in the relationship between a pupil's exam score at age 16 and their gender?

Answer

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3. Calculate the test statistic: $10968.689 - 10967.750 = 0.939$
4. Compare to the χ^2 distribution with 2 degrees of freedom $p = 0.63$

Exam scores example

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Answer

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3. Calculate the test statistic: $10968.689 - 10967.750 = 0.939$
4. Compare to the χ^2 distribution with 2 degrees of freedom
   $p = 0.63$
5. We conclude that there are no differences between schools in the relationship between a pupil's gender and their exam score at age 16

Negative variances

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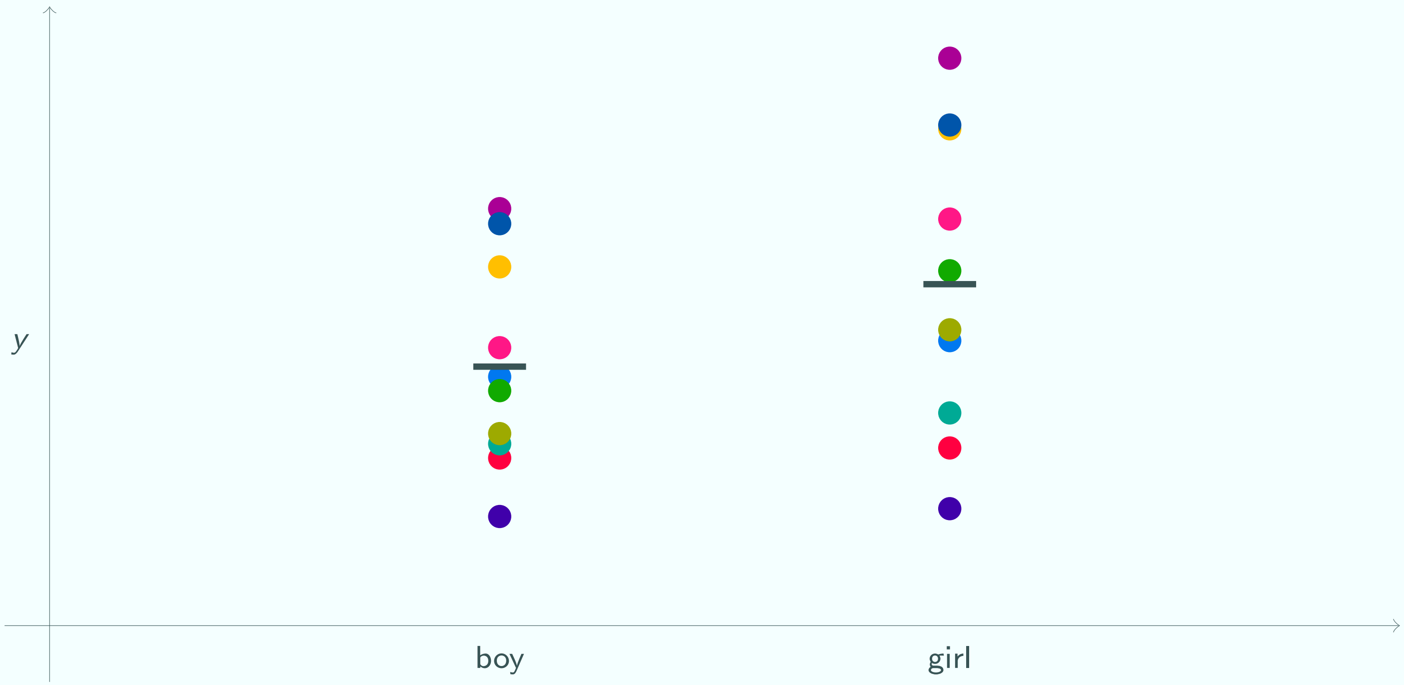
- This seems strange because a variance can't be negative
- In this case, the variance (and covariance) were not significant, so we don't need to worry
- In general, it is actually possible to get a significant and negative variance
- This actually does make sense due to the second interpretation of random slope models which we will see later
- Sometimes the final estimated variance is not negative but it goes negative during estimation so we need to allow negative variances

Random slopes and categorical variables

- It is possible to put a random slope on a categorical variable such as gender
- Often called a random coefficient rather than random slope
- Random slopes on continuous variables can also be called random coefficients



