Random Intercept Models

Hedonism example

Our questions in the last session

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Our questions in the last session

- Are there differences between countries in hedonism?

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- Are there differences between countries in hedonism?
- Are some countries more hedonistic than others?

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- Do differences between countries in hedonism remain after controlling for individual age?

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- Do differences between countries in hedonism remain after controlling for individual age?
- Are some countries more hedonistic than others after controlling for individual age?
- How much of the variation in hedonism is due to country differences after controlling for individual age?
- What is the relationship between an individual's hedonism and their age?

Exam scores example

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- Are there differences between schools in exam scores at age 16?

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- Do differences between schools in exam scores at age 16 remain after controlling for exam score at age 11?

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- Are there differences between schools in pupils' progress between age 11 and 16?

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- So we would like to control for previous exam score

Using a single-level regression model

- We usually do this by fitting a regression model:

$$y_i = \beta_0 + \beta_1 x_i + e_i \quad e_i \sim \mathrm{N}(0, \sigma_e^2)$$

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Data where observations in the same group are related, e.g.

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Random intercept model

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j + e_{ij} \quad e_{ij} \sim \mathrm{N}(0, \sigma_e^2)$$
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Fixed and random part

The random intercept model has two parts:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j + e_{ij}$$

Fixed and random part

The random intercept model has two parts:

- a "fixed part"

$$y_{ij} = \overbrace{\beta_0 + \beta_1 x_{ij}}^{\text{fixed part}} + u_j + e_{ij}$$

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$$\beta_0, \beta_1, \ldots$$

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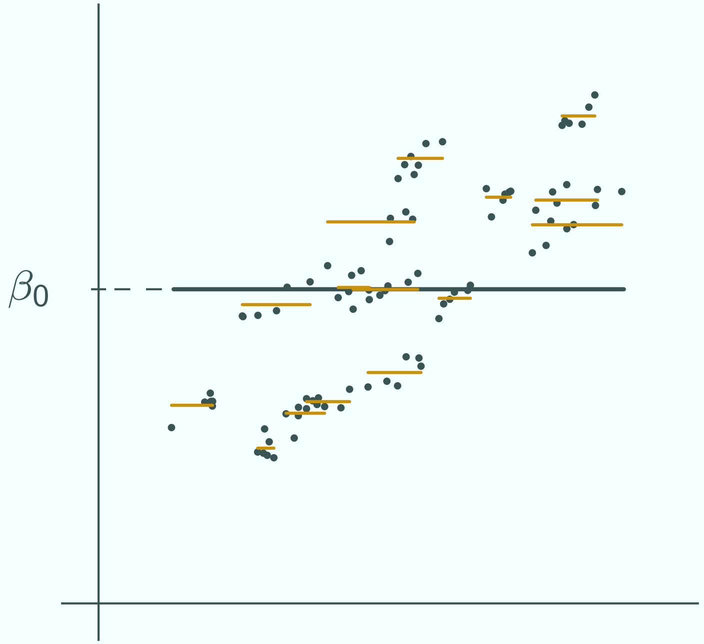
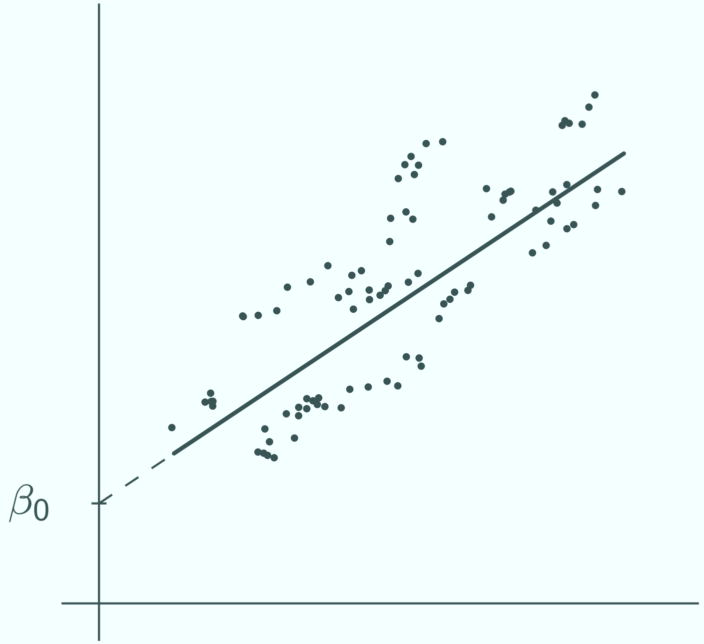
- The "random part" is random in the same way that the error term e_i of the single level regression model is random:
  - the u_j and e_{ij} are allowed to vary
  - some unmeasured processes are generating the u_j and e_{ij}

What does the model look like?

Variance components model



Single level regression model



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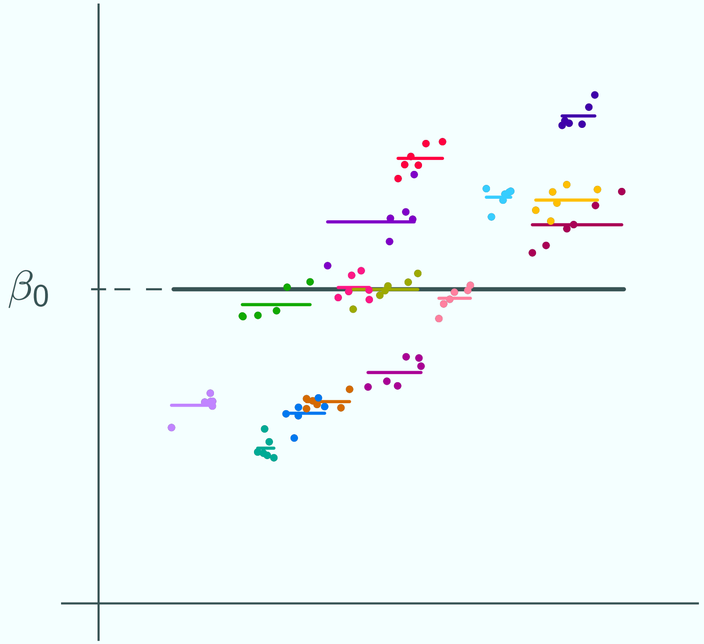

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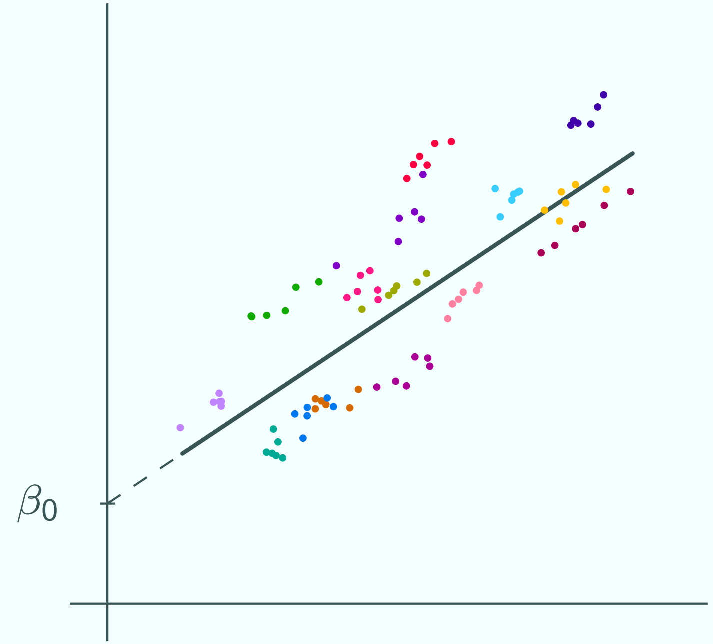

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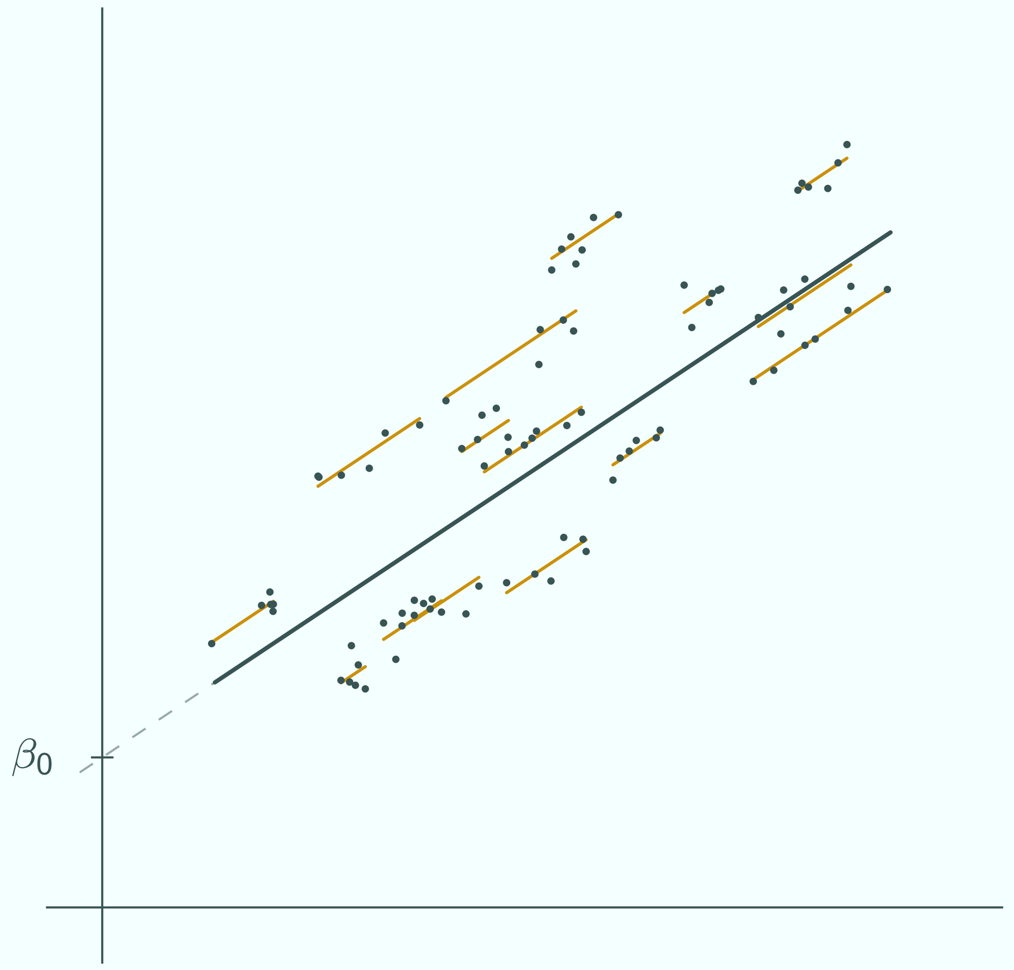

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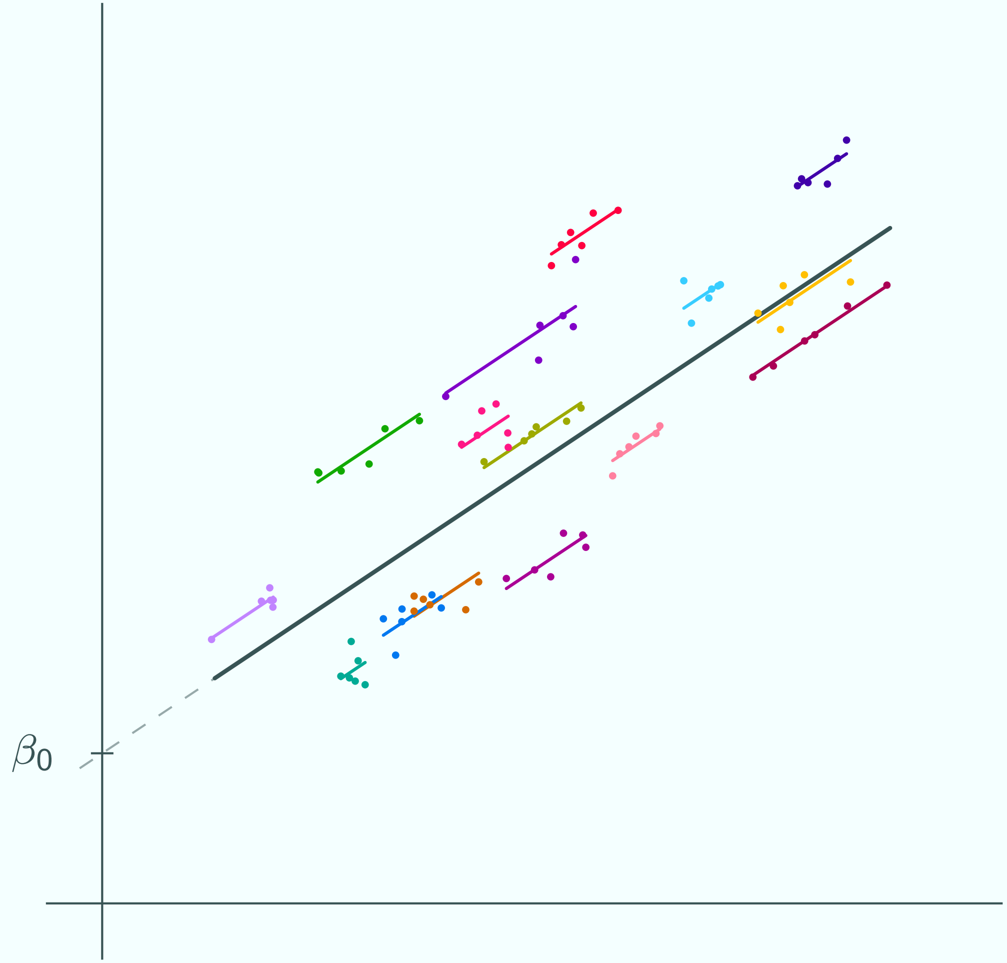

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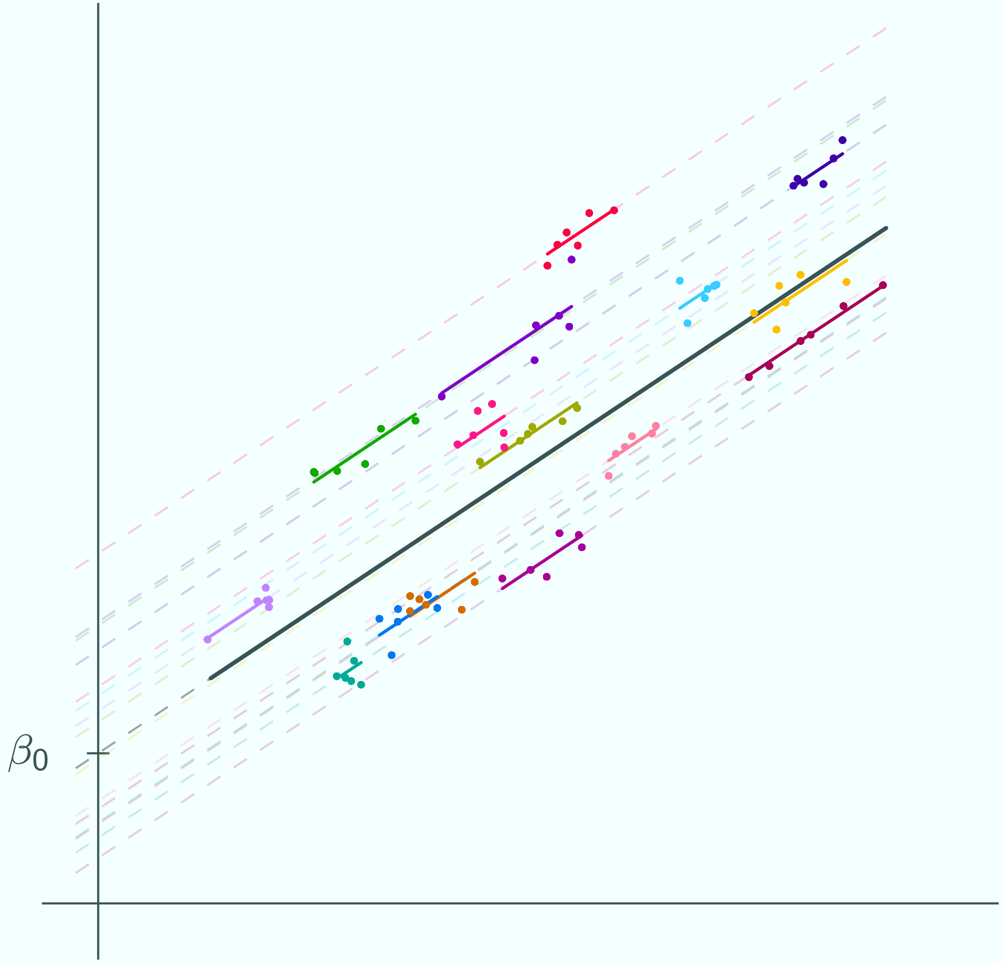

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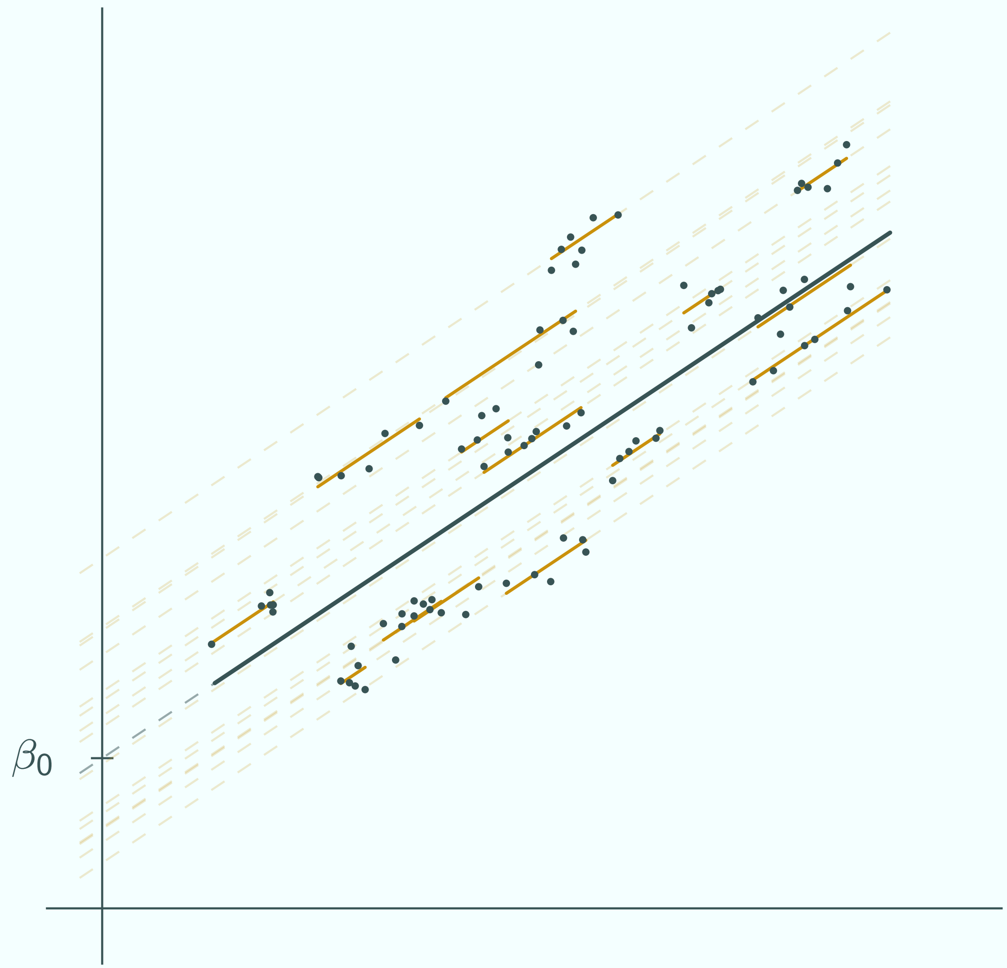

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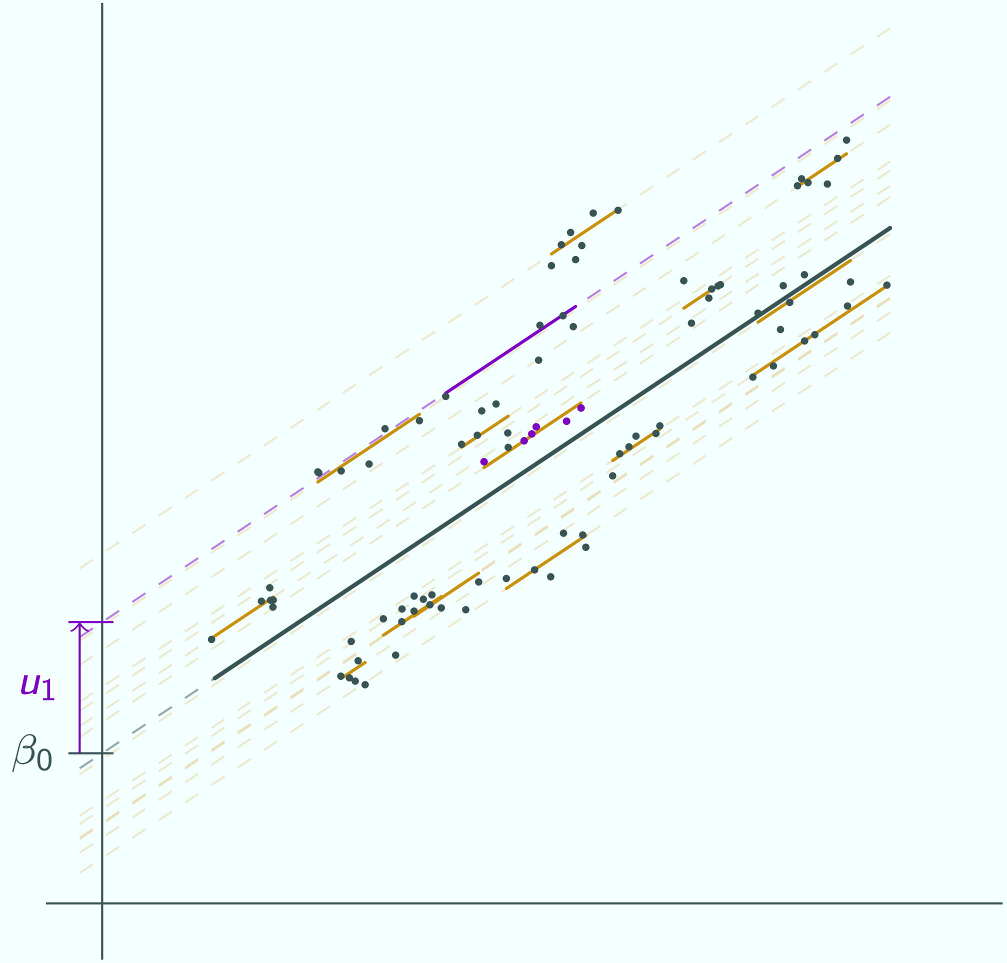

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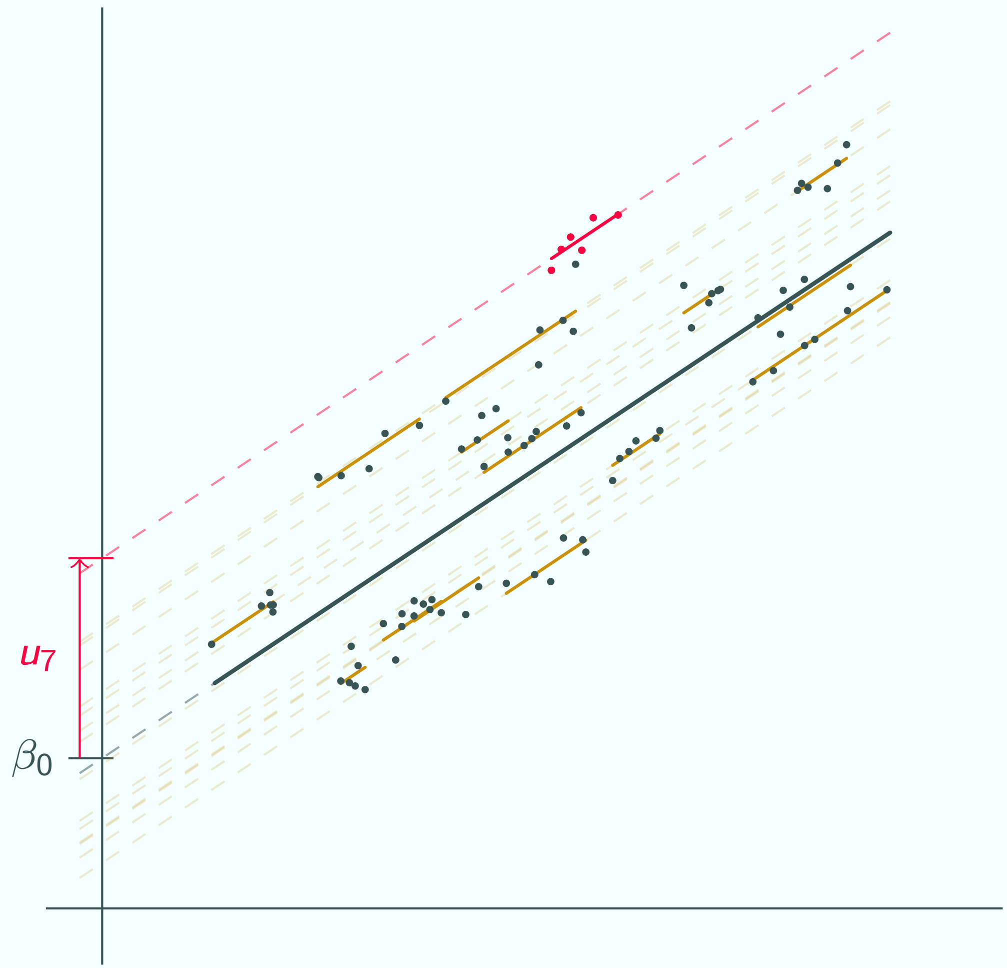

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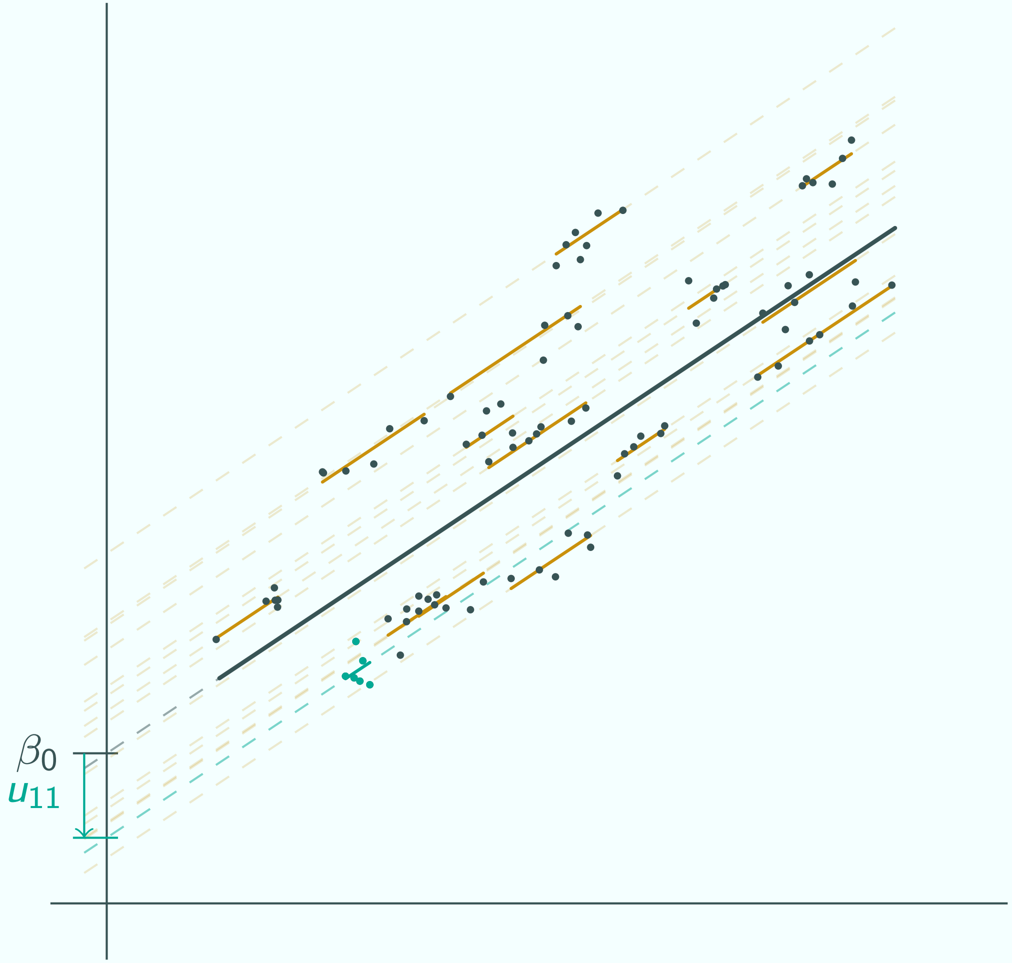

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Overall line

Like the single level regression model, the overall average line has equation $\beta_0 + \beta_1 x_{ij}$

Group lines

The 'random intercept'

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Like the variance components model, each group has its own line, parallel to the overall average line

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- For the single level regression model, the intercept is just β_0

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- This is a parameter from the fixed part of the model

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- For each group line the intercept is $\beta_0 + u_j$

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- This involves a parameter from the random part and so it is allowed to vary $\rightarrow$ 'random intercept'

Interpreting the parameters

Fixed part

Interpretation is as for a single level regression model

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- σ_u^2 is the unexplained variation at level 2

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Random part

Interpretation is as for a variance components model

Note that again the parameters we estimate are σ_u^2 and σ_e^2, not u_j and e_{ij}

- σ_u^2 is the unexplained variation at level 2
  - e.g. the variation in hedonism due to differences between countries after controlling for age

Interpreting the parameters

Fixed part

Interpretation is as for a single level regression model

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- σ_e^2 is the unexplained variation at level 1
  - e.g. the variation in hedonism due to differences between individuals after controlling for age

Hypothesis testing

- Hypothesis testing is an important part of interpretation

Fixed part

Random part

Hypothesis testing

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- We don't only want to know the size of the fixed effects and the amount of variance at each level

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- We don't only want to know the size of the fixed effects and the amount of variance at each level
- We also want to know whether the fixed effects are significant

Fixed part

Random part

Hypothesis testing

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- We also want to know whether the fixed effects are significant
- and whether there is a significant amount of variance at level 2

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- Divide the coefficient by its standard error to get

$$z = \frac{\beta_1}{\text{s.e.}(\beta_1)}$$

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- We CAN'T just divide σ_u^2 by s.e.(σ_u^2) and compare the modulus with 1.96

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Random part

- We CAN'T just divide σ_u^2 by s.e.(σ_u^2) and compare the modulus with 1.96
- Instead we have to fit the model with and without u_j and do a likelihood ratio test to see whether σ_u^2 is significant

Likelihood ratio test

- $$\begin{array}{rll} \text{We fit} & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + u_j + e_{ij} & (1) \\ \text{and} & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + e_{ij} & (0) \end{array}$$
  and note the likelihoods

Likelihood ratio test

- We fit $y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + u_j + e_{ij}$ ①
  and $y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + e_{ij}$ ⓪
  and note the likelihoods
- The test statistic is 2(log(likelihood(①)) − log(likelihood(⓪)))

Likelihood ratio test

- $$\begin{array}{rl} \text{We fit} & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + u_j + e_{ij} \quad ① \\ \text{and} & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + e_{ij} \quad ⓪ \end{array}$$
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  - MLwiN gives $-2 \times \log(\text{likelihood})$ in the **Equations** window
  - So we just take
    (MLwiN's value for ⓪) − (MLwiN's value for ①)

Likelihood ratio test

- $$\begin{aligned} \text{We fit} \quad & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + u_j + e_{ij} \quad & (1) \\ \text{and} \quad & y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + e_{ij} & (0) \end{aligned}$$
  and note the likelihoods
- The test statistic is 2(log(likelihood((1))) − log(likelihood((0))))
  - MLwiN gives $-2 \times$ log(likelihood) in the **Equations** window
  - So we just take
    (MLwiN's value for (0)) − (MLwiN's value for (1))
- The null hypothesis is that $\sigma_u^2 = 0$ and so we don't need u_j in the model

Likelihood ratio test

- We fit $y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + u_j + e_{ij}$ ①
  and $y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \ldots + e_{ij}$ ⓪
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- The test statistic is $2(\log(\text{likelihood}(①)) - \log(\text{likelihood}(⓪)))$
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- We compare the test statistic to the $\chi^2_{(1)}$ distribution

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  - Some people divide the corresponding p-value by 2 (since $\sigma_u^2 \geqslant 0$)

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- We compare the test statistic to the $\chi^2_{(1)}$ distribution
  - Some people divide the corresponding *p*-value by 2 (since $\sigma_u^2 \geqslant 0$)
- There is 1 degree of freedom because there is one more parameter, σ_u^2, in ① compared to ⓪

Exam scores example

Question

Do differences between schools in exam scores at age 16 remain after controlling for exam score at age 11?

Answer

Exam scores example

Question

Do differences between schools in exam scores at age 16 remain after controlling for exam score at age 11?

Answer

1. Fit model with random intercept and note the $-2 \times \log(\text{likelihood})$ value:

Exam scores example

1

Equations

$$\text{normexam}_{ij} = \beta_{0j} + 0.563(0.012)\text{standlrt}_{ij} + e_{ij}$$

$$\beta_{0j} = 0.002(0.040) + u_{0j}$$

$$u_{0j} \sim \mathrm{N}(0, \sigma^2_{u0}) \quad \sigma^2_{u0} = 0.092(0.018)$$

$$e_{ij} \sim \mathrm{N}(0, \sigma^2_e) \quad \sigma^2_e = 0.566(0.013)$$

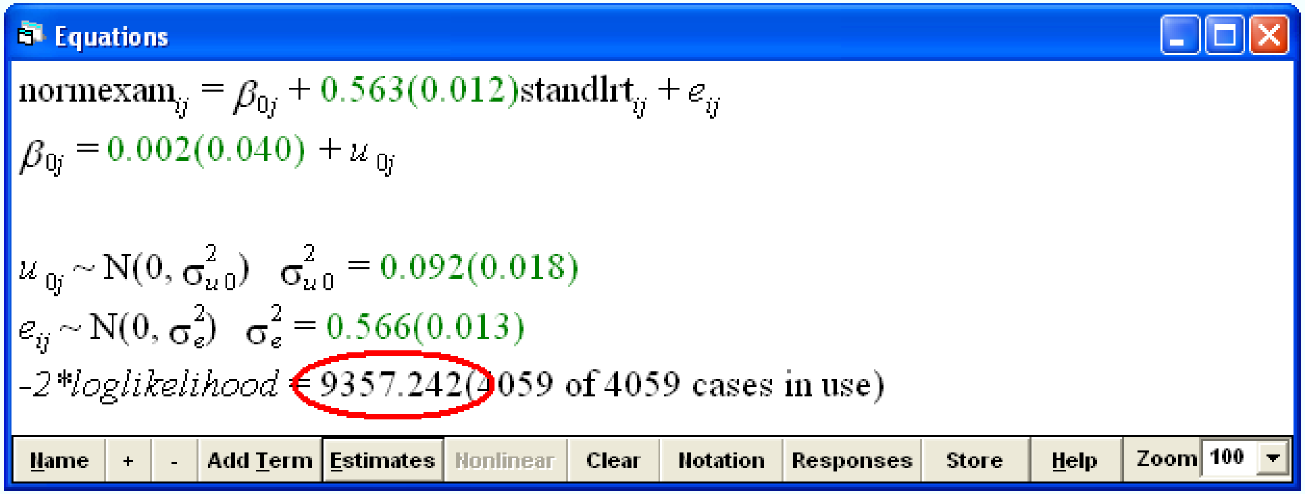
$-2*loglikelihood = 9357.242$(4059 of 4059 cases in use)

Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

2

Exam scores example

1



2

Exam scores example

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Do differences between schools in exam scores at age 16 remain after controlling for exam score at age 11?

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1. Fit model with random intercept and note the $-2 \times \log(\text{likelihood})$ value: 9357.242

Exam scores example

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2

Equations

$$\text{normexam}_{ij} = -0.001(0.013) + 0.595(0.013)\text{standlrt}_{ij} + e_{ij}$$

$$e_{ij} \sim \text{N}(0, \sigma^2_e) \quad \sigma^2_e = 0.648(0.014)$$

$-2*loglikelihood = 9760.509$(4059 of 4059 cases in use)

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Exam scores example

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Exam scores example

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1. Fit model with random intercept and note the $-2 \times \log(\text{likelihood})$ value: 9357.242
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3. Form the test statistic:

Exam scores example

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Do differences between schools in exam scores at age 16 remain after controlling for exam score at age 11?

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3. Form the test statistic: $9760.509 - 9357.242 = 403.267$

Exam scores example

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1. Fit model with random intercept and note the $-2 \times \log(\text{likelihood})$ value: 9357.242
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4. Compare against the χ^2 distribution with 1 degree of freedom

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3. Form the test statistic: $9760.509 - 9357.242 = 403.267$
4. Compare against the χ^2 distribution with 1 degree of freedom $p = 1.0709 \times 10^{-89}$
5. We conclude that there are differences between schools in exam scores at age 16 after controlling for exam score at age 11

What questions can we answer?

We can use the model to answer two kinds of question:

About variables

About levels

Often, we will look at both kinds of question, even if our main focus is on one or the other

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About levels

- e.g. "How much variation in hedonism is due to country differences (after controlling for age)?"
- This is a question about variances
- It is answered using the random part: the level 2 variance σ_u^2

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Examples of research

Muijs (2003)

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Question: Do pupils make more progress in maths when receiving support from numeracy support assistants?

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Levels		Variance	
2	school	15.82	51.4%
1	pupil	14.93	48.6%

Examples of research

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Levels		Variance	
2	school	15.82	51.4%
1	pupil	14.93	48.6%

Adding in a variable for receiving support from an assistant to a model including pupil background characteristics and prior achievement made almost no difference to the school or pupil level variance, but adding in teacher effectiveness reduced the school level variance by 1.91 (12%) (while pupil level variance remained roughly the same)

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	Levels	Variance
2	individual	Not reported
1	occasion	Not reported

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1	occasion	Not reported

The multilevel structure arises because we expect two measurements from the same person at different times to be similar (more similar than two measurements from different people)

Examples of research

Goldstein et al. (2007)

See also the Gallery of Multilevel papers on CMM's website
`http://www.bris.ac.uk/cmm/gallery`

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Levels		Variance	
2	junior school	0.05	14.9%
1	pupil	0.30	85.0%

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Levels		Variance	
2	junior school	0.05	14.9%
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The school level variance is found to be significant, so the authors conclude that which junior school a pupil attends does affect their progress in maths between KS1 and KS2

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Adding more explanatory variables

Extending our questions

- How much variation in pupils' progress is due to differences between schools after we control for pupil background characteristics such as SES, gender and ethnicity?

Adding more explanatory variables

Extending our questions

- How much variation in pupils' progress is due to differences between schools after we control for pupil background characteristics such as SES, gender and ethnicity?
- What are the relationships between hedonism and an individual's age, income, education and gender?