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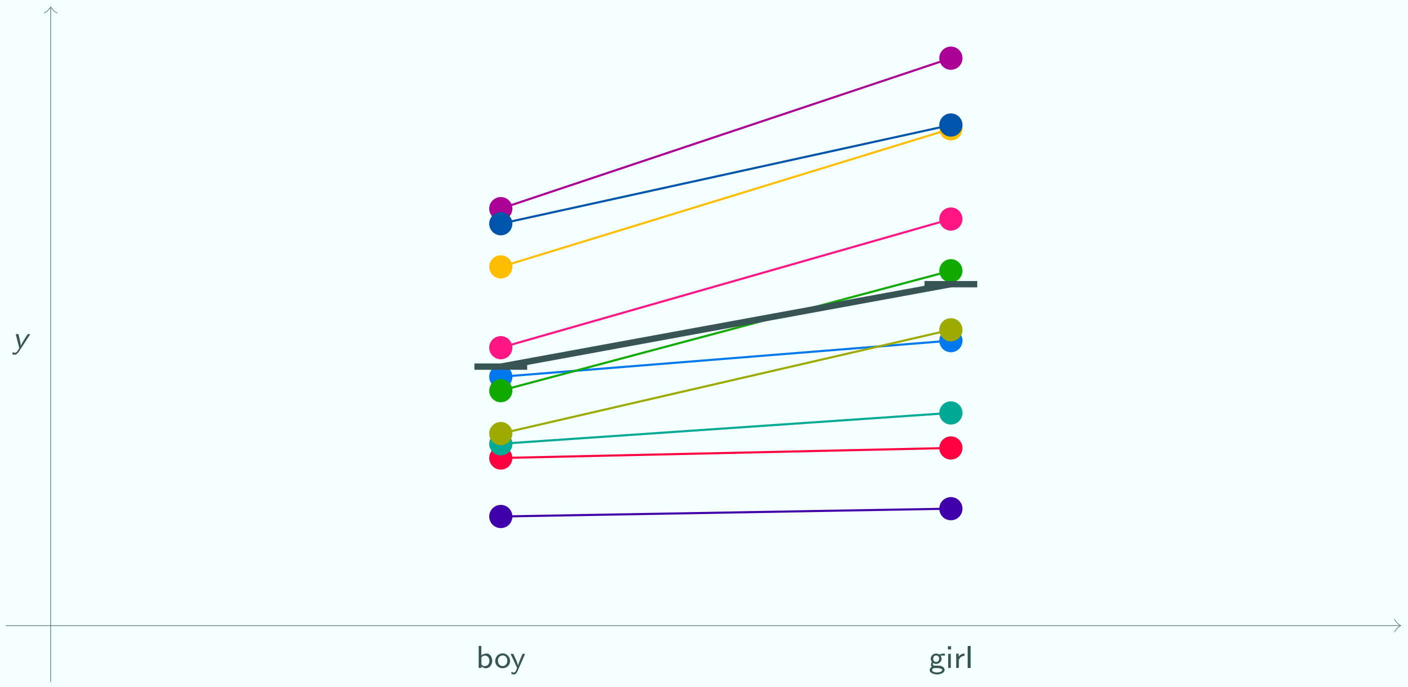
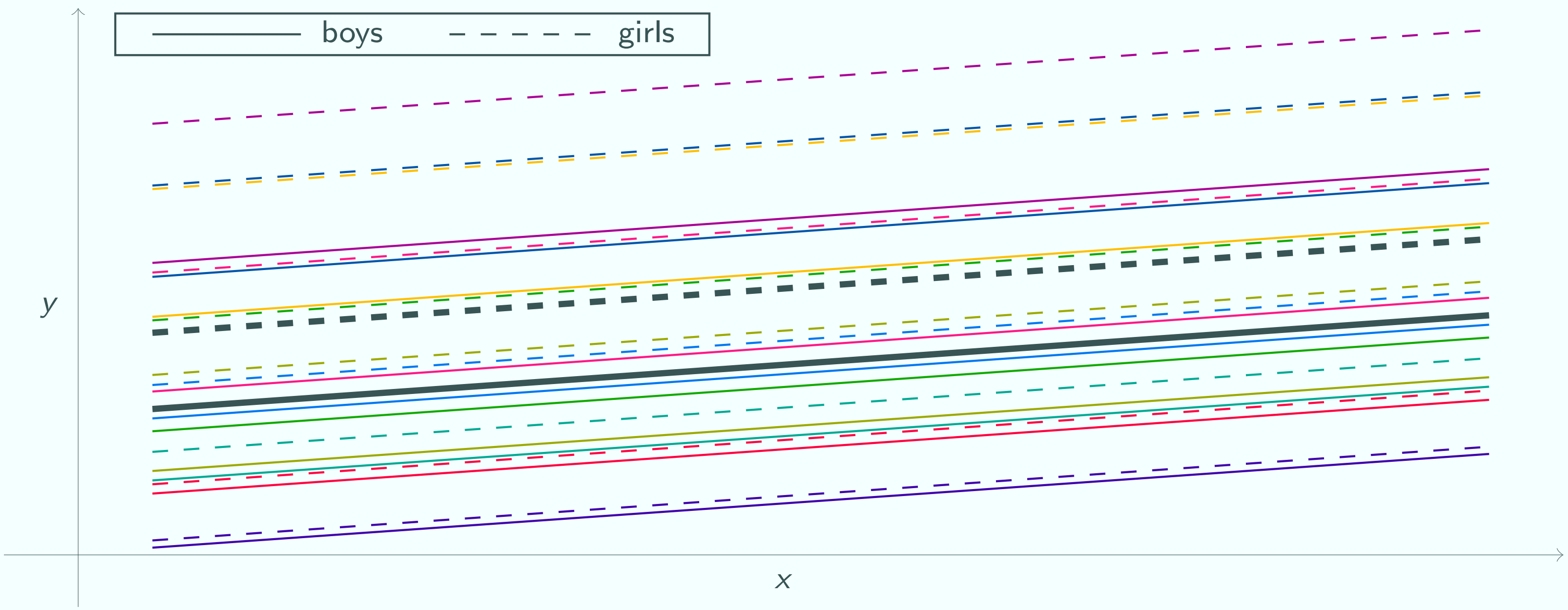


Random slopes and categorical variables

Model with a continous explanatory variable and gender; random coefficient on gender only; plotting against the continuous variable