Adding more explanatory variables

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- We can easily add in more explanatory variables, as for a single level regression model

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + \beta_4 x_{2ij} x_{3ij} + \ldots + u_j + e_{ij}$$

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- Notice that we can also include interactions in the usual way

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- Variables can be continuous or categorical

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- Notice that we can also include interactions in the usual way
- Variables can be continuous or categorical
- Variables can be defined at a higher level (e.g. school mean intake score) (see Contextual effects session)

Adding more explanatory variables

Level 2 variance can increase

- When we add in a (level 1) variable, the variation at level 2 may decrease or increase (or stay the same)

Adding more explanatory variables

Level 2 variance can increase

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- The variance components model has less variation at level 2 than the random intercept model

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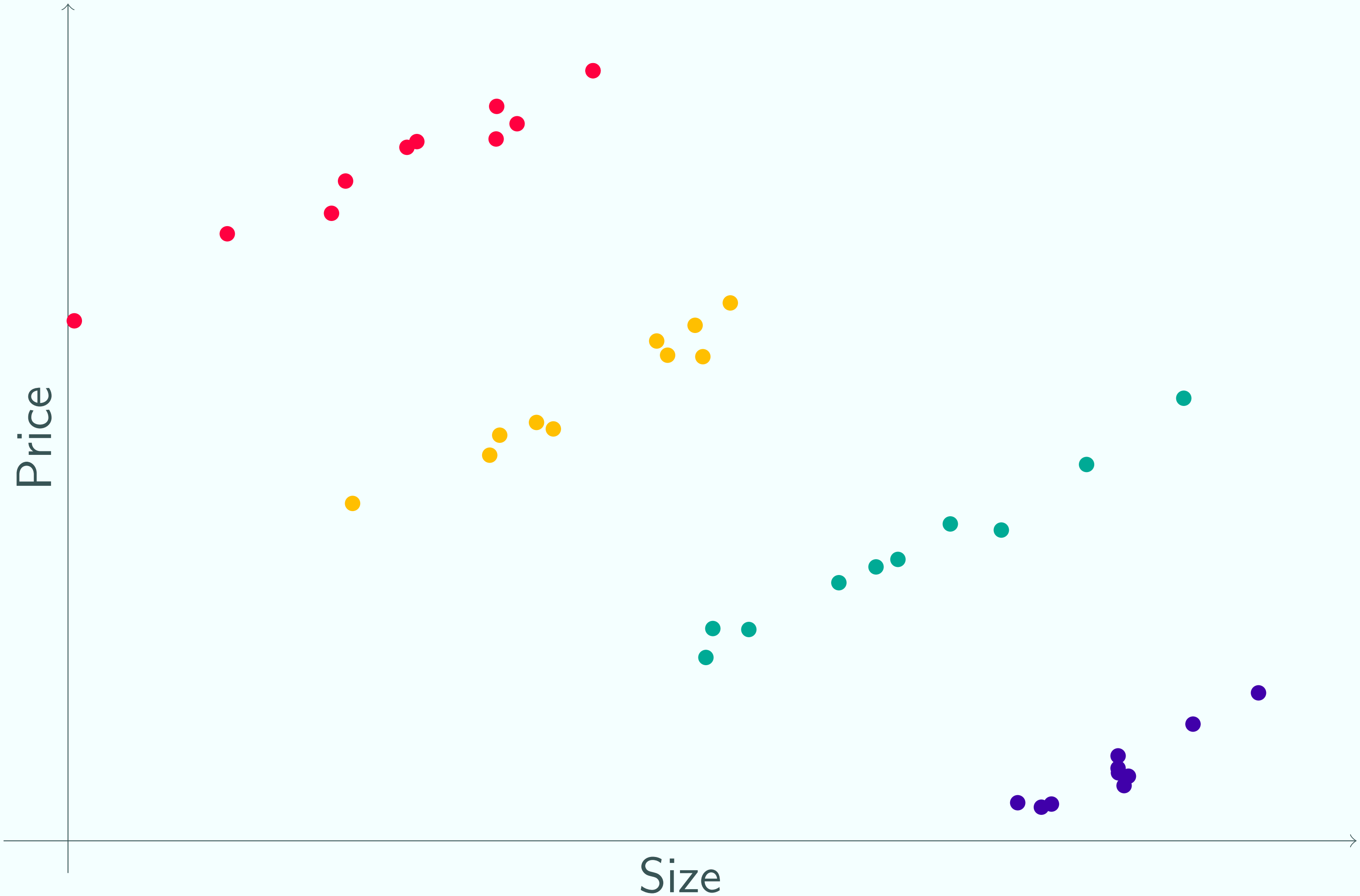
House prices in 4 different areas of a city



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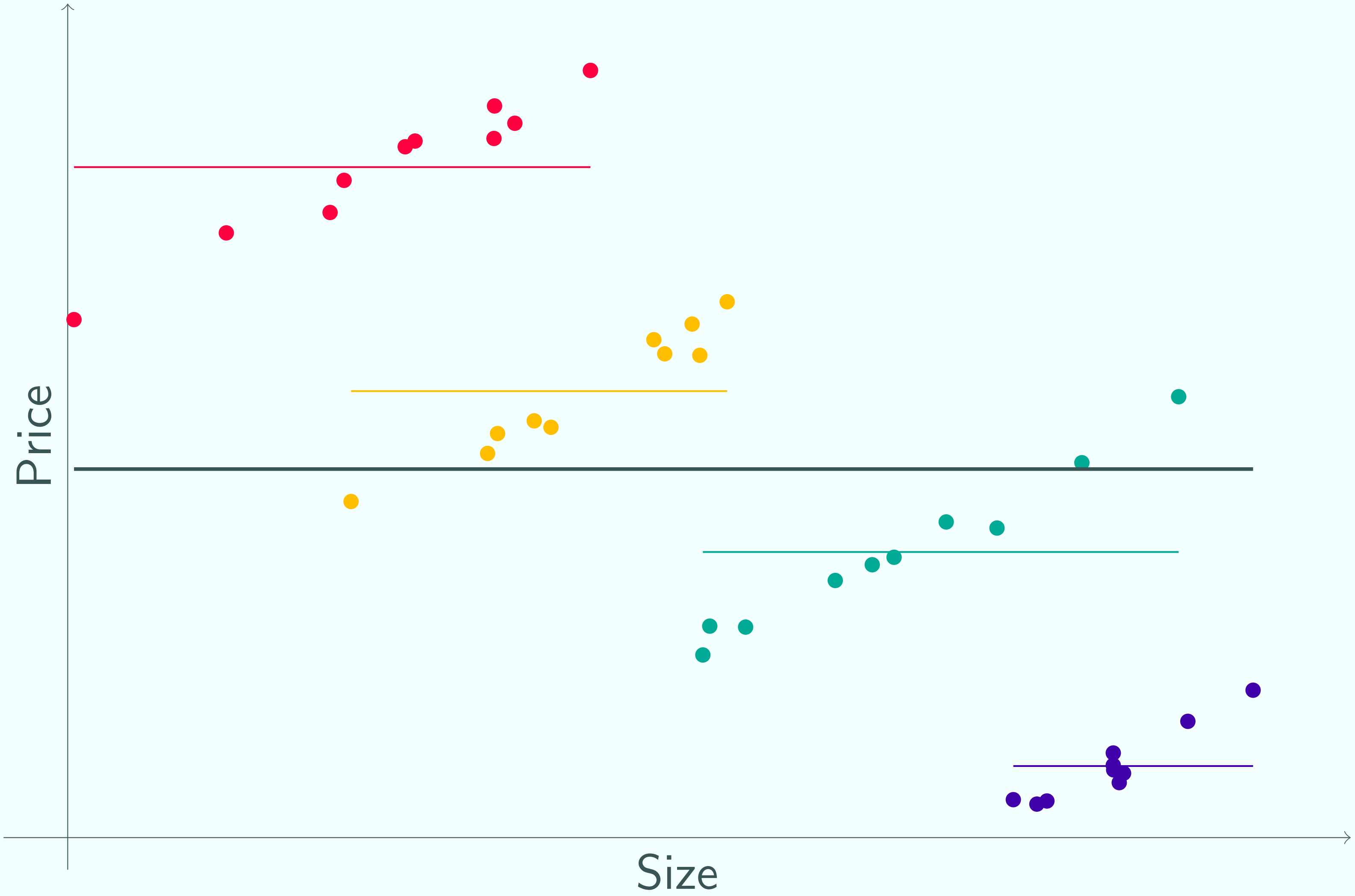

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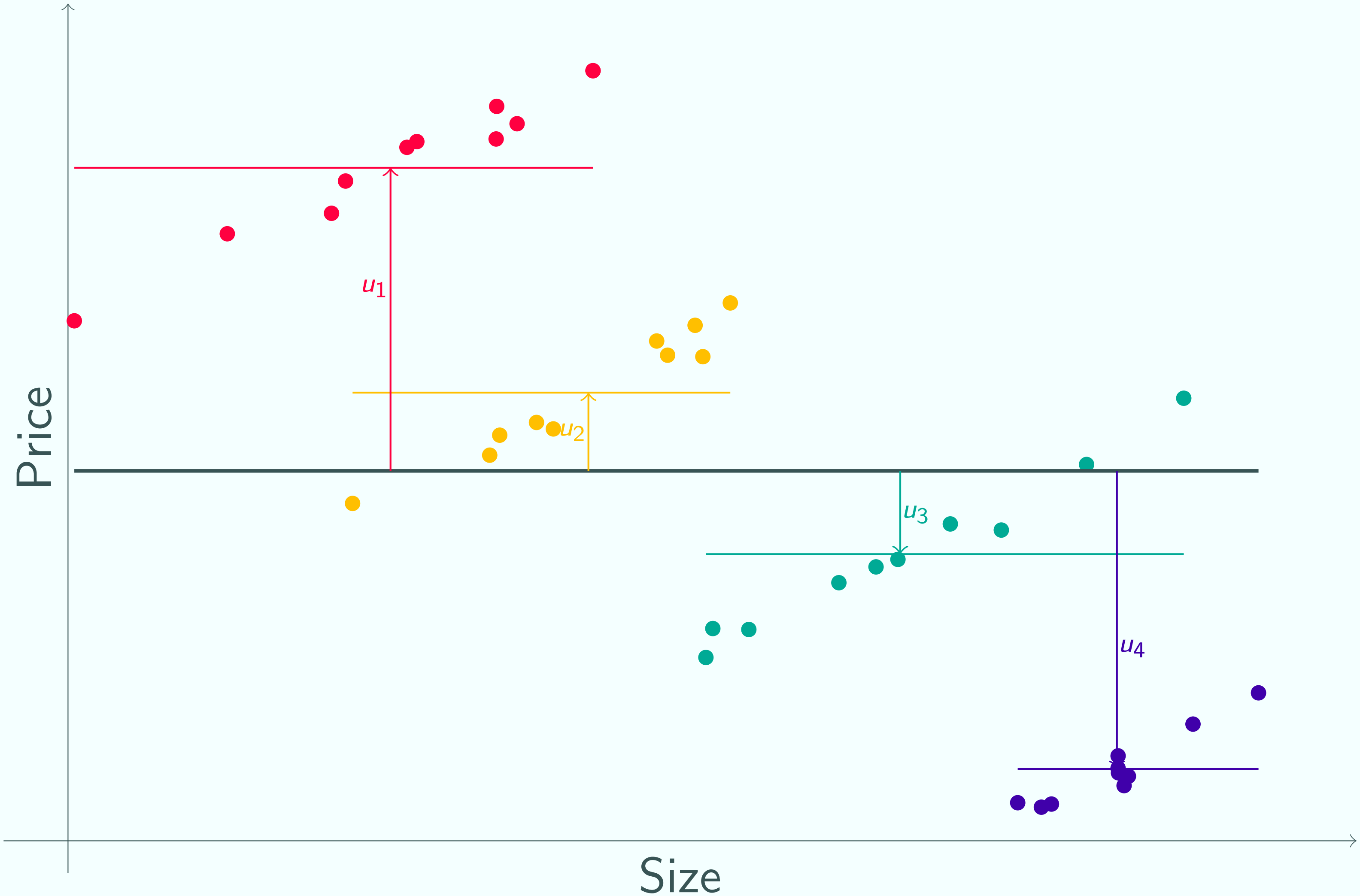

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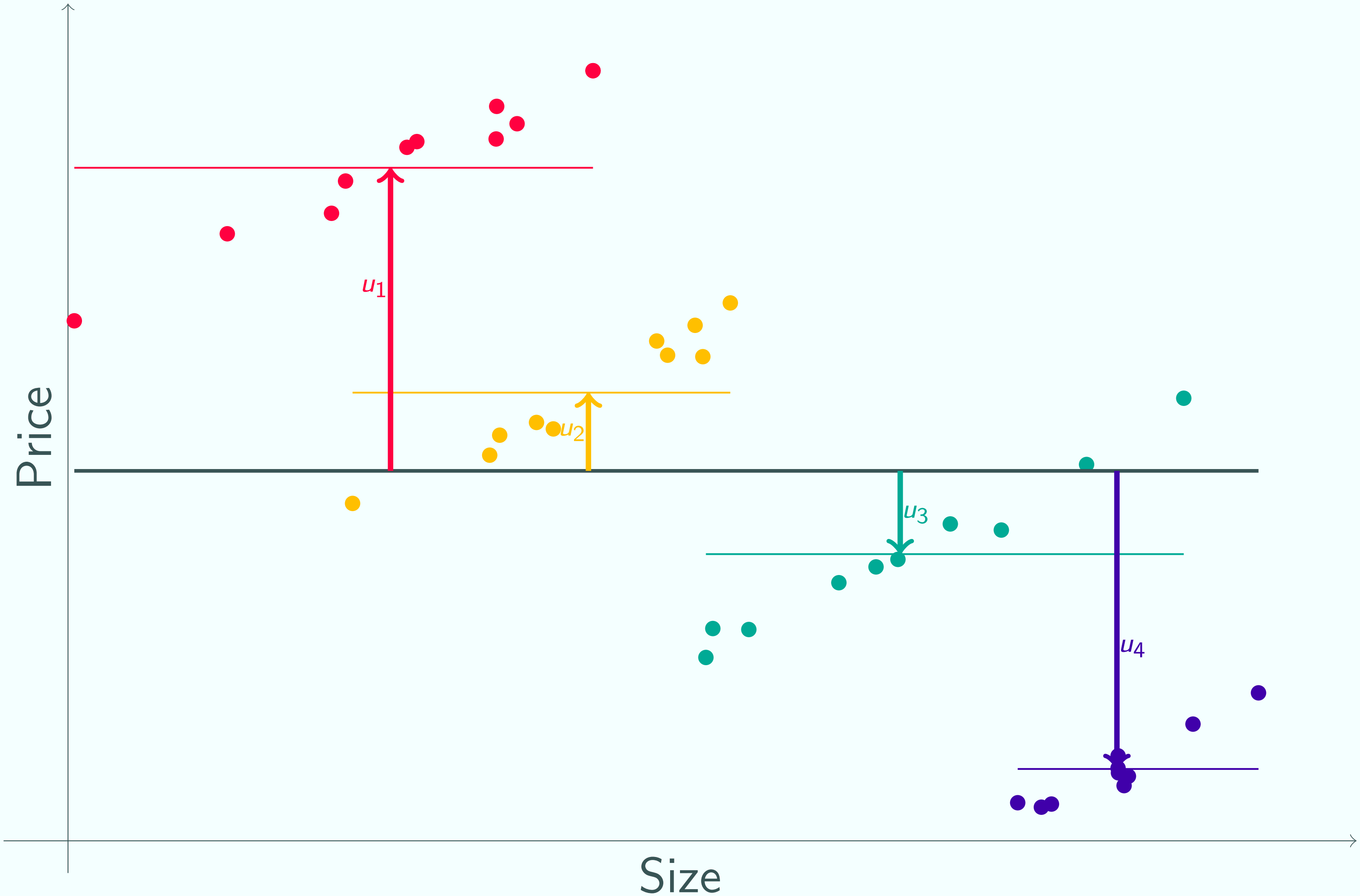

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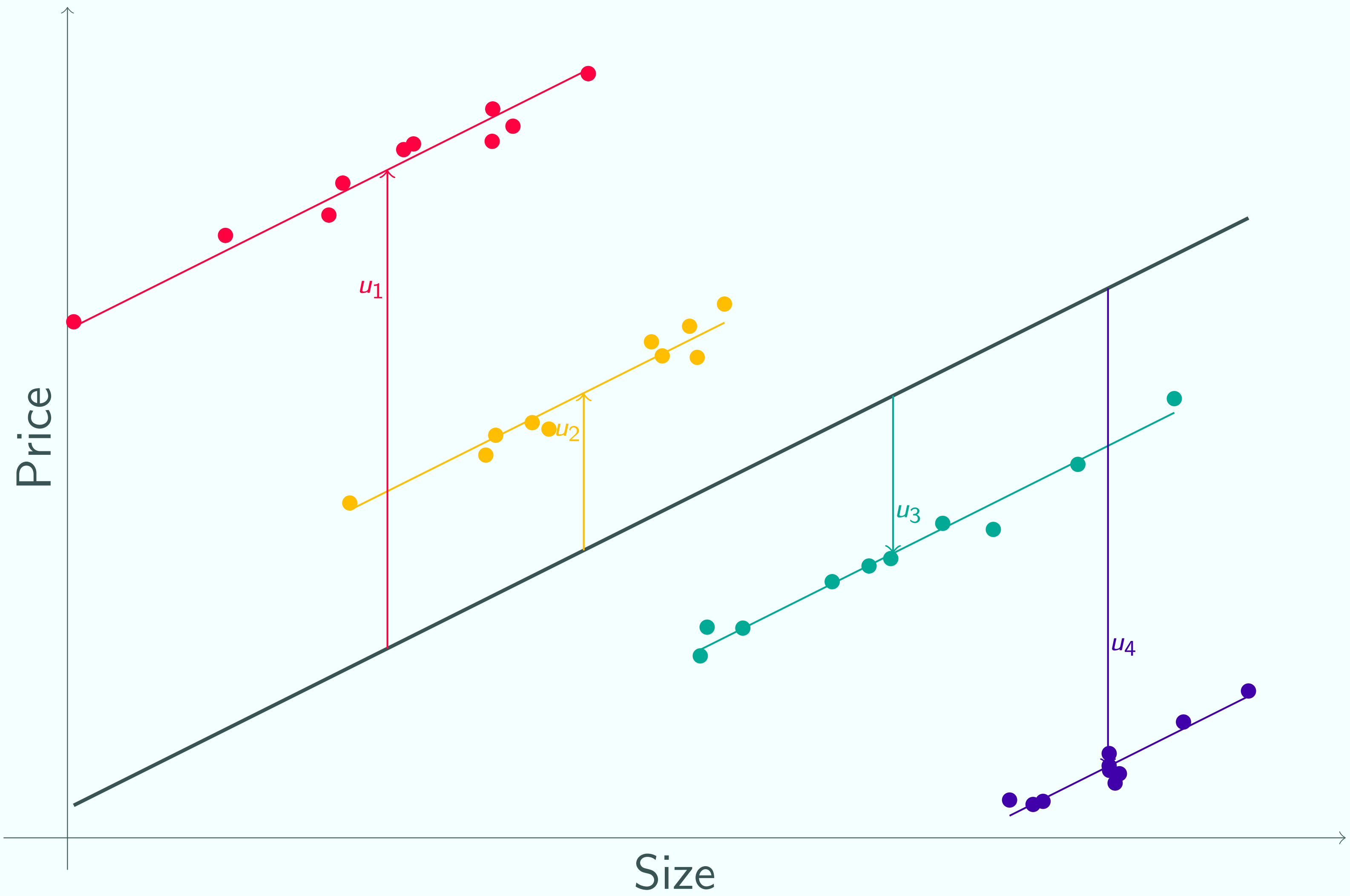