- Random coefficient means distance between the line for girls and the line for boys differs from group to group
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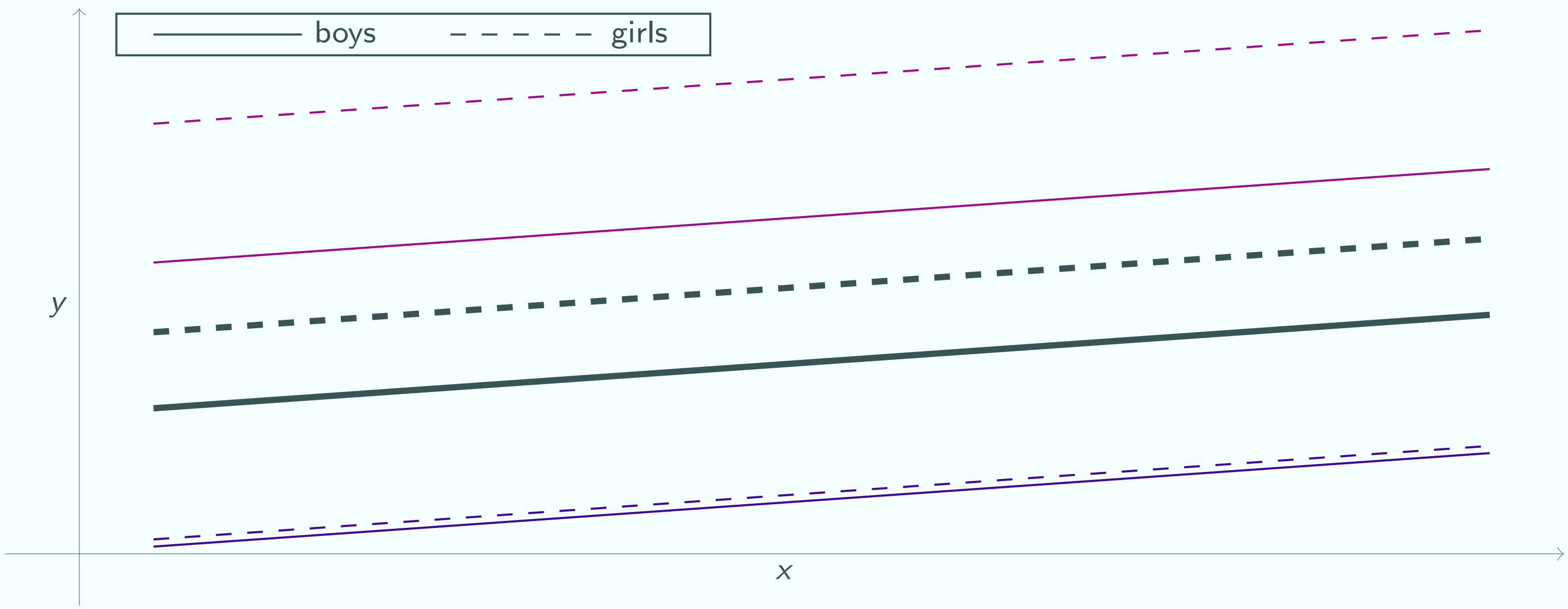
Picking out 2 groups for clarity



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Examples of research questions

Clark et al. (1999)

Is there a large amount of variability between subjects in the rate of change in MMSE score?

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Correlation between slopes and intercepts was 0.33 so subjects with higher intercepts have less decline in MMSE
MMSE is used to assess patients diagnosed with Alzheimer's and the authors were interested in whether it is a good measure. They decided not, largely because of the variability in slope between subjects

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The authors comment that the variability in slopes that they find could be due to ceiling effects of the post-test

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Do districts vary in their sensitivity of land value to climate (mean maximum July temperature over 30 years, JULTMX)?

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Examples of research questions

Polsky and Easterling (2001)

Do districts vary in their sensitivity of land value to climate (mean maximum July temperature over 30 years, JULTMX)?

Levels:	2	district	Random slope on:	JULTMX
	1	county	Answer:	Yes

The authors go on to fit a model that shows that counties in districts with more variability in temperature from year to year benefit more from high July temperatures

Jex and Bliese (1999)

Is there variability across army companies in the relationship between hours worked and psychological strain?

Levels:	2	company	Random slope on:	hours worked
	1	soldier	Answer:	Yes

See also the Gallery of Multilevel Papers on the CMM website!

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Levels:	2	district	Random slope on:	JULTMX
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Levels:	2	company	Random slope on:	hours worked
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The authors go on to examine whether the variabilty could be explained by differing beliefs in the efficacy of the company across companies, but conclude that it cannot

See also the Gallery of Multilevel Papers on the CMM website!

Calculating the total variance

Level 1

- We only have one random term at level 1, e_{0ij}

Level 2

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The variance partitioning coefficient now also depends on x_{1ij}

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The variance partitioning coefficient now also depends on x_{1ij}

$$\mathrm{VPC} = \frac{\text{level 2 variance}}{\text{total residual variance}} = \frac{\sigma^2_{u0} + 2\sigma_{u01}x_{1ij} + \sigma^2_{u1}x^2_{1ij}}{\sigma^2_{u0} + 2\sigma_{u01}x_{1ij} + \sigma^2_{u1}x^2_{1ij} + \sigma^2_{e0}}$$

Exam scores example

Question

How does the amount of variation in exam scores at age 16 due to school differences change as a function of exam score age 11?

Answer

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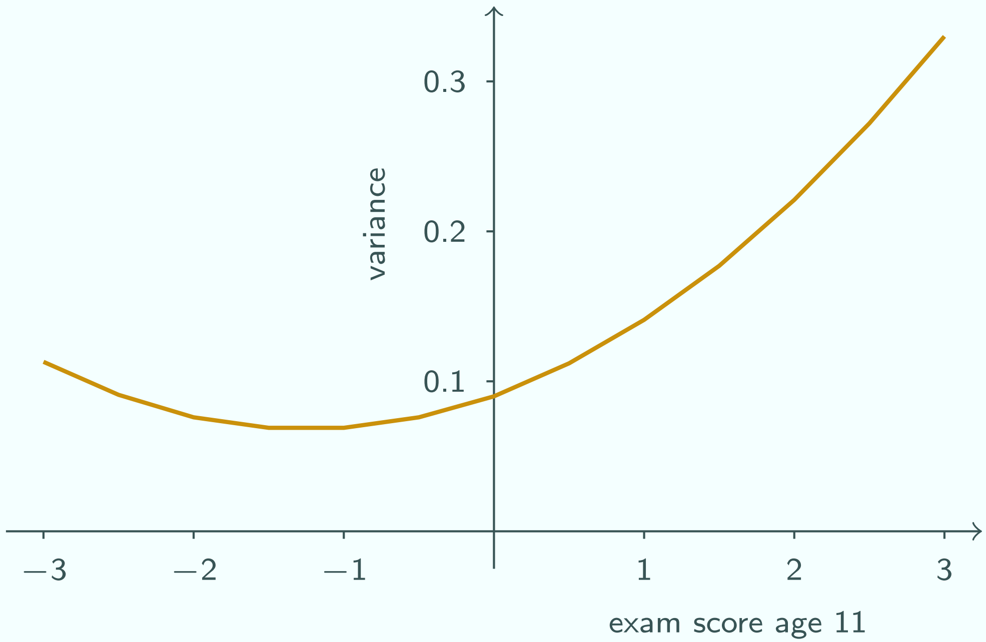
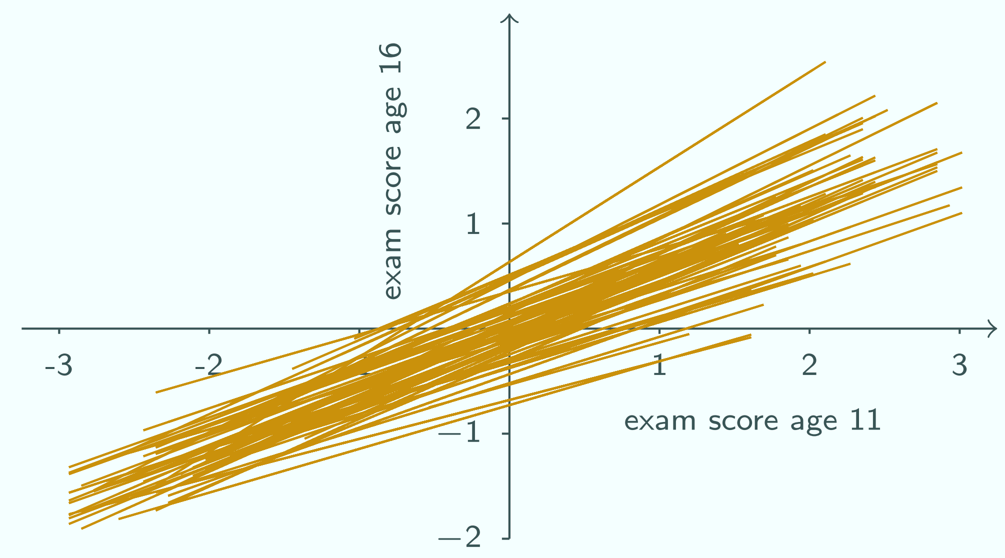
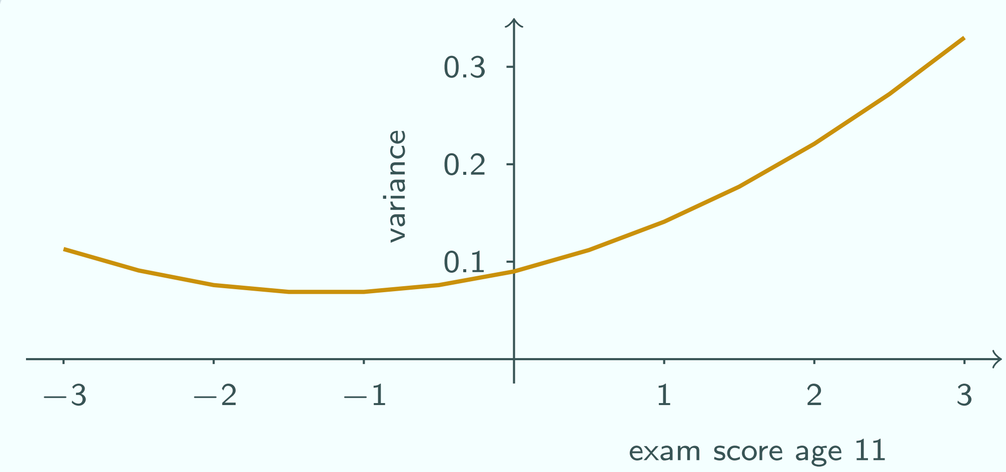
Answer

1. Fit a model with a random slope on exam score age 11
2. Calculate the level 2 variance
3. Plot:



Why does the variance depend on x?