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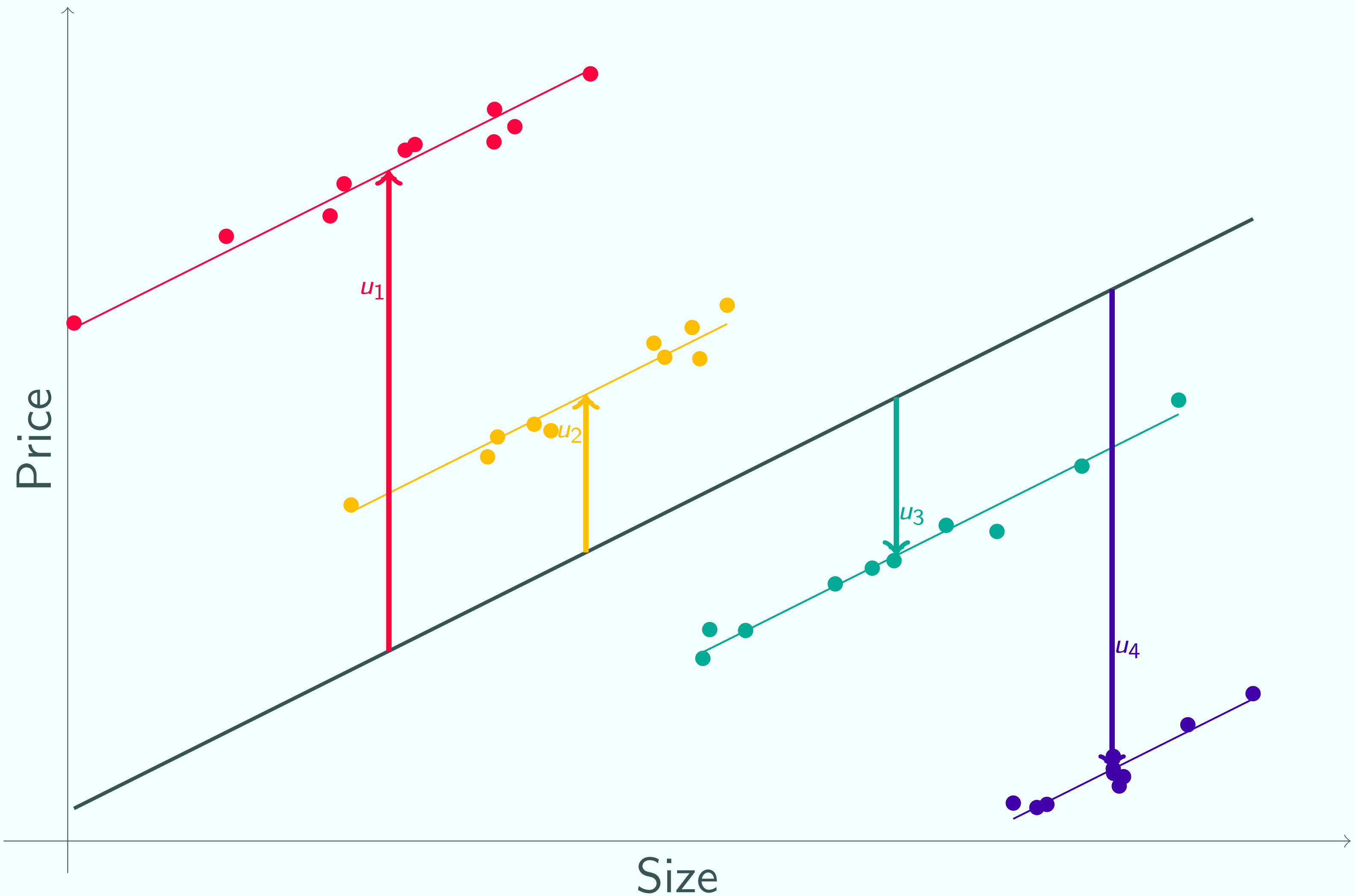

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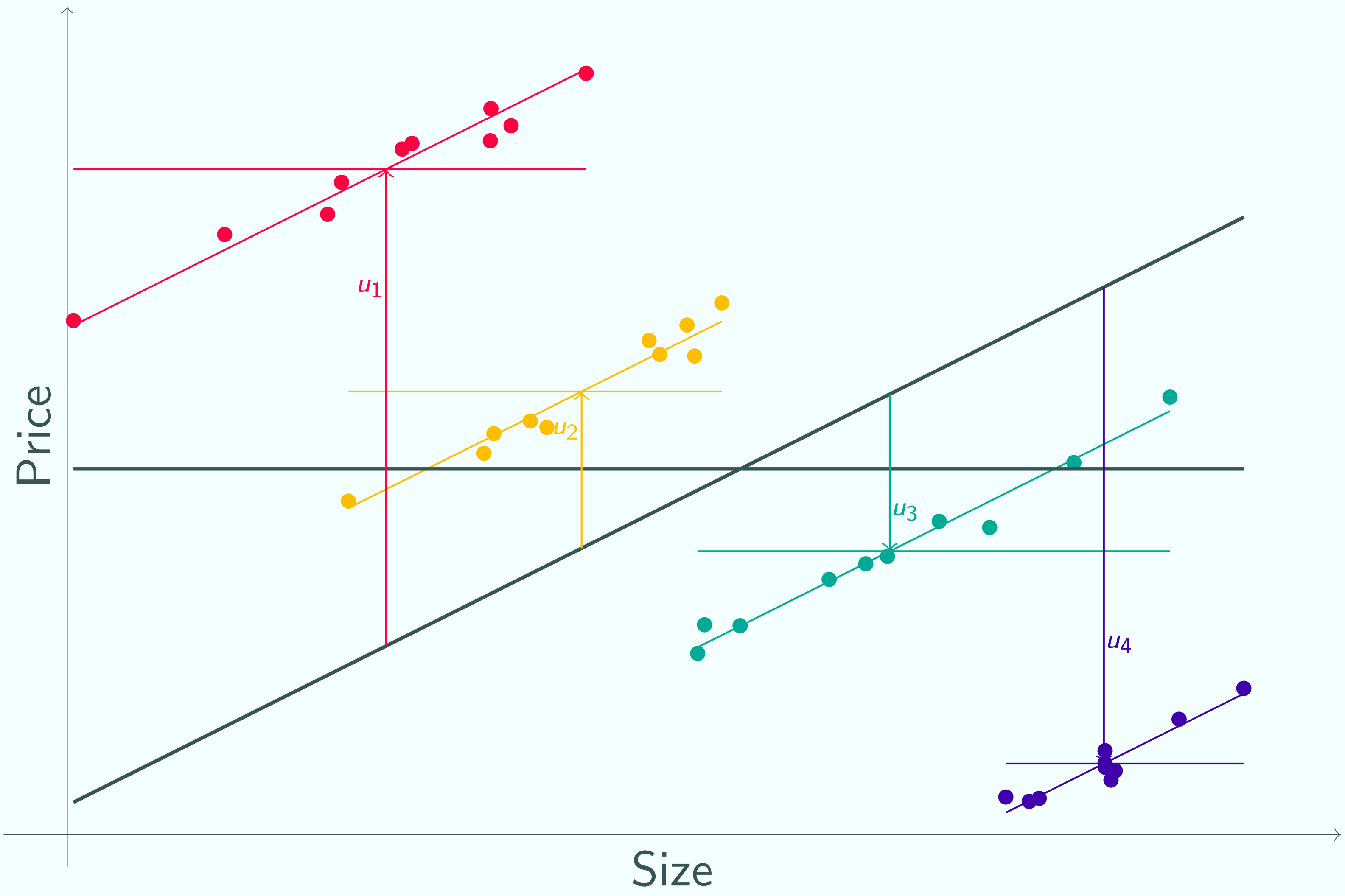

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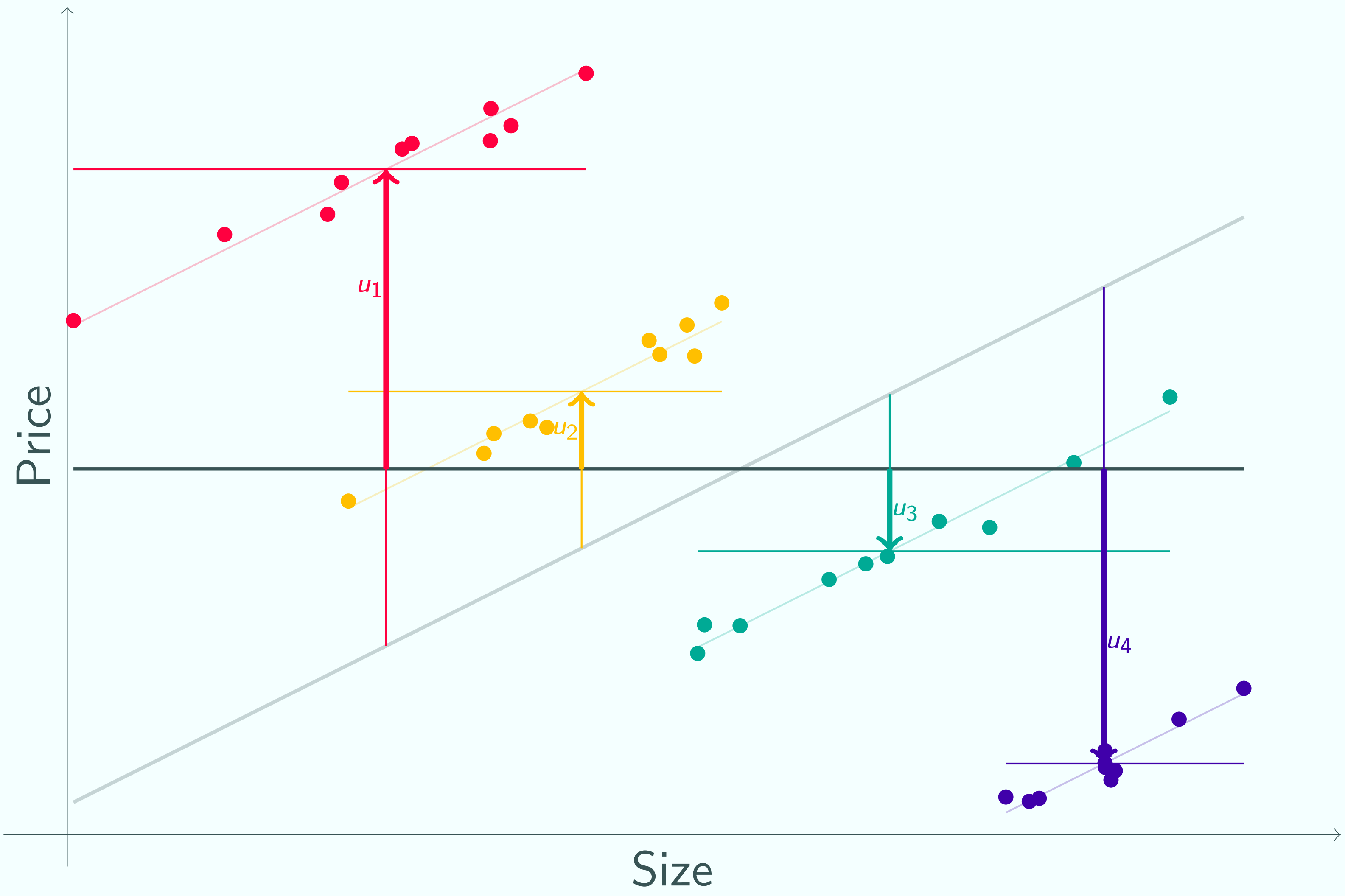

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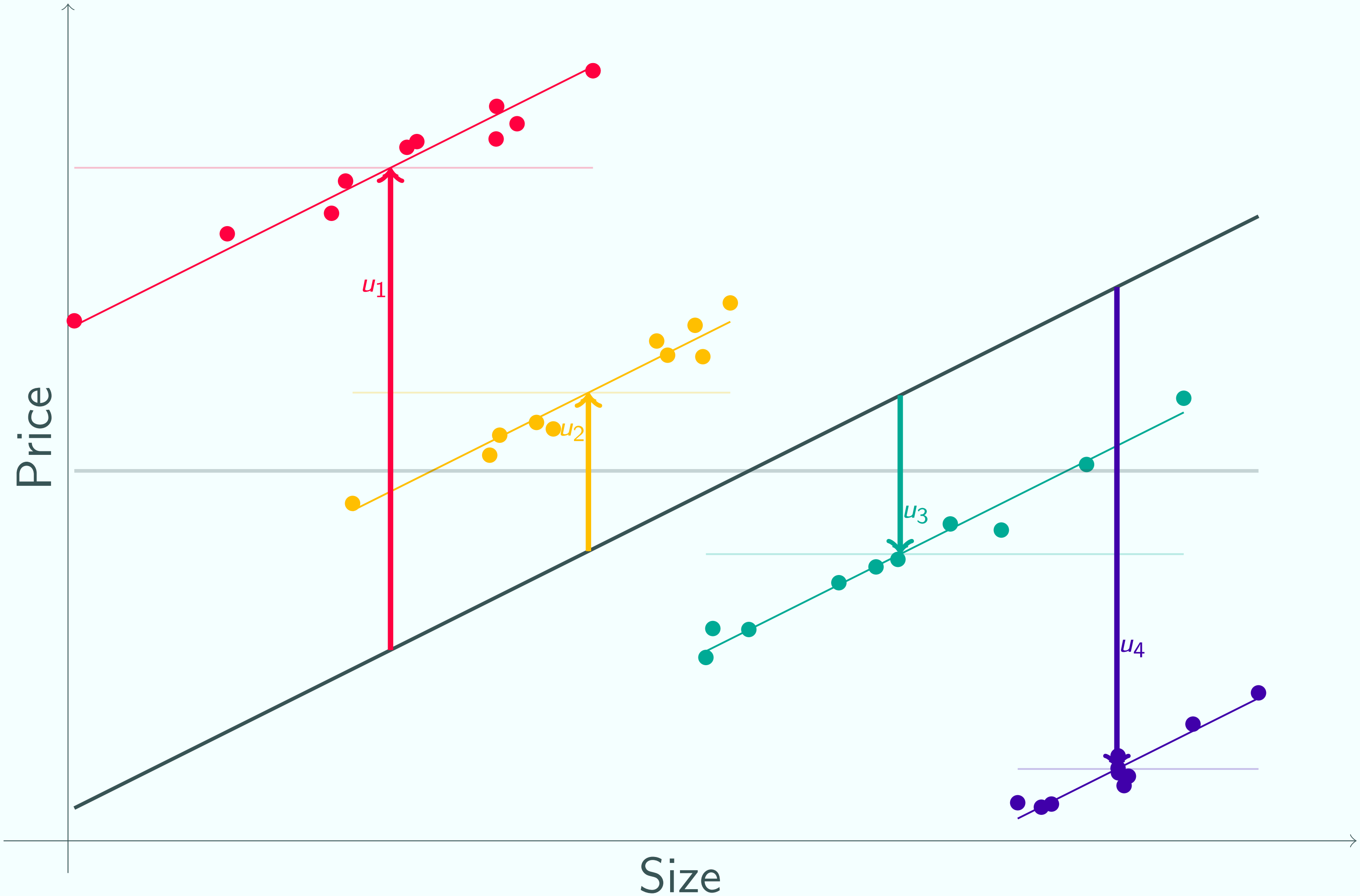

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Hospital is probably a level

If we have data on treatment outcomes for patients in 100 UK hospitals:

Ethnicity is not a level

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- The categories of ethnicity have a special meaning
- We expect different results for different categories
- We probably don't have many different categories in any case

Exercises: Session 1

Variance partitioning coefficients

- For variance components models, we saw that the VPC is a useful way to see how the variance divides up

Calculating the VPC

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- This is just the same as for the variance components model

Exam scores example

Question

How much of the variation in pupils' progress between age 11 and 16 is due to school differences?

Answer

Exam scores example

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Answer

1. Fit our random intercepts model and note the variances

Equations

$$\text{normexam}_{ij} = \beta_{0j} + 0.563(0.012)\text{standlrt}_{ij} + e_{ij}$$

$$\beta_{0j} = 0.002(0.040) + u_{0j}$$

$$u_{0j} \sim \text{N}(0, \sigma^2_{u0}) \quad \sigma^2_{u0} = 0.092(0.018)$$

$$e_{ij} \sim \text{N}(0, \sigma^2_e) \quad \sigma^2_e = 0.566(0.013)$$

$$-2{*}loglikelihood = 9357.242(4059 \text{ of } 4059 \text{ cases in use})$$

Name | + | - | Add Term | Estimates | Nonlinear | Clear | Notation | Responses | Store | Help | Zoom 100

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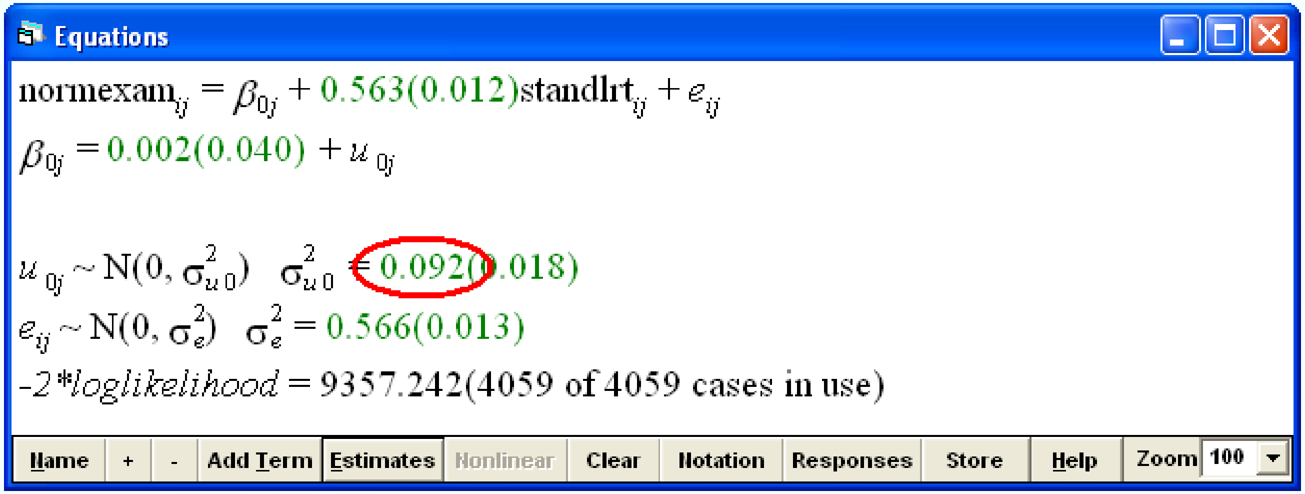

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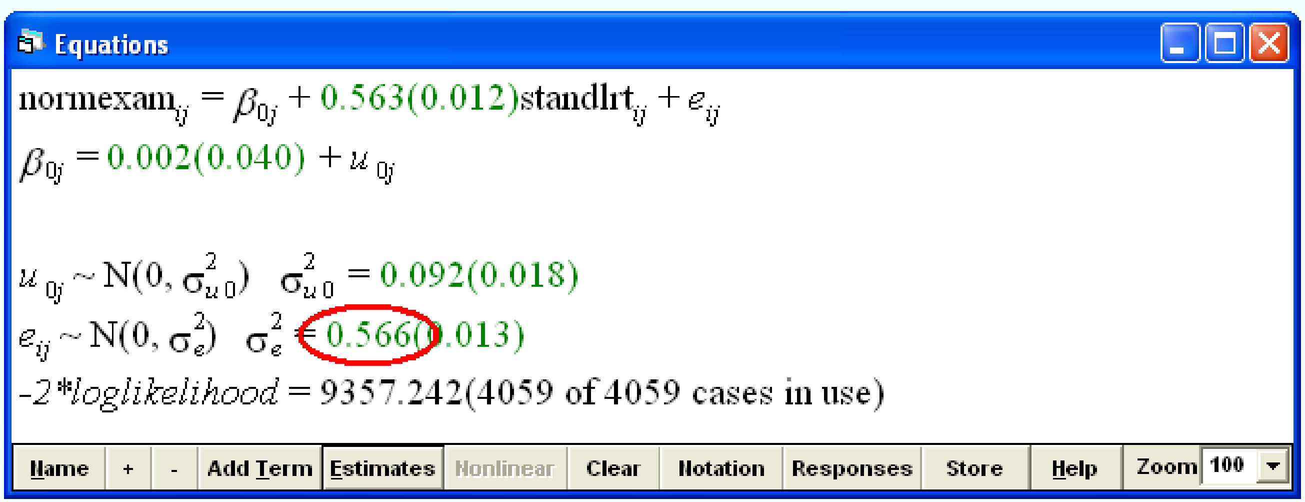

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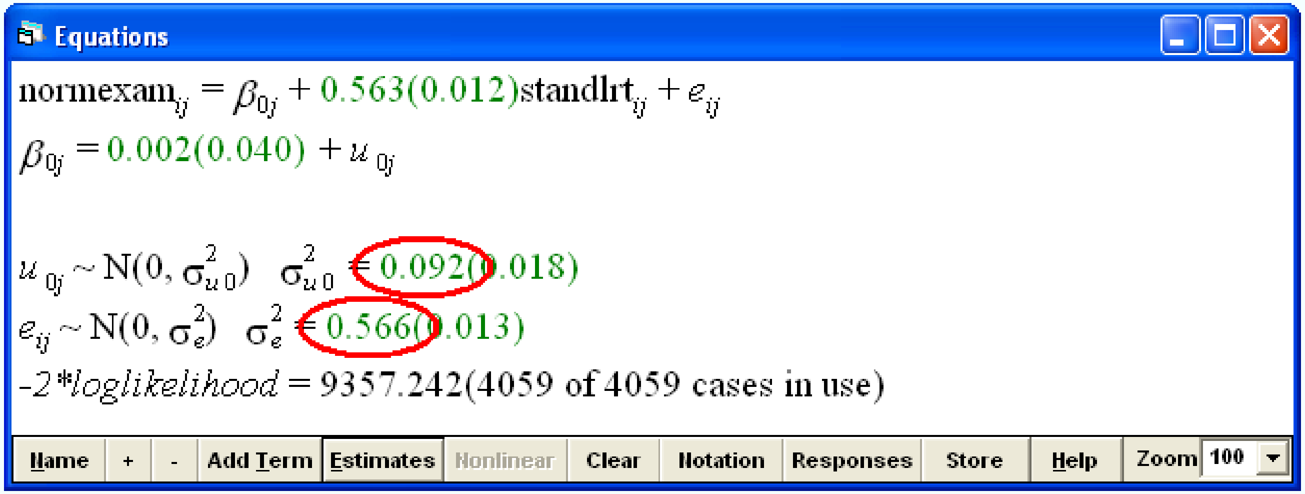

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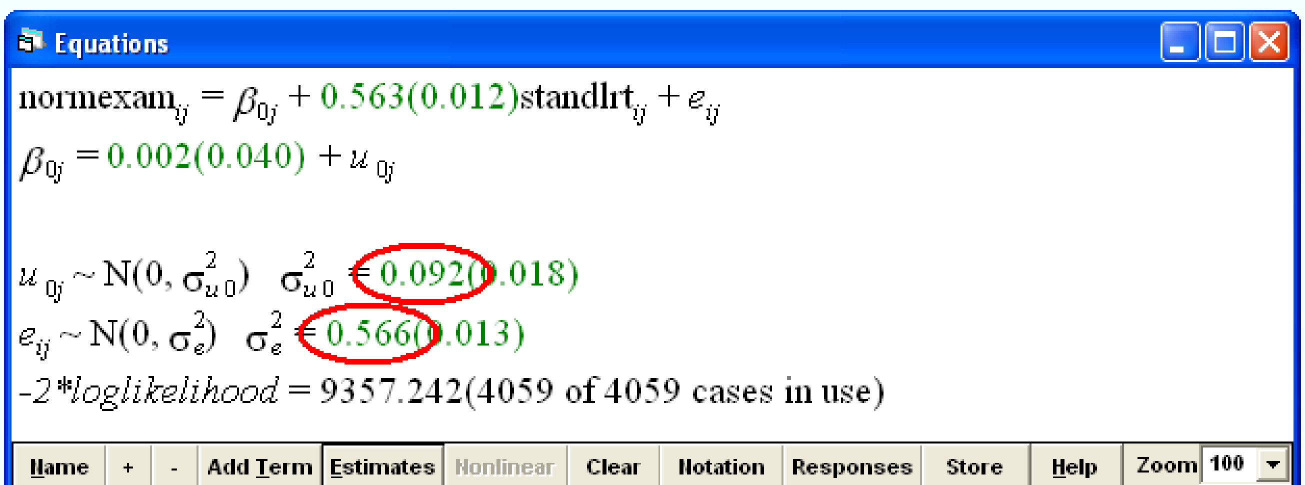
   0.092/(0.092+0.566) = 13.9%



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Another way to think of ρ is that it measures the clustering

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Large $\rho \Rightarrow$ a lot of clustering

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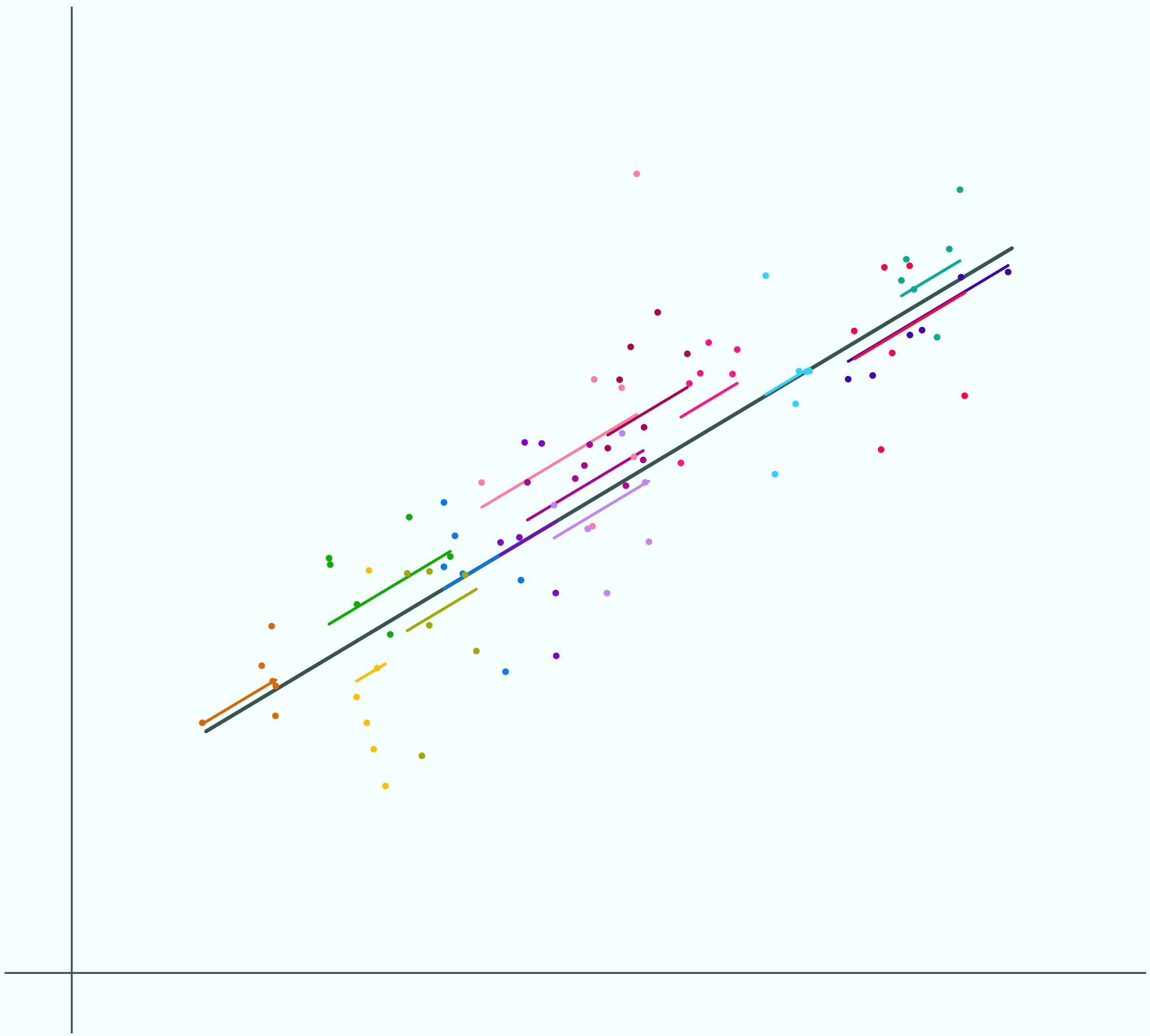

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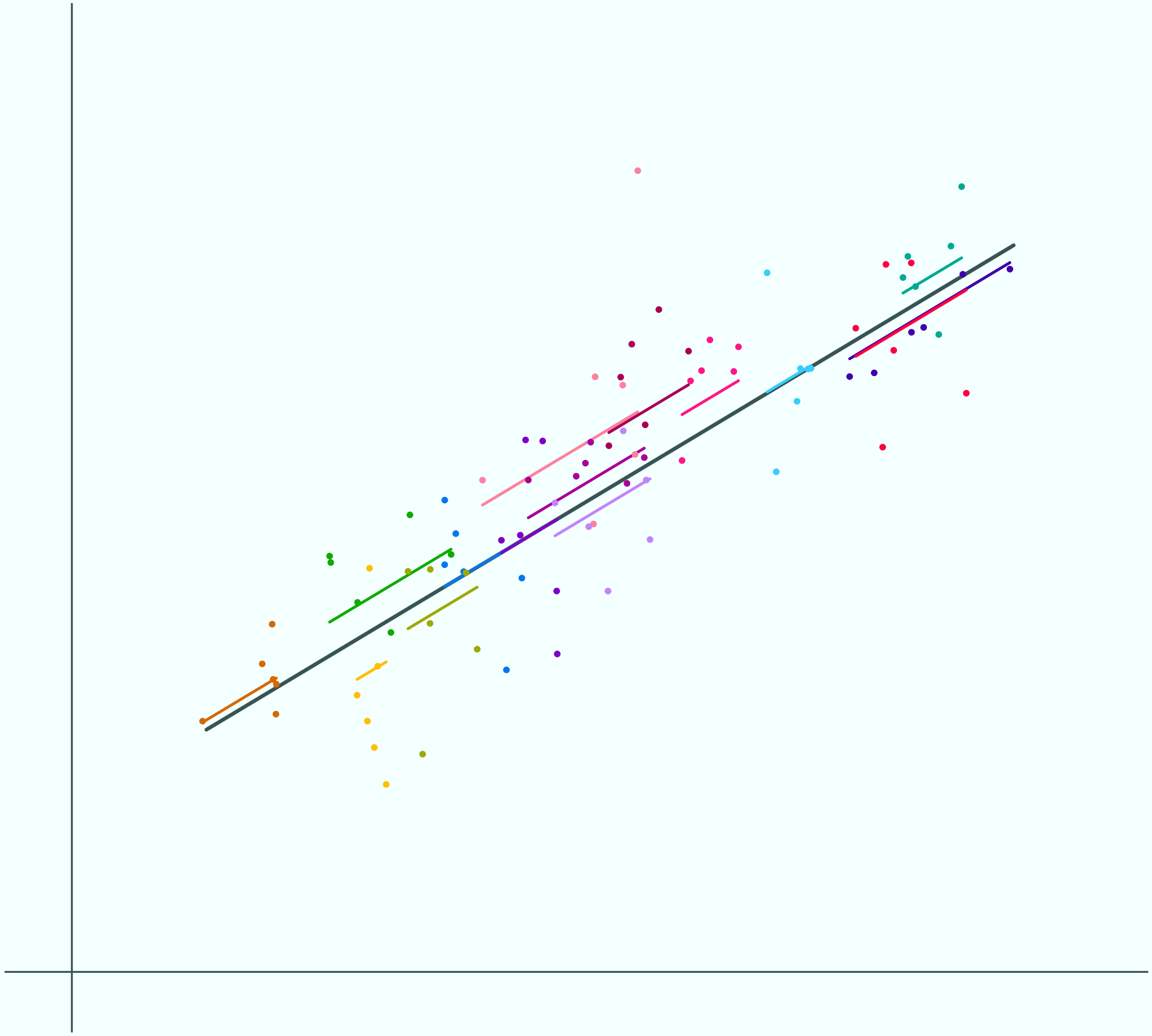
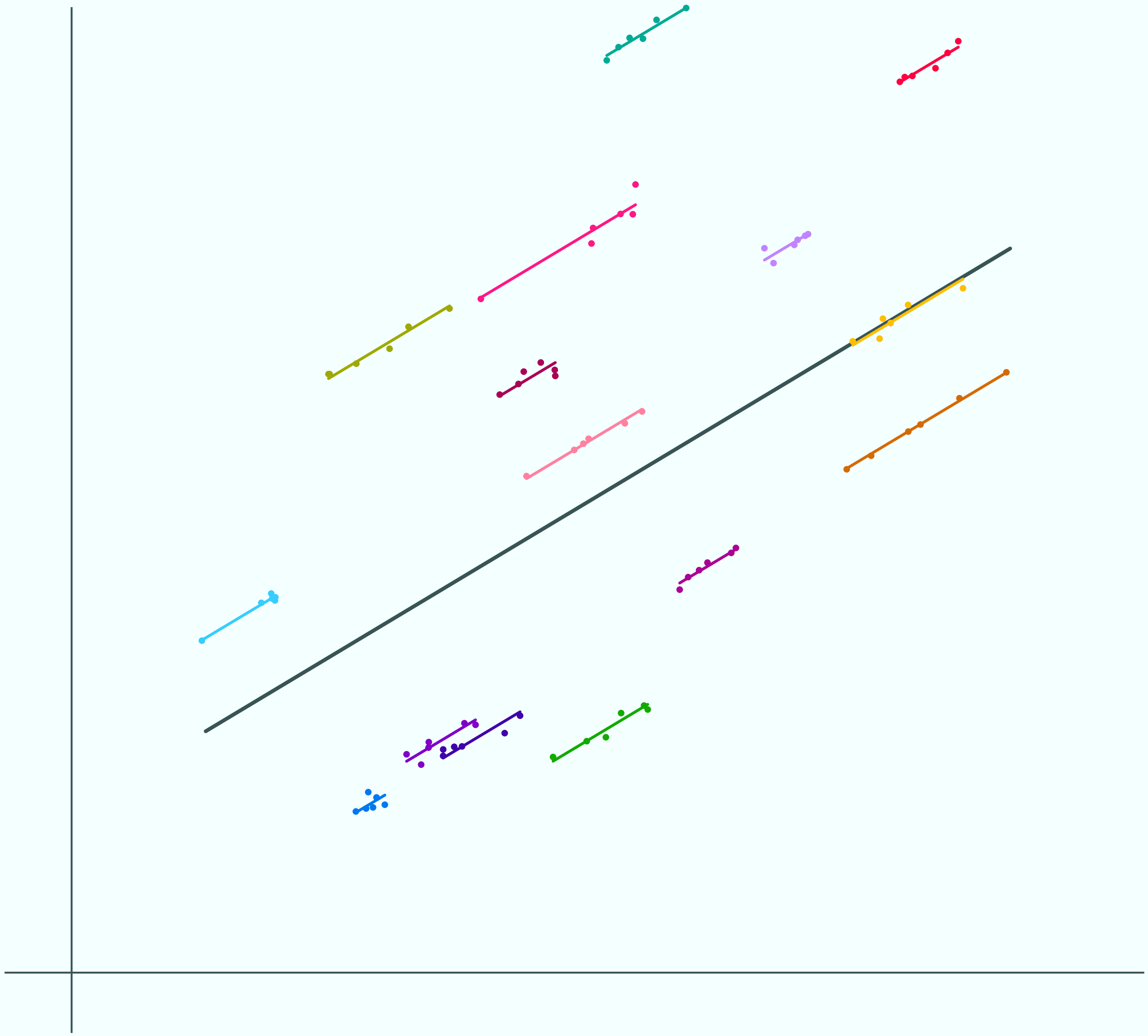
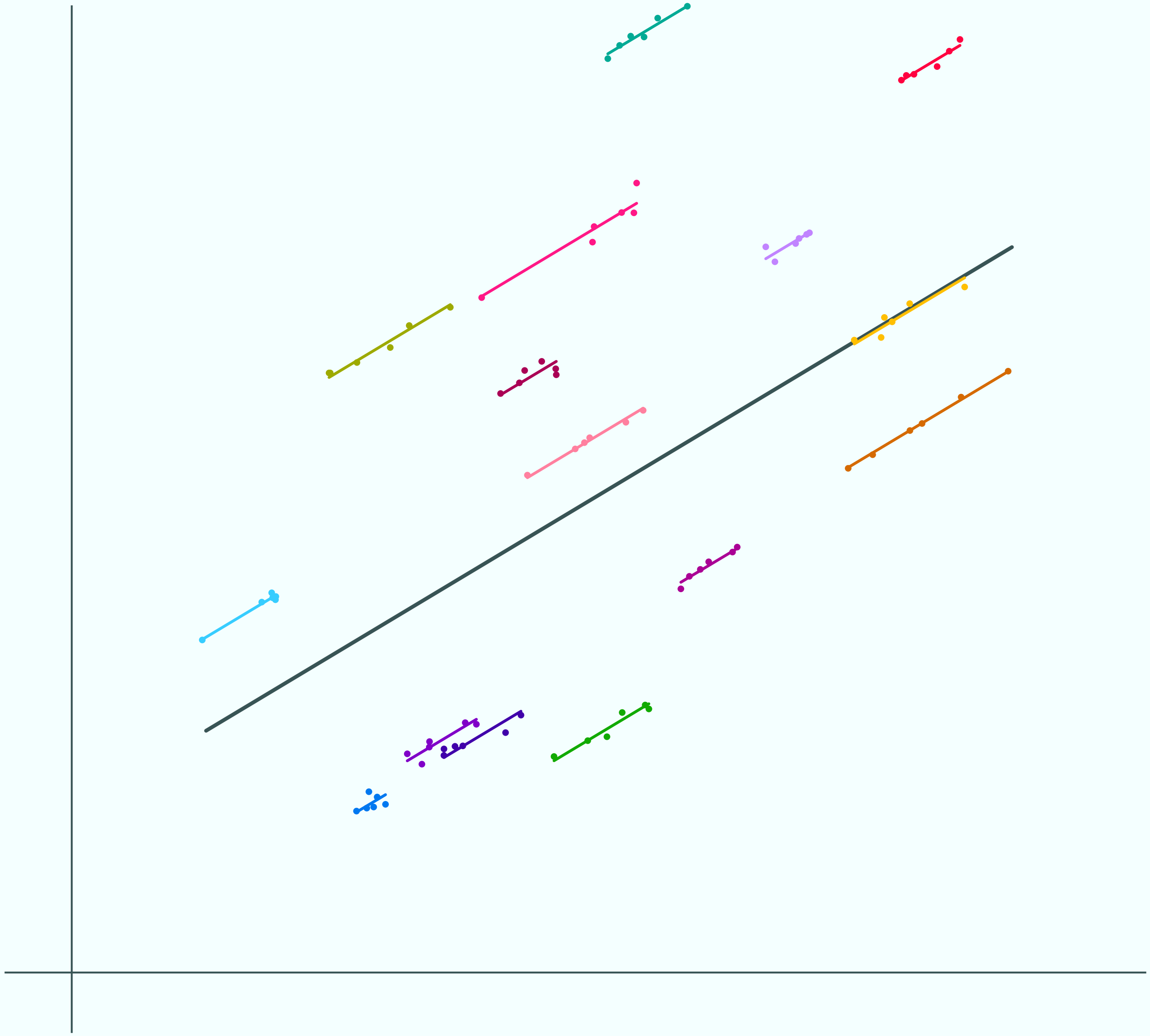
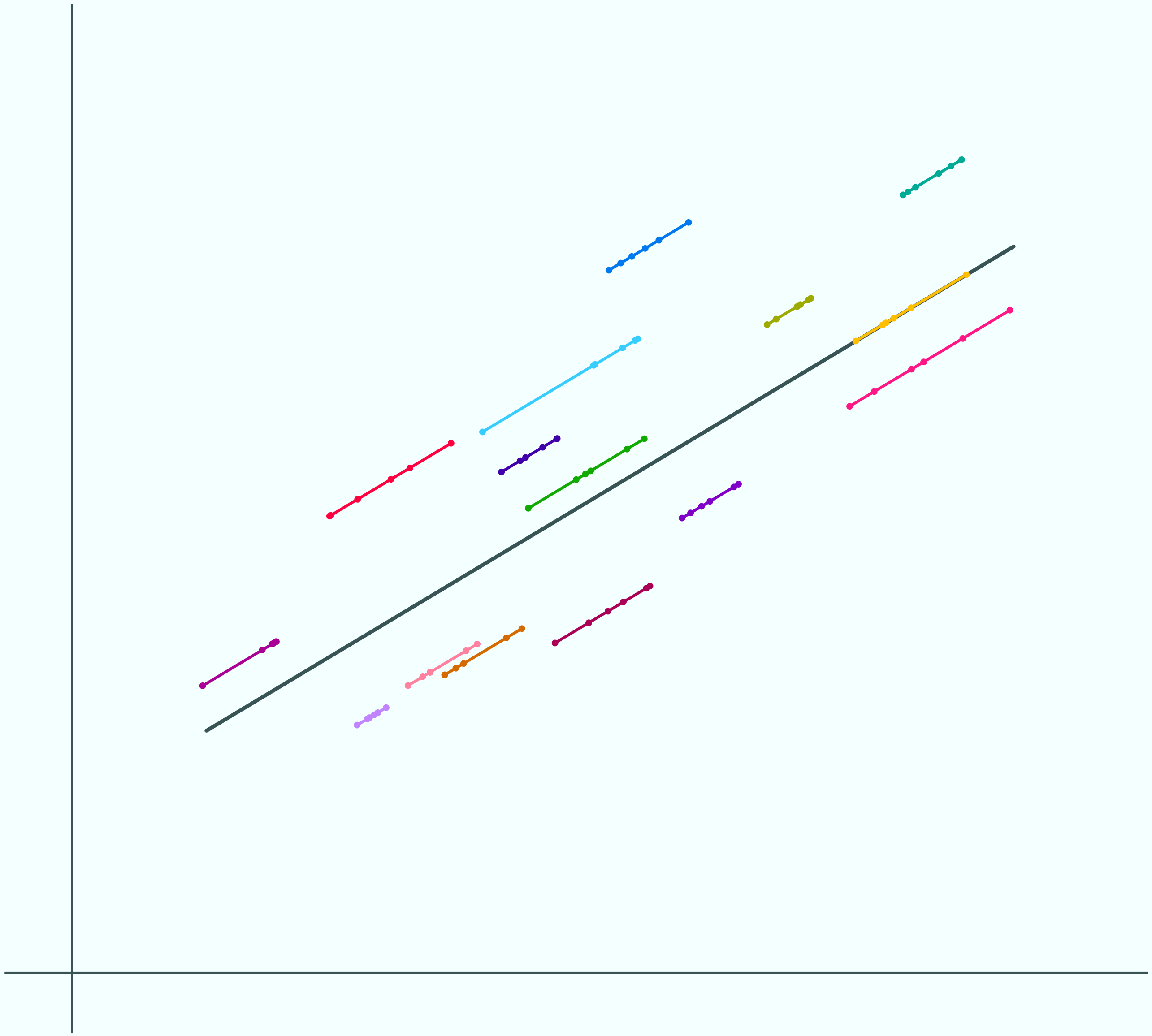