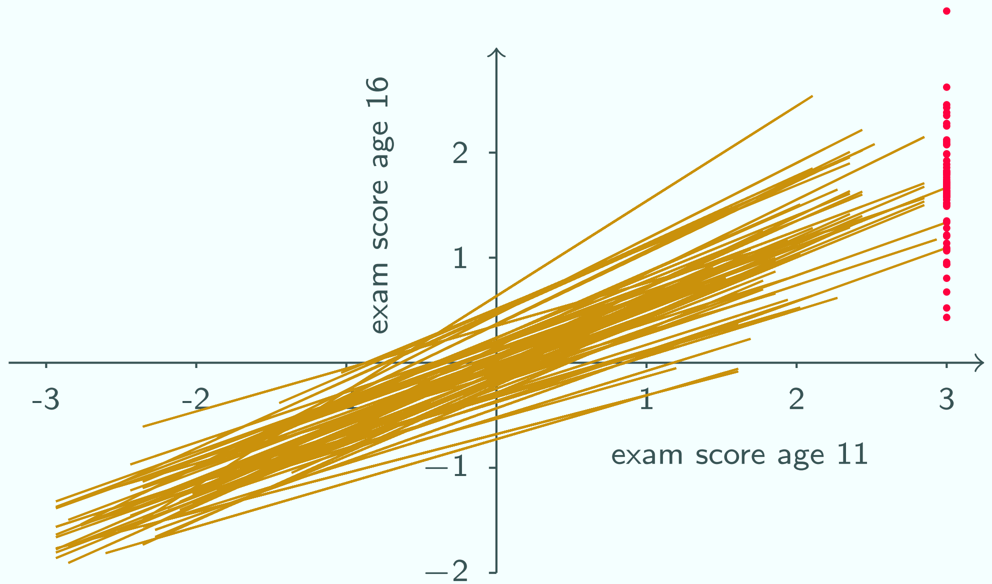


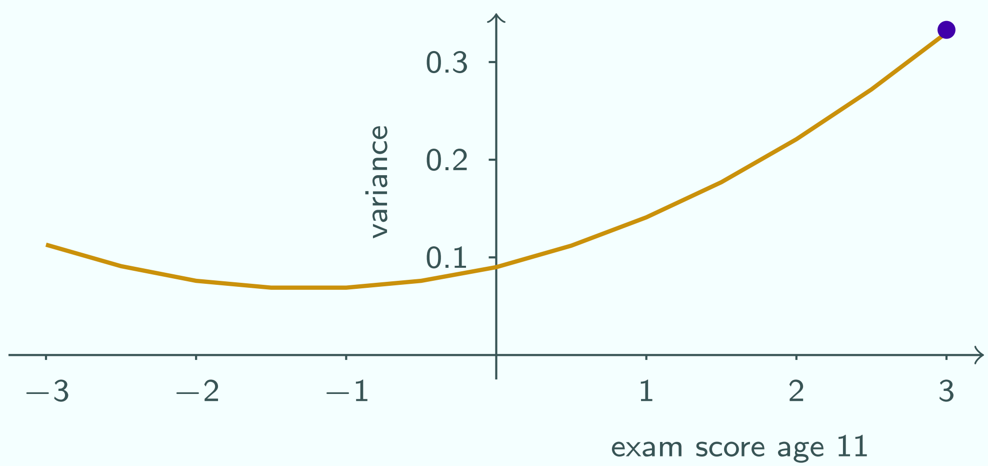


Why does the variance depend on x?

At $x = 3$

- The school lines are spread out
- There are greater differences between schools
- The school level variance is higher

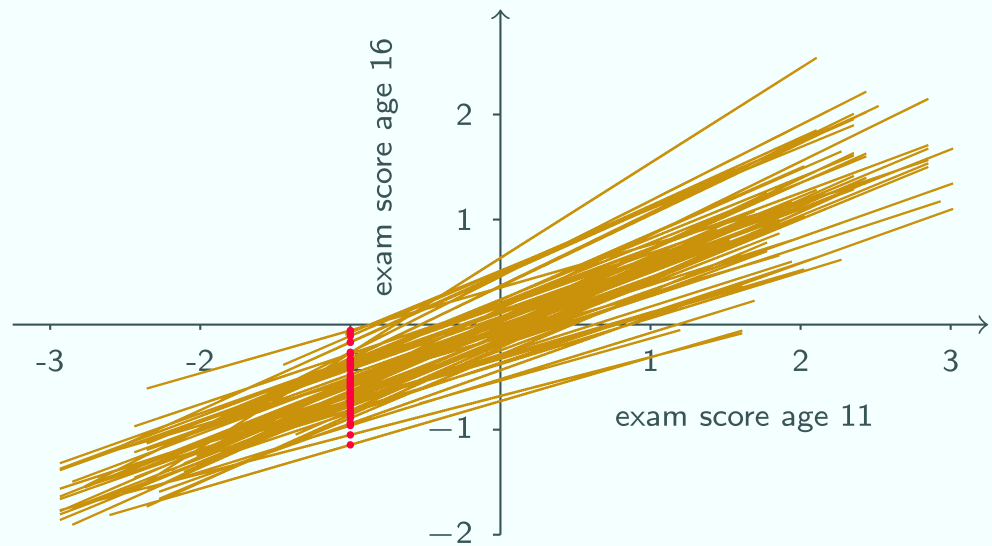


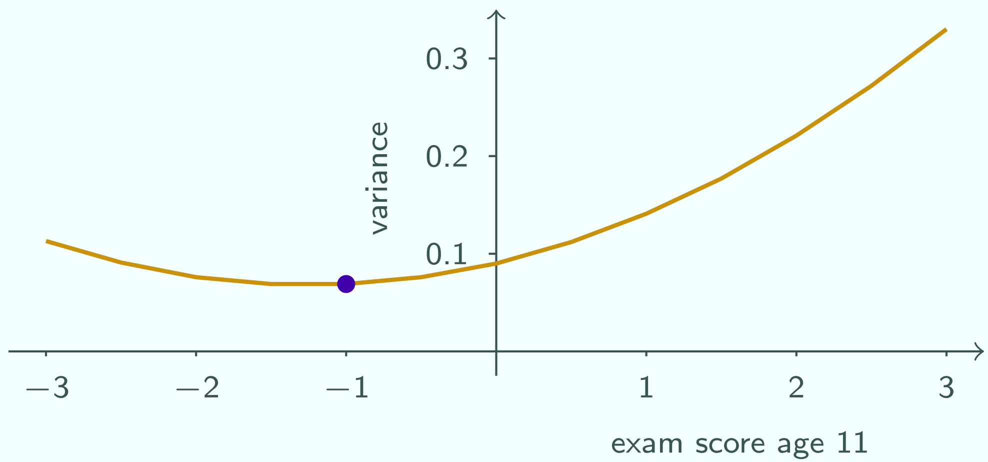


Why does the variance depend on x?

At $x = -1$

- The school lines are closer together
- There are smaller differences between schools
- The school level variance is lower





Assumptions of random part

Random slope model

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + u_{0j} + u_{1j} x_{1ij} + e_{0ij}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim \mathrm{N}(0, \Omega_u), \quad \Omega_u = \begin{bmatrix} \sigma^2_{u0} & \\ \sigma_{u01} & \sigma^2_{u1} \end{bmatrix}$$

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We have all the same assumptions as for the random intercept model, plus:

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V, the correlation matrix

Random slope model

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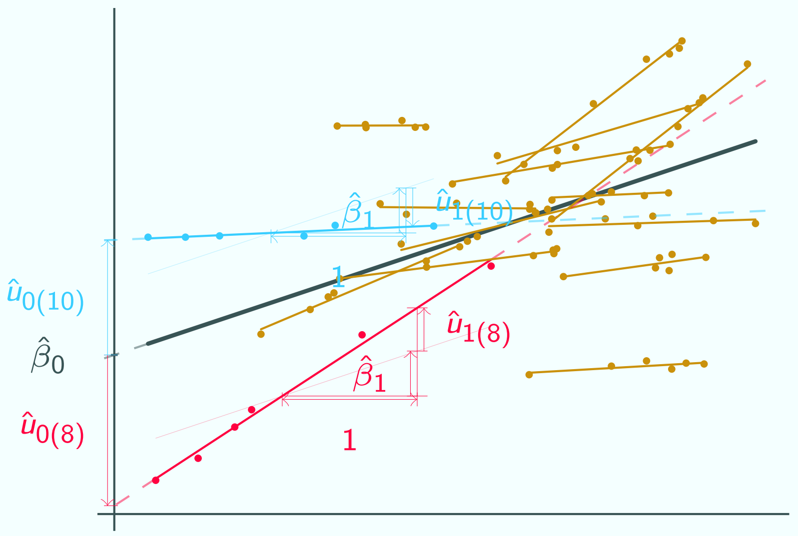
- The important thing is to recognise that it depends on the two values of x_1, as well as σ^2_{u1}, σ^2_{u0} and σ_{u01}

Residuals

With a random slope model we have several sets of level 2 residuals:

- a set of intercept residuals
- and a set of residuals for each set of random slopes

Each set of residuals is shrunk (using very complicated formulae!)



Prediction: visualising the model

Overall regression line

- Prediction from the fixed part gives the overall regression line

Group lines

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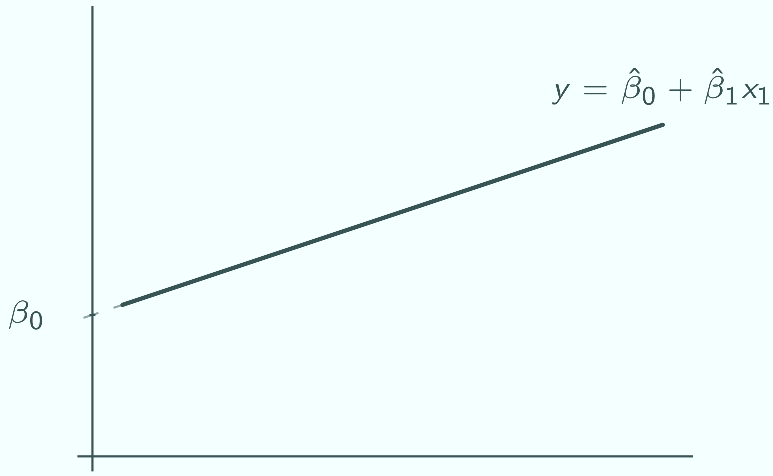