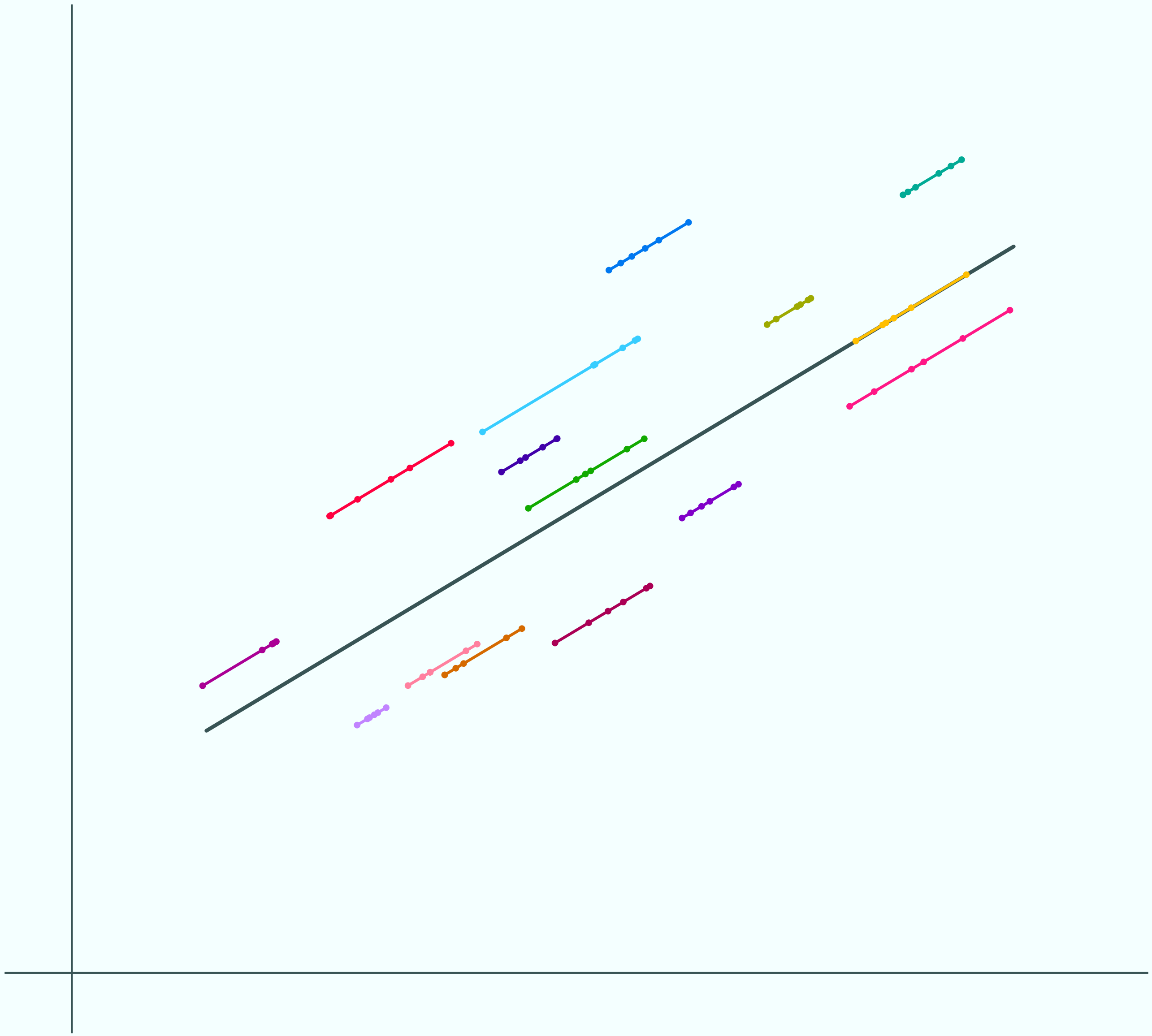


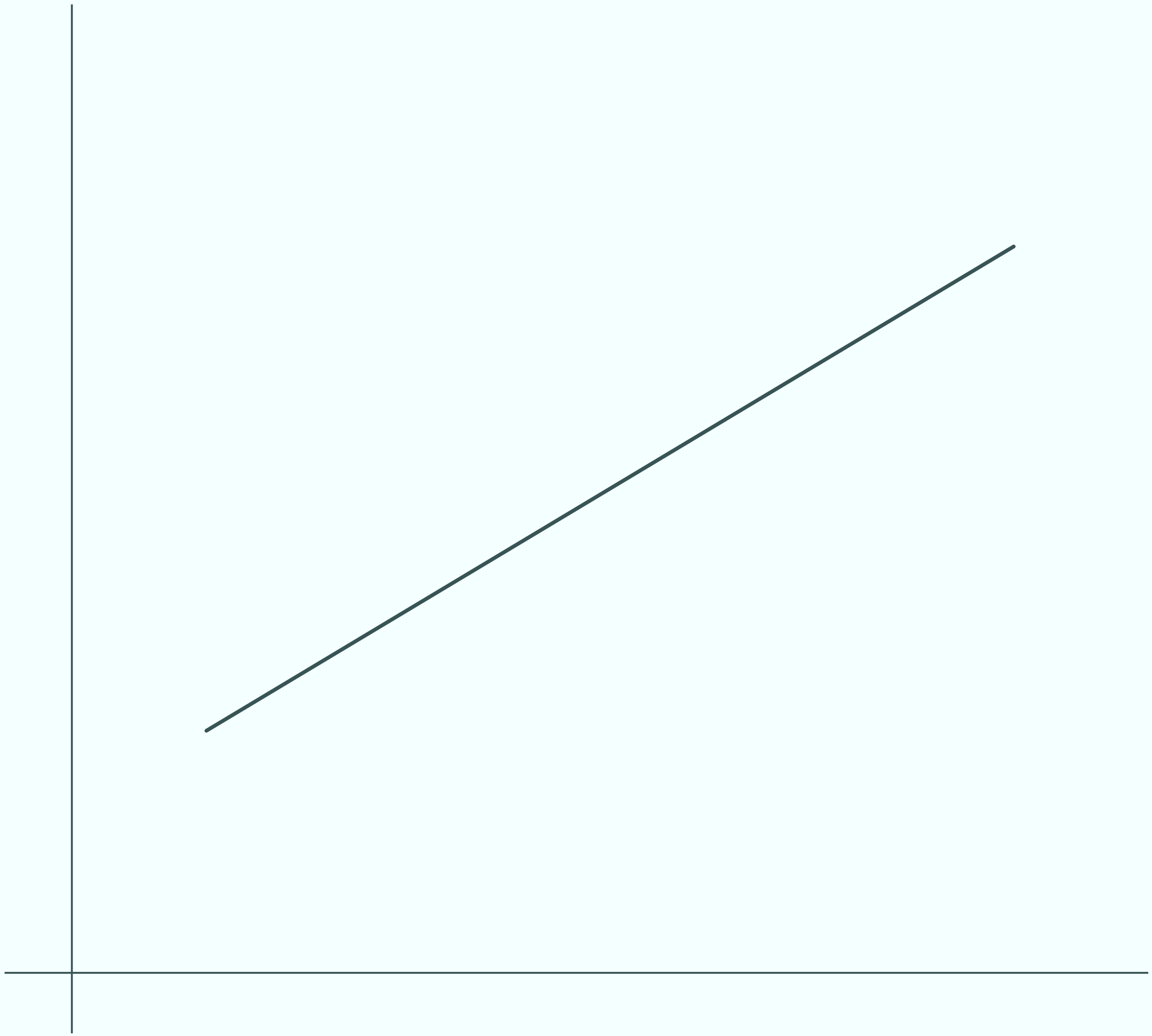
A large value of ρ

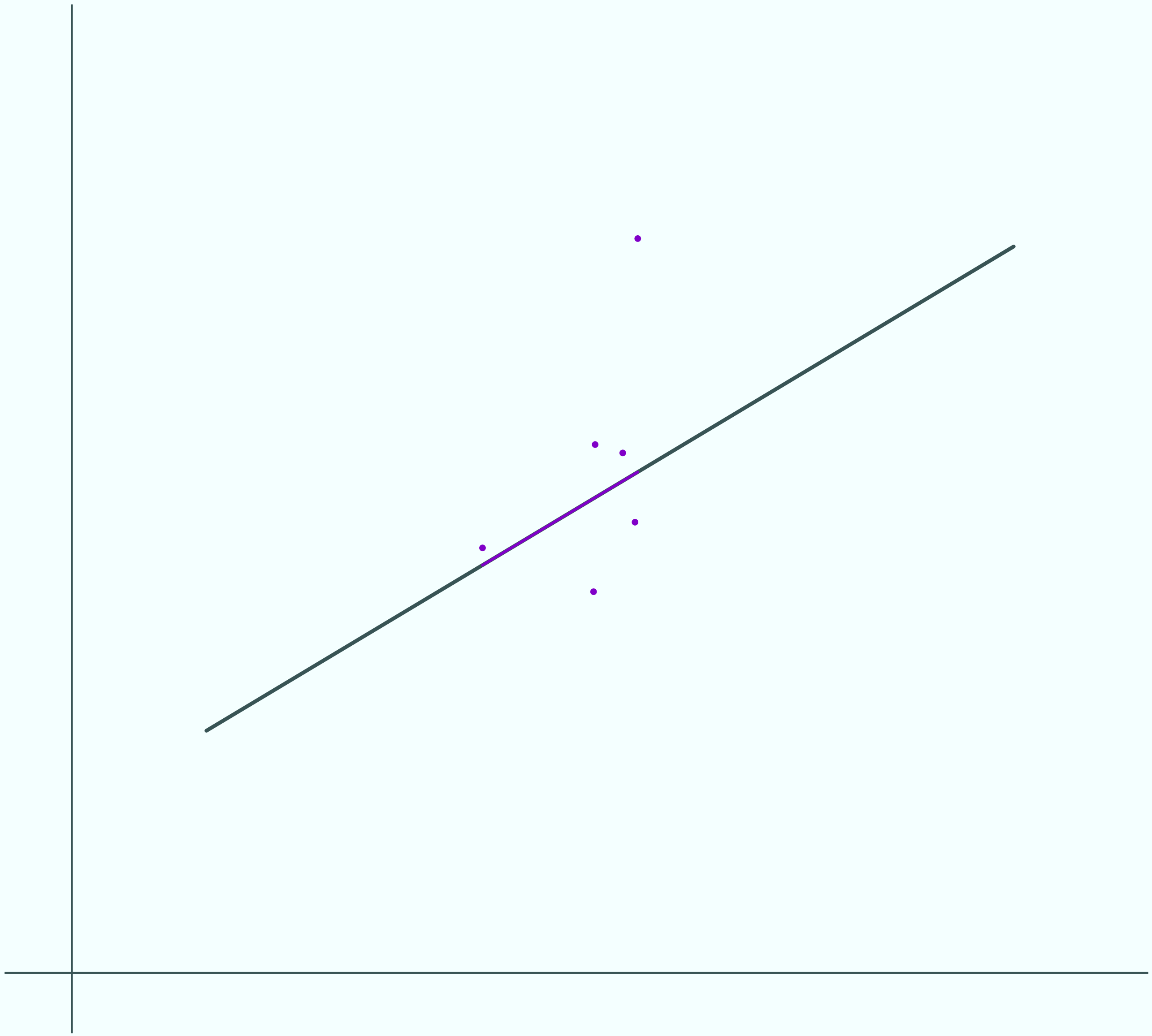


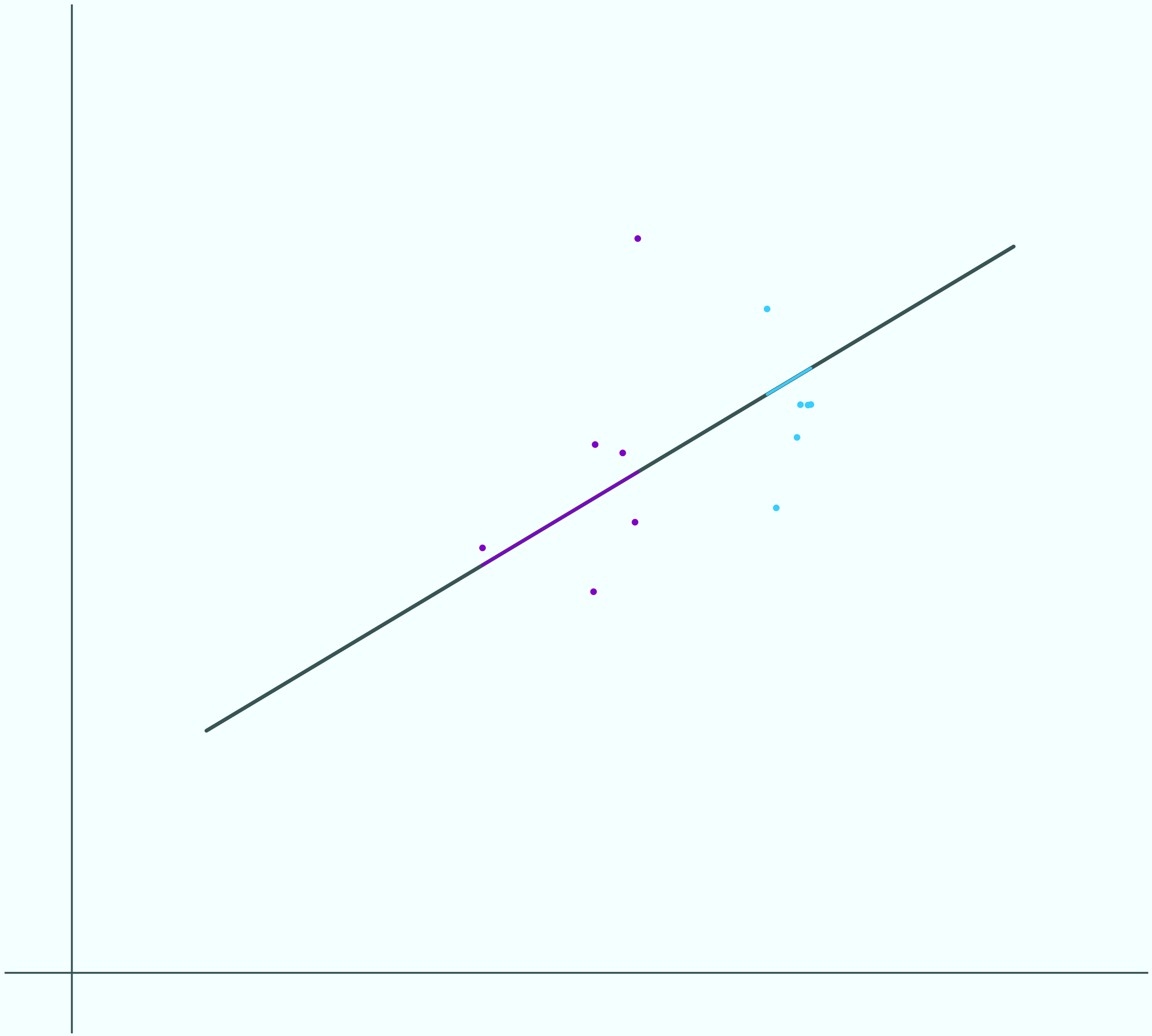


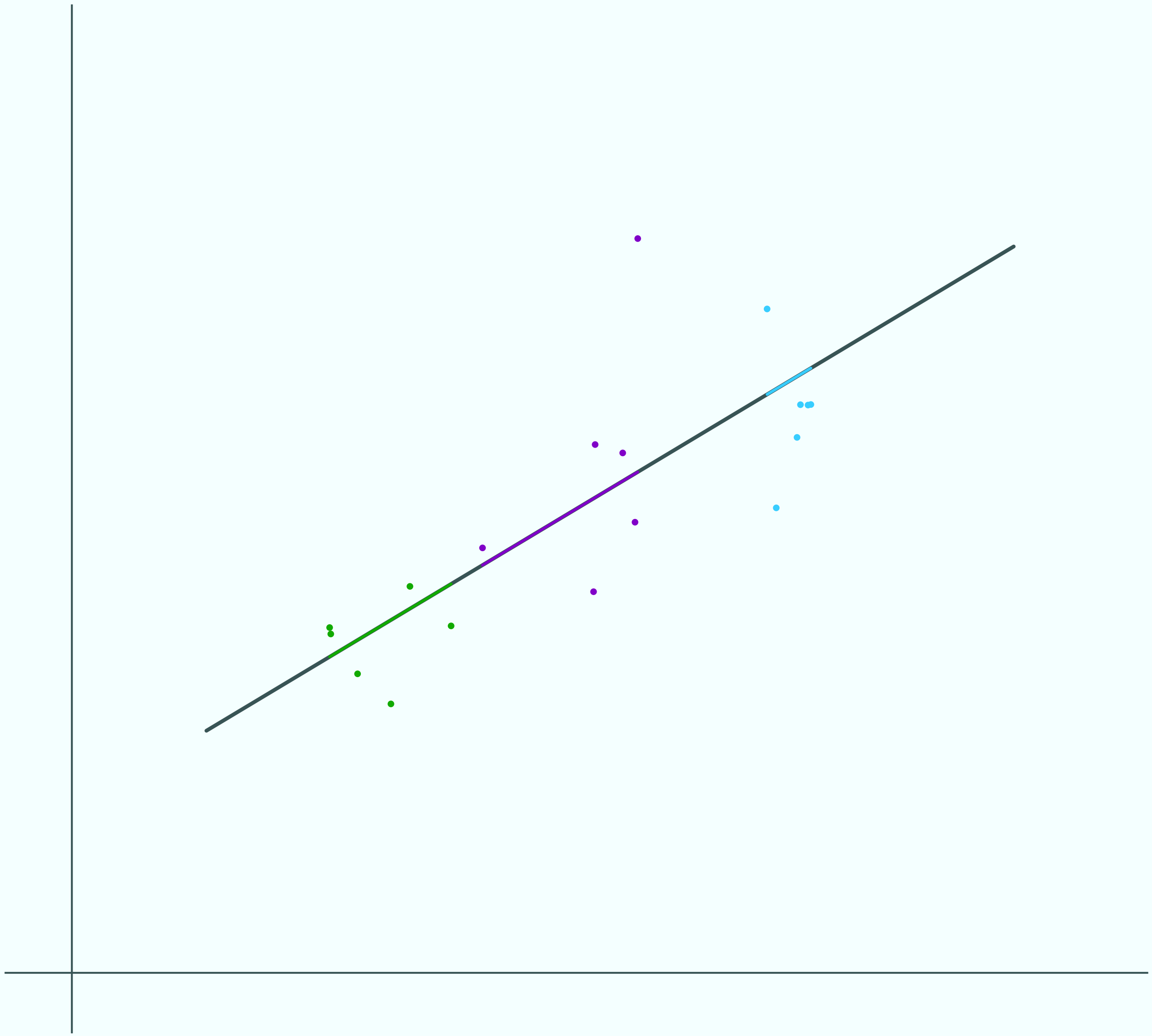
$\rho = 1$

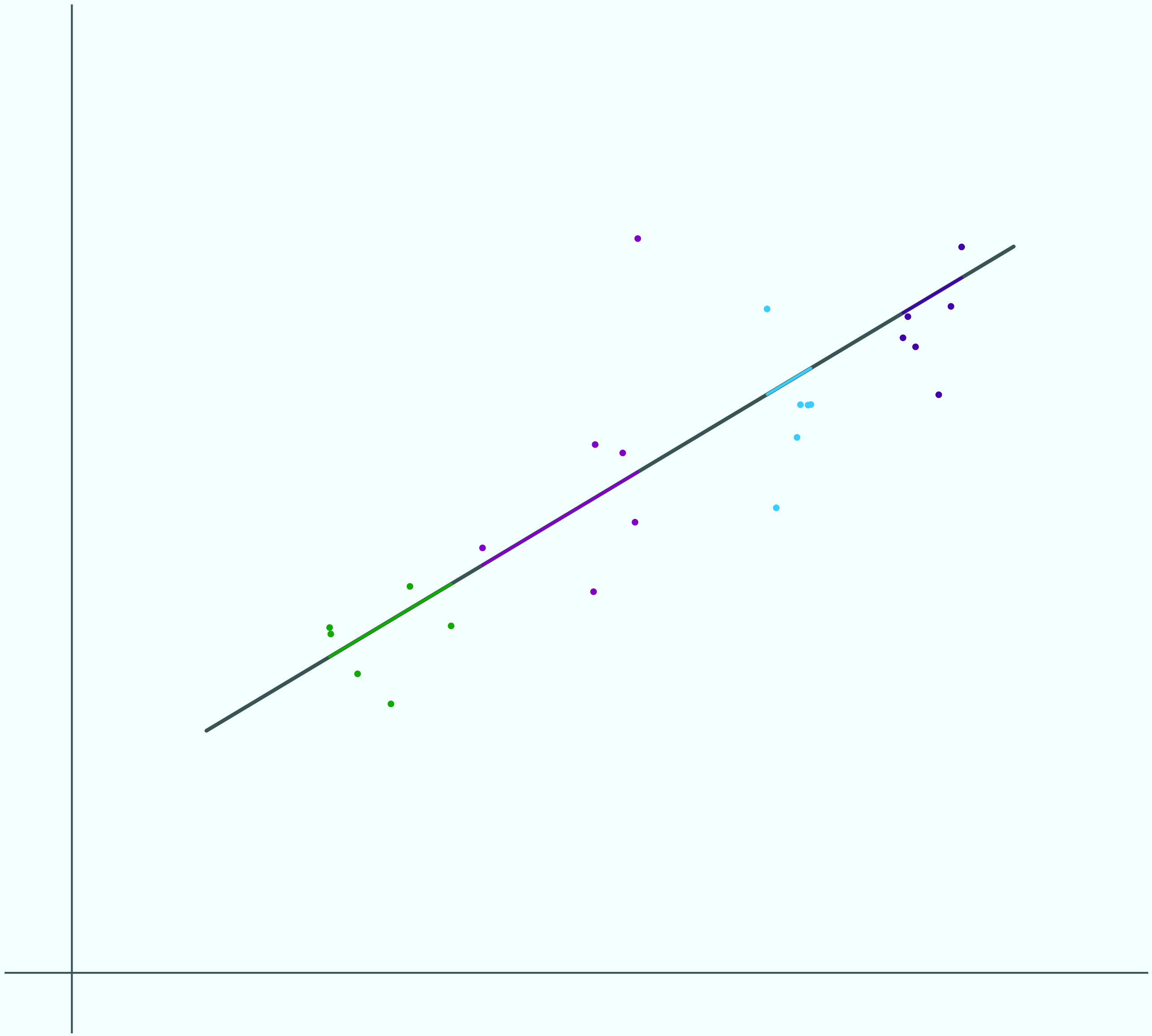


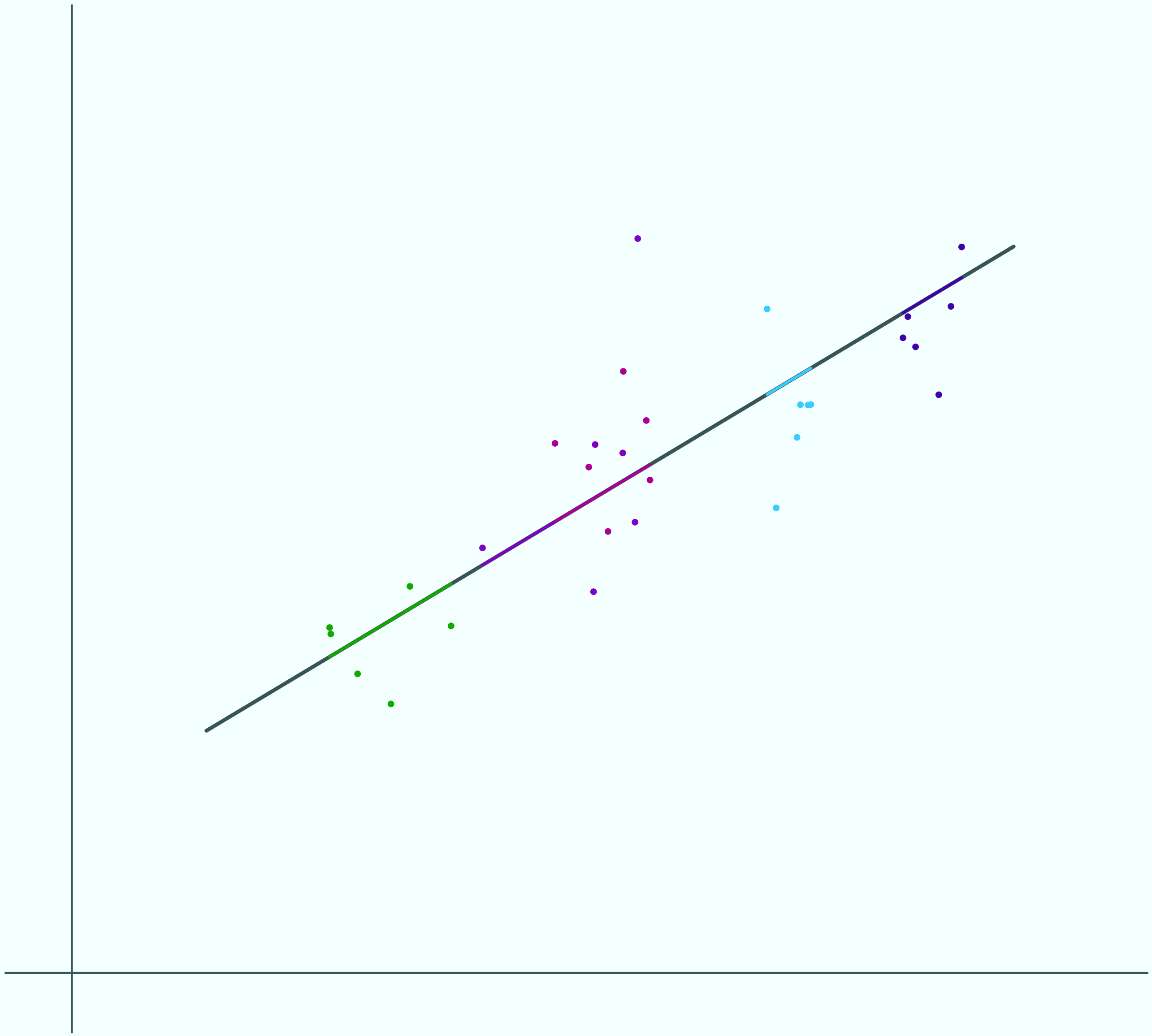


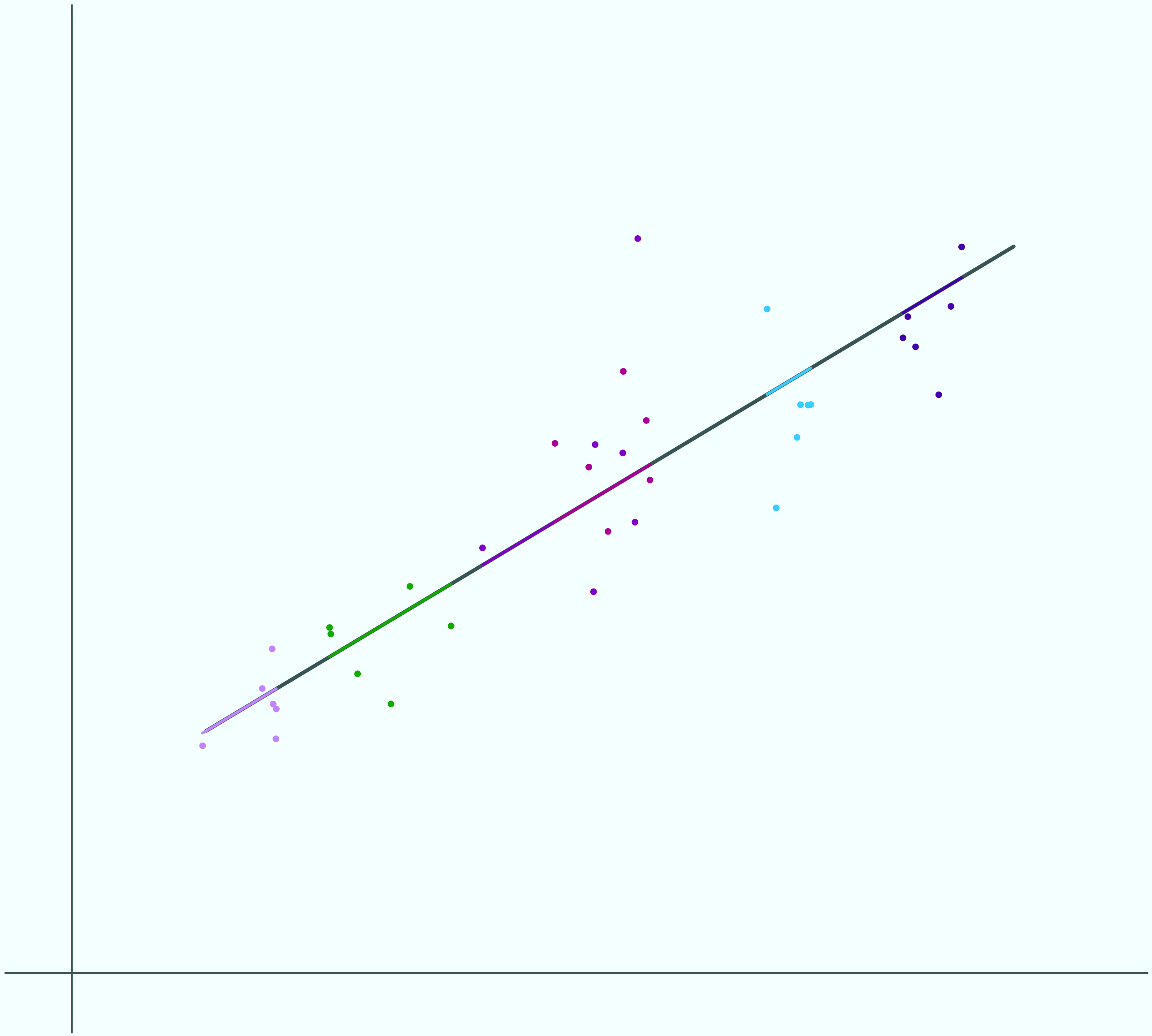


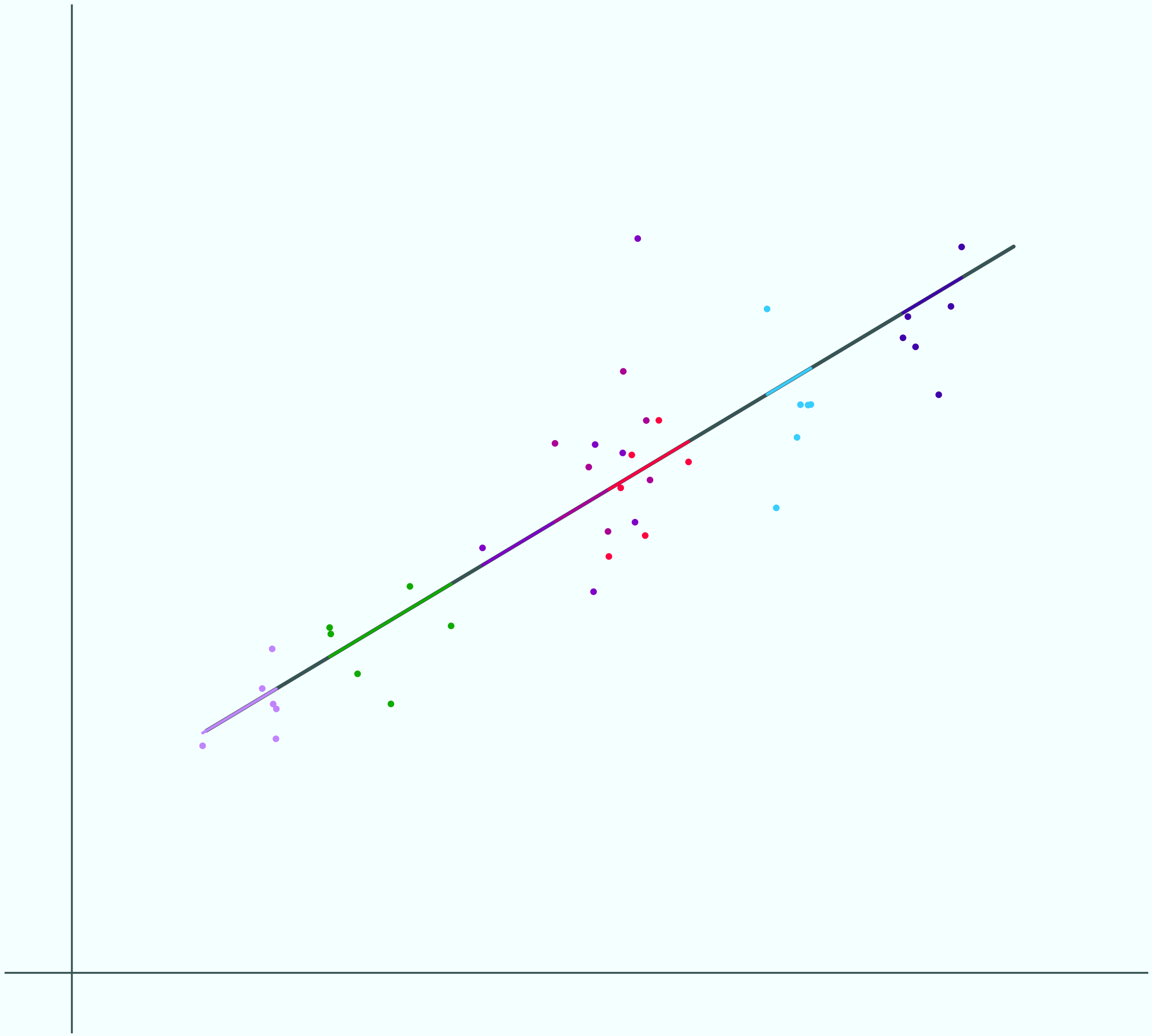


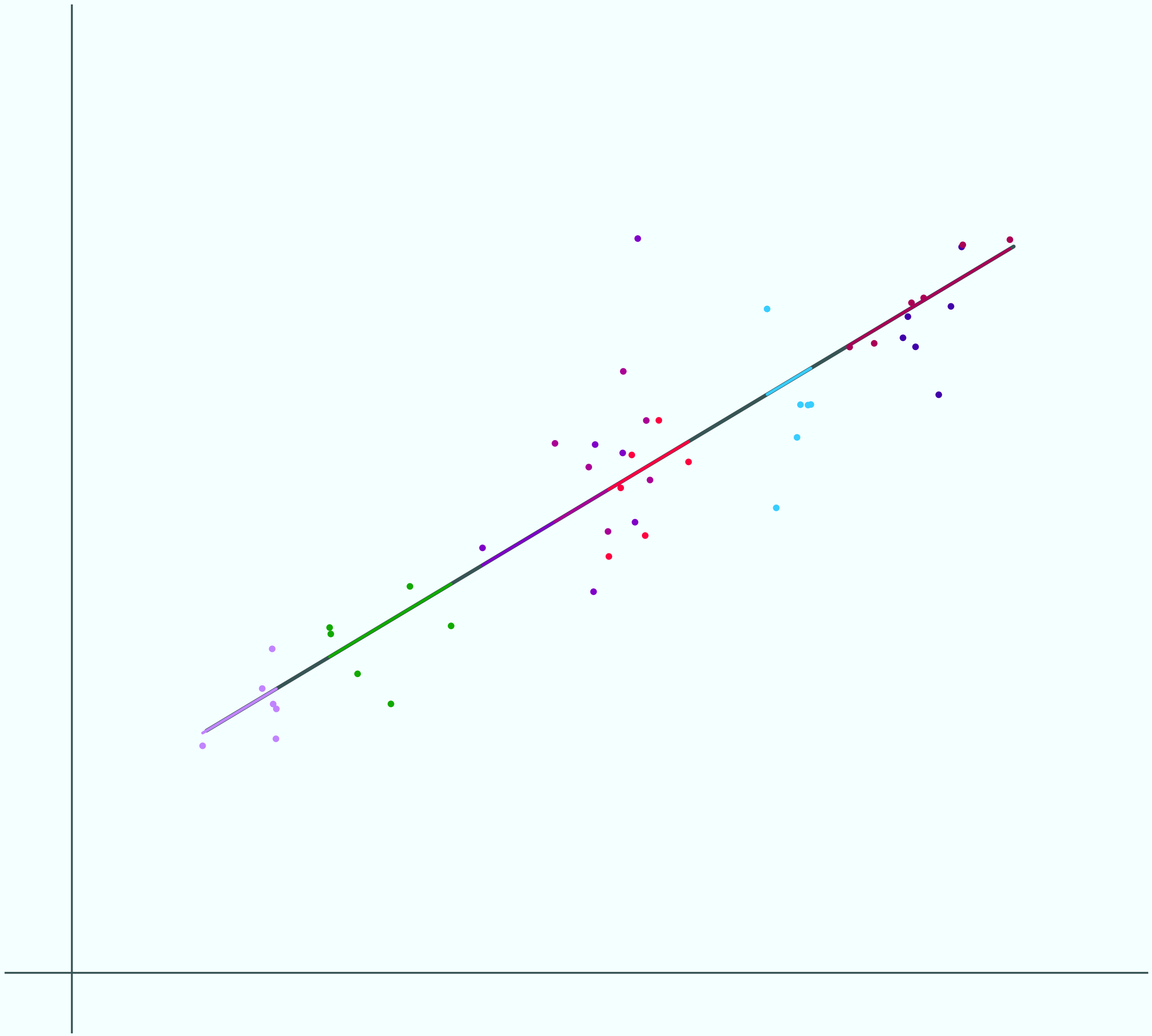