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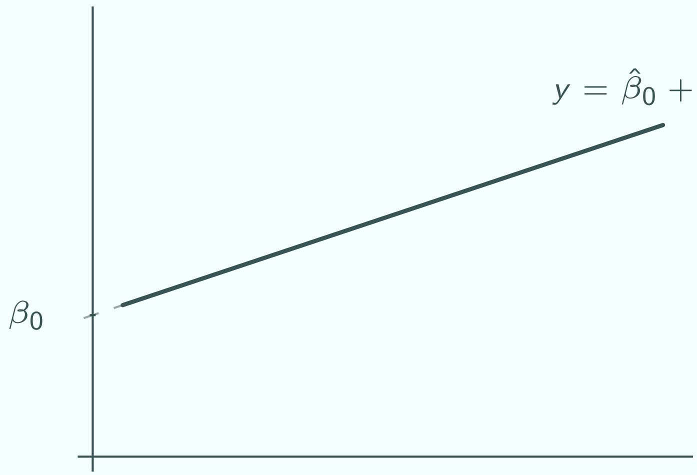

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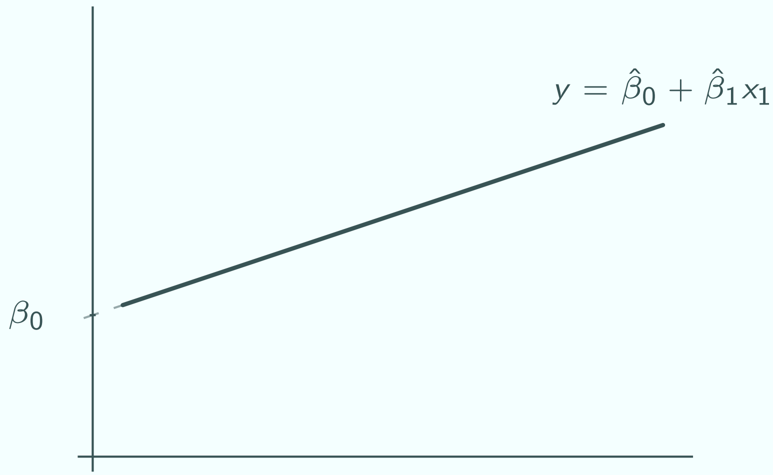

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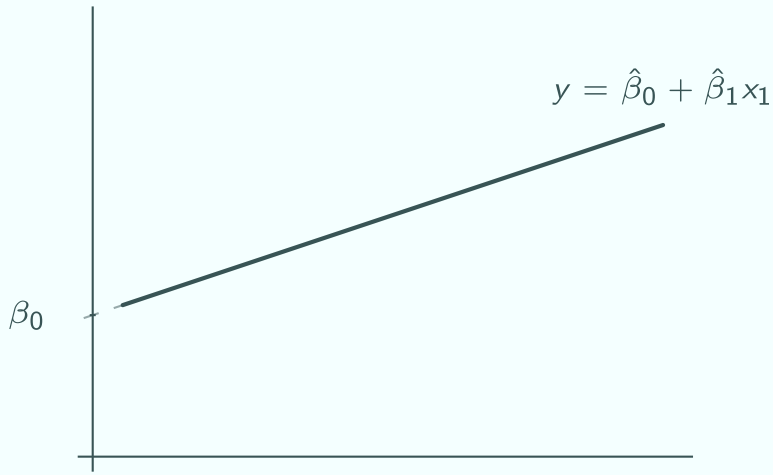

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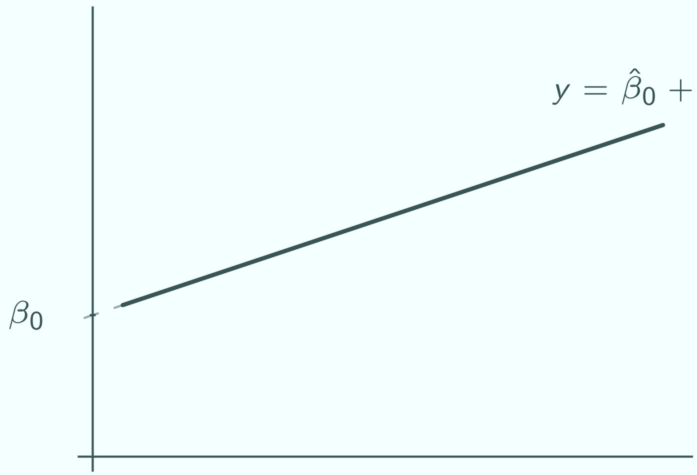

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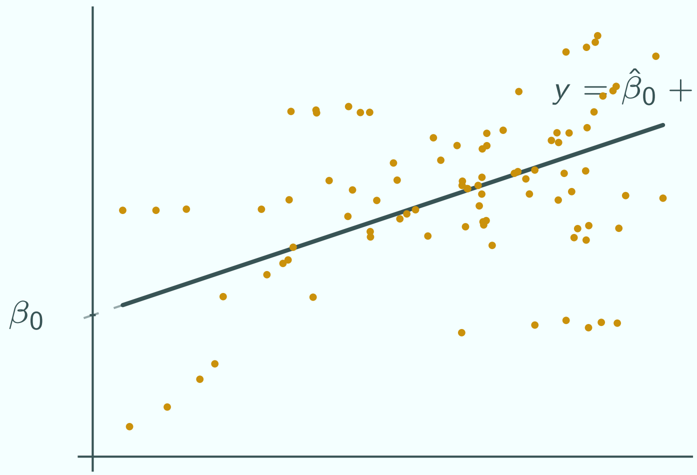