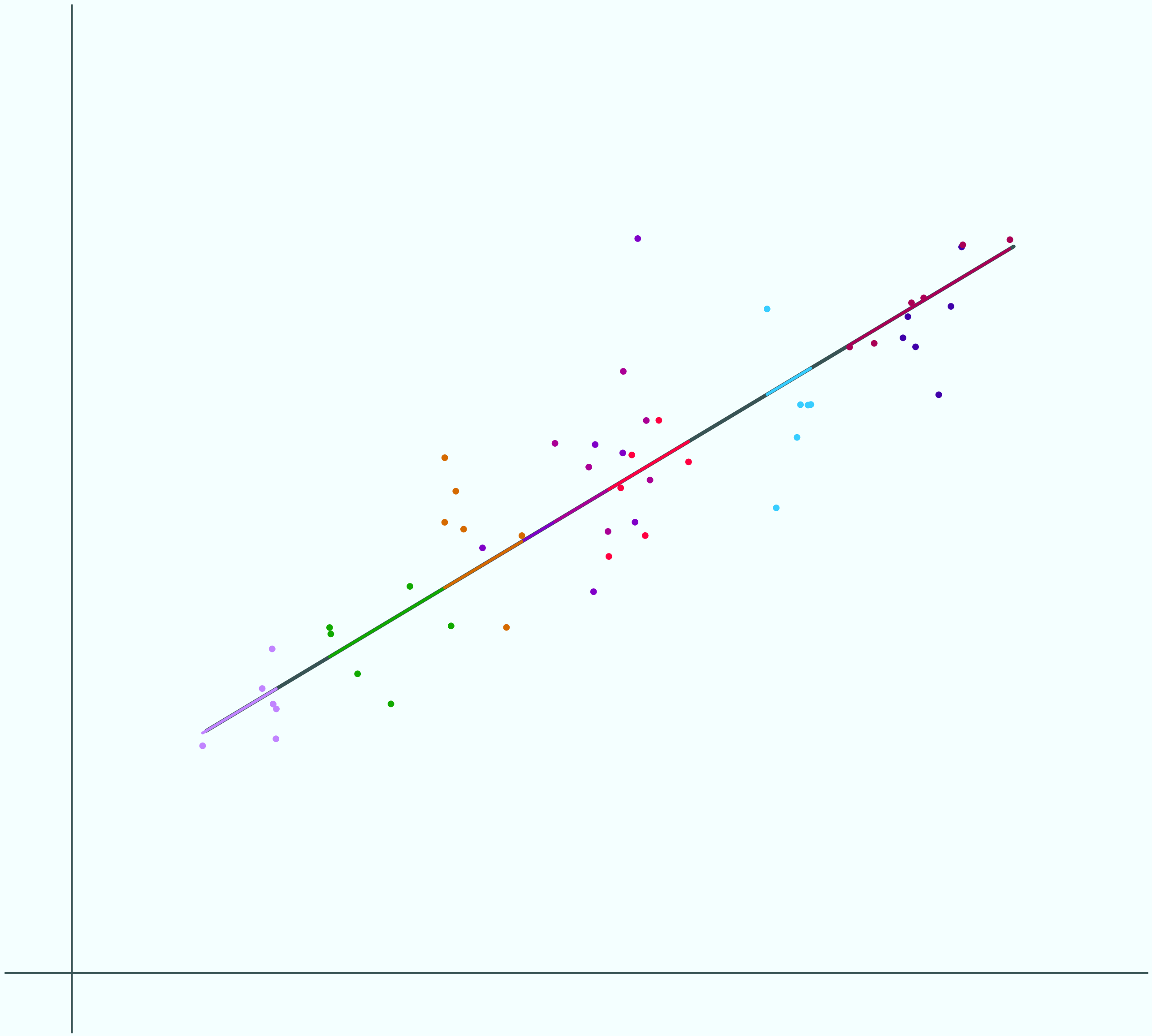


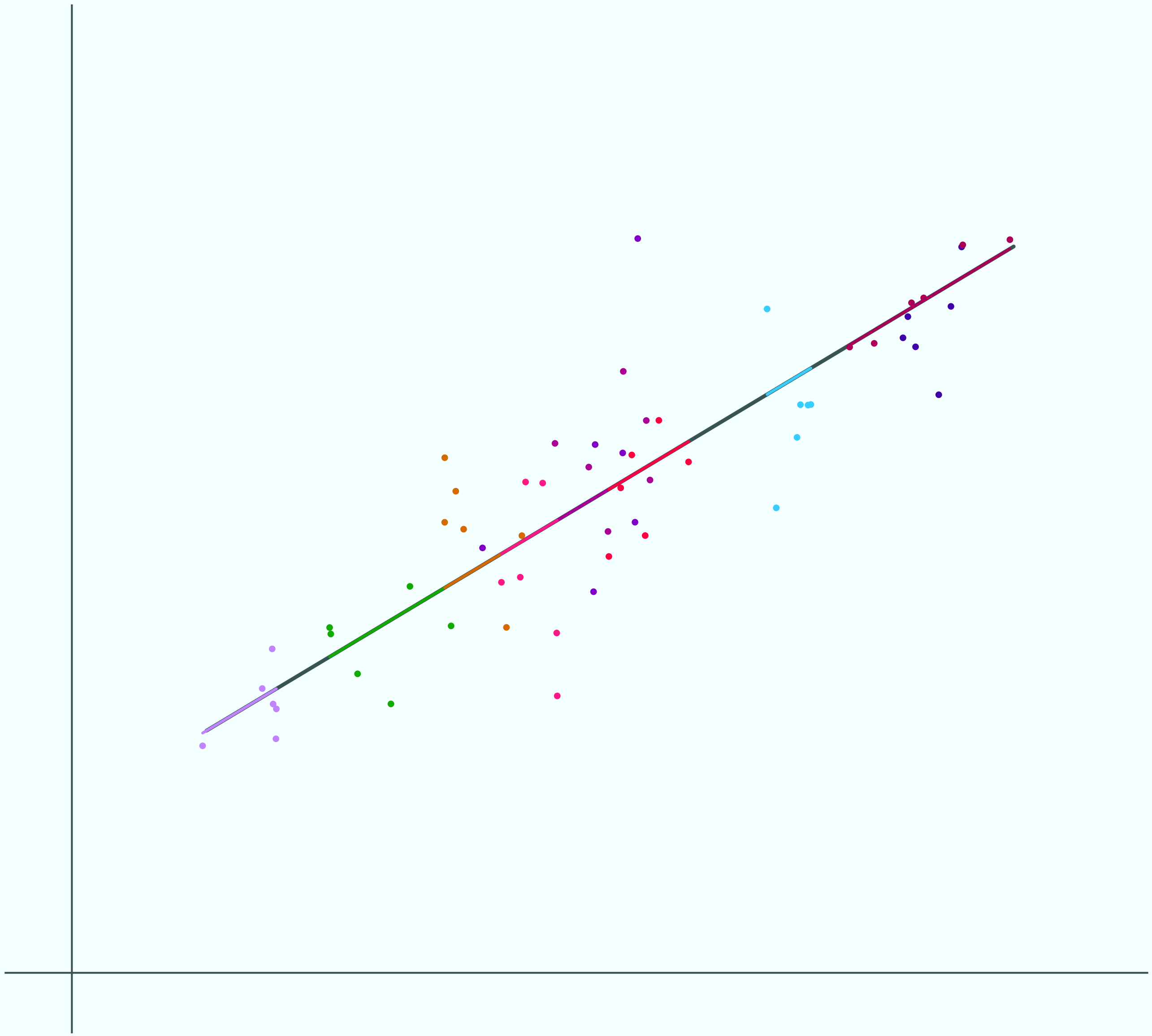


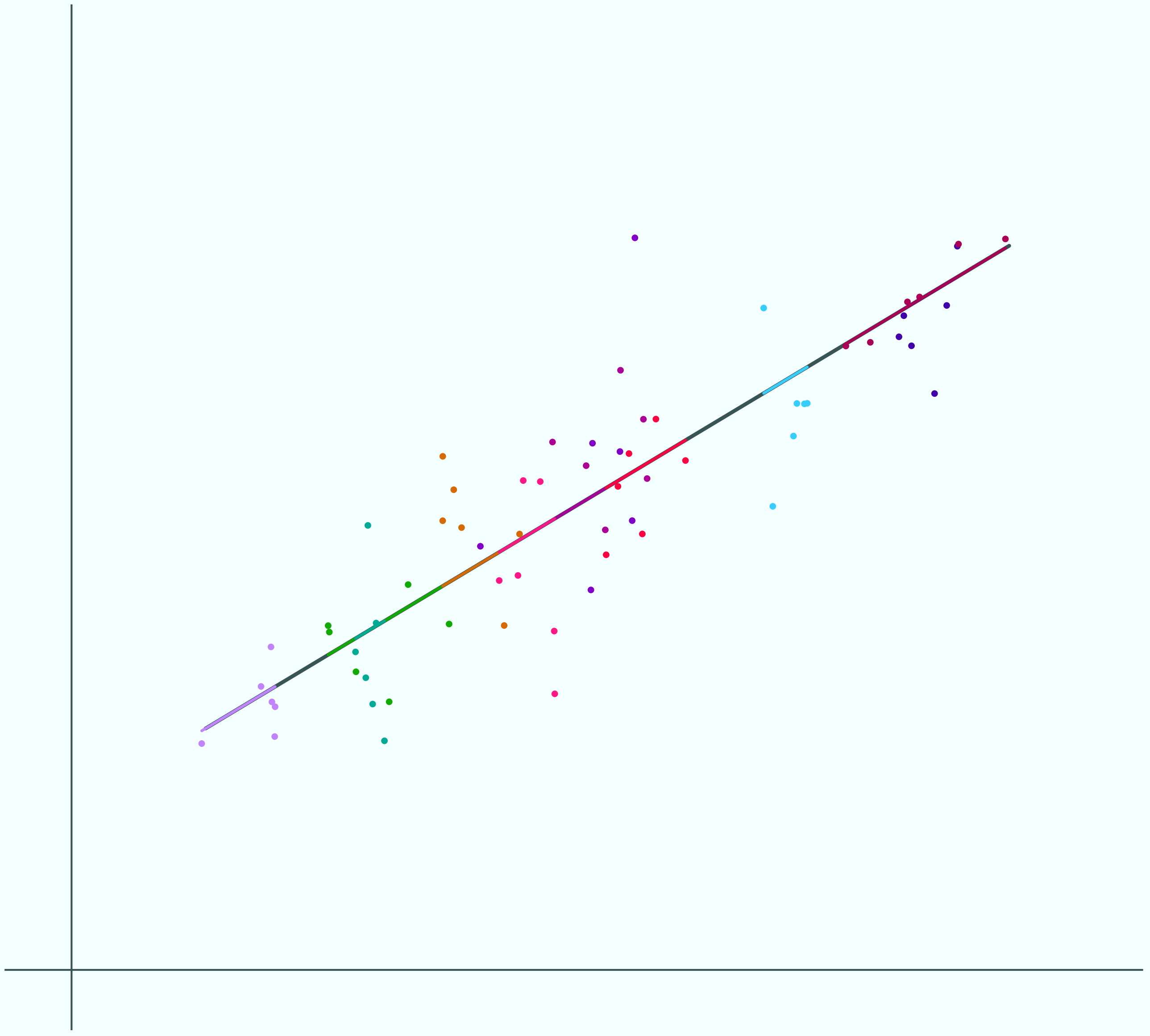


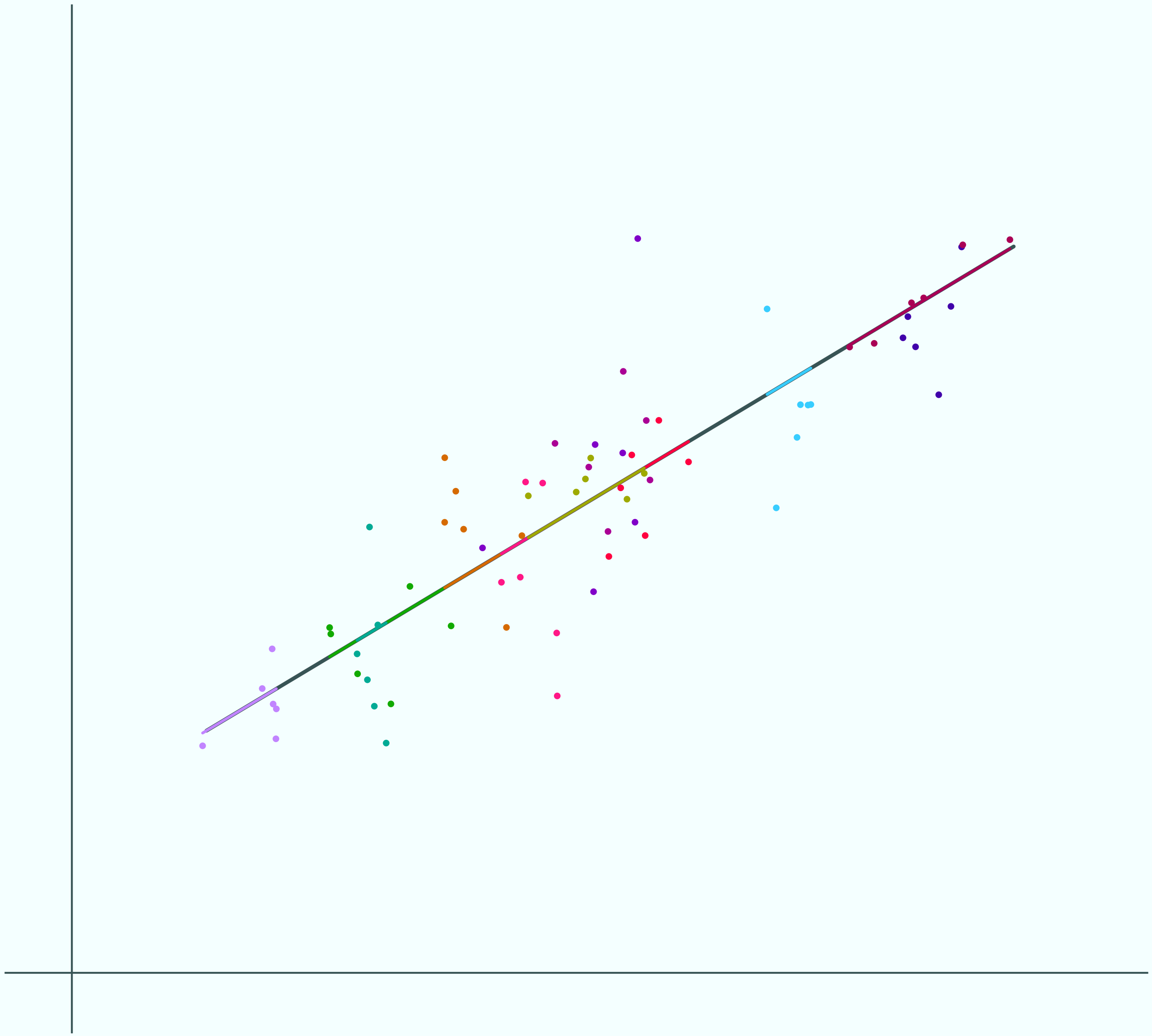


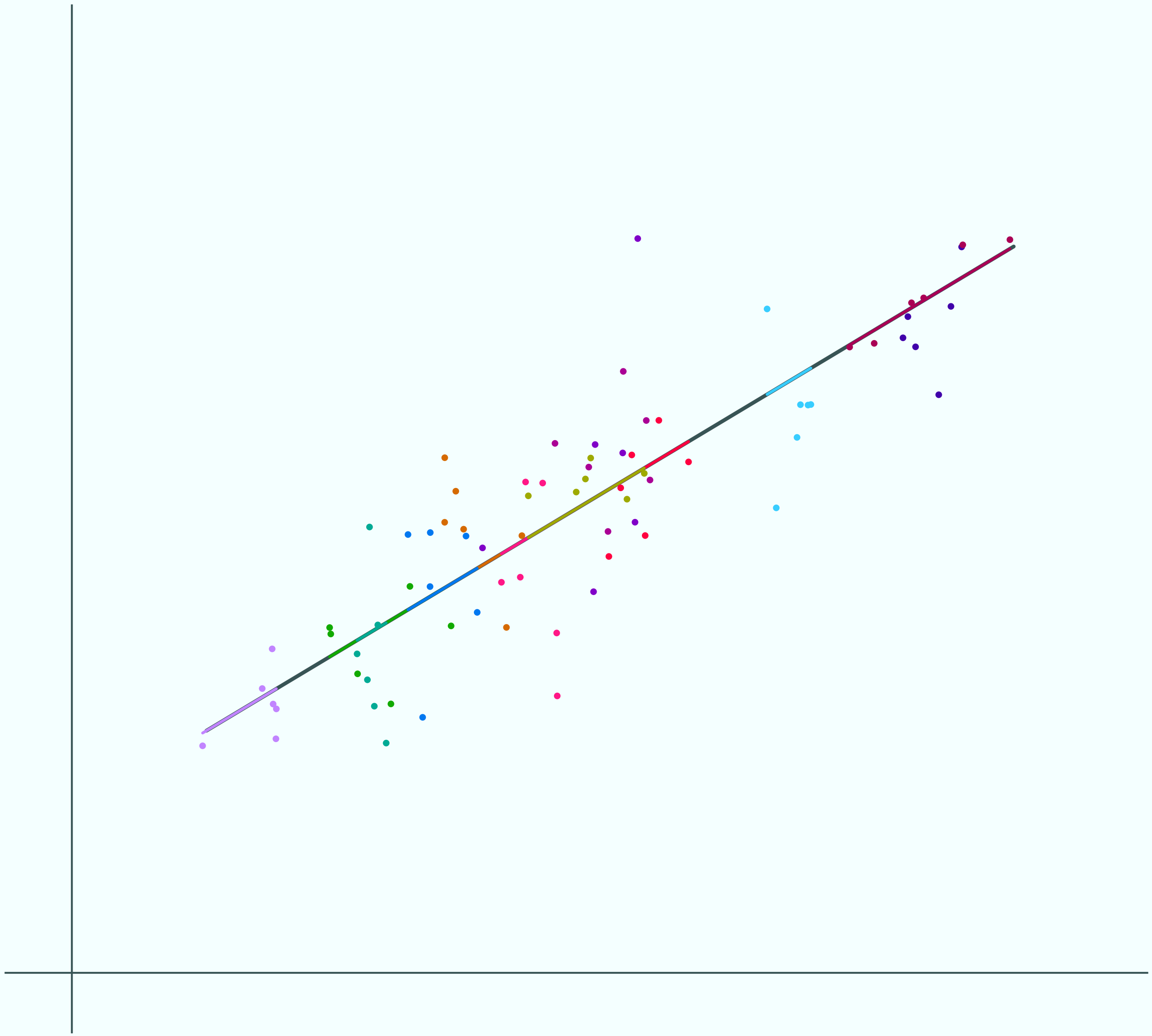