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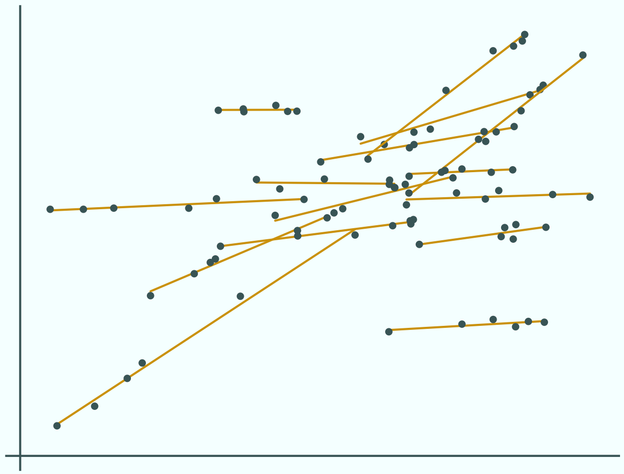

Prediction: visualising the model

Combined predictions

Putting the prediction from the fixed part and the prediction from the random part together on the same graph, we get:

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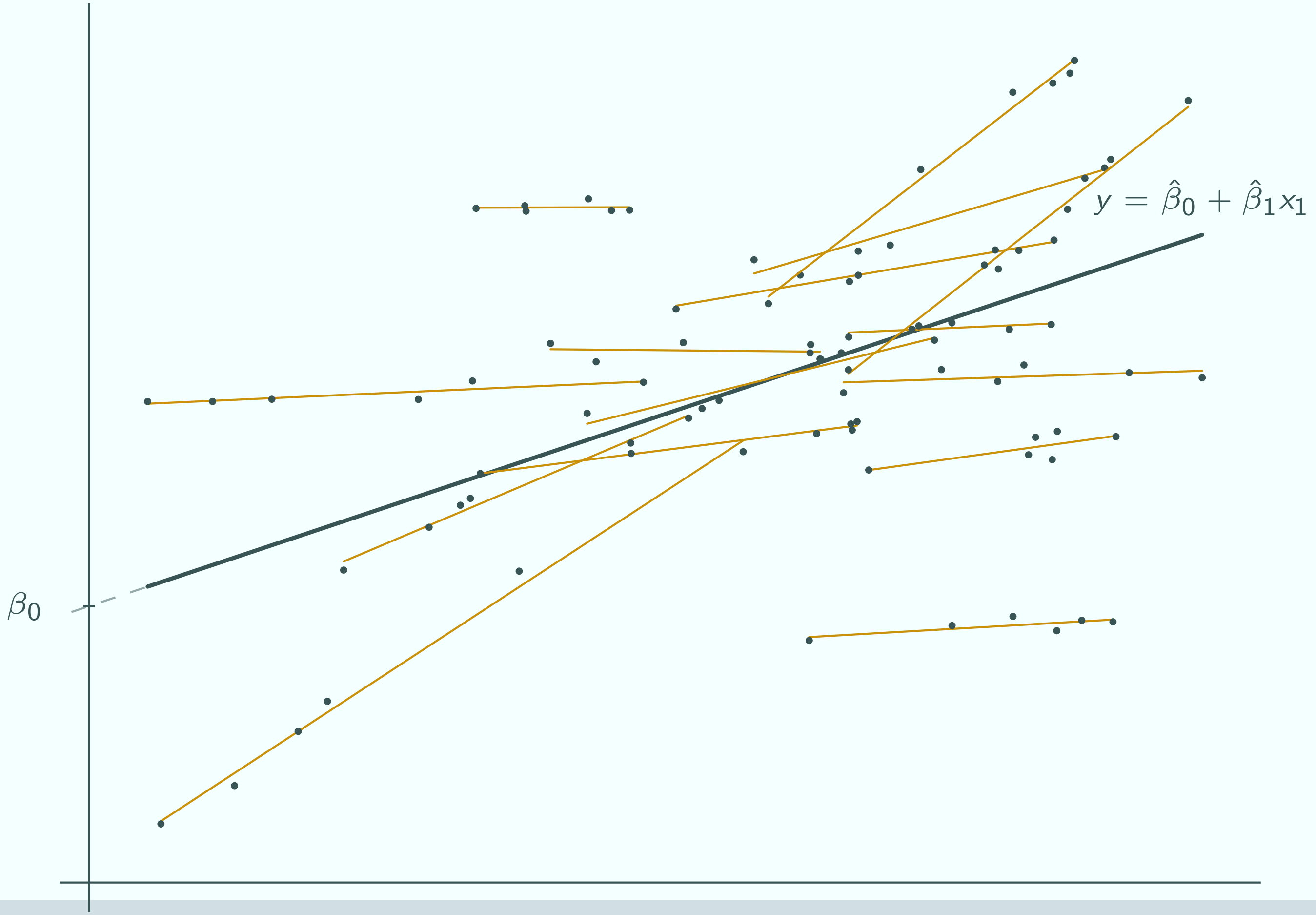

Random slope models and random intercepts

Terminology

- The random slope model

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + u_{0j} + u_{1j} x_{1ij} + e_{0ij}$$

has a random intercept as well as a random slope

Do we always add a random intercept?

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- However, usually when we use the term ‘random intercept model’ we mean a model that has only a random intercept, and no random slope

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- Random slopes can be fitted to interaction terms as well

Exercises

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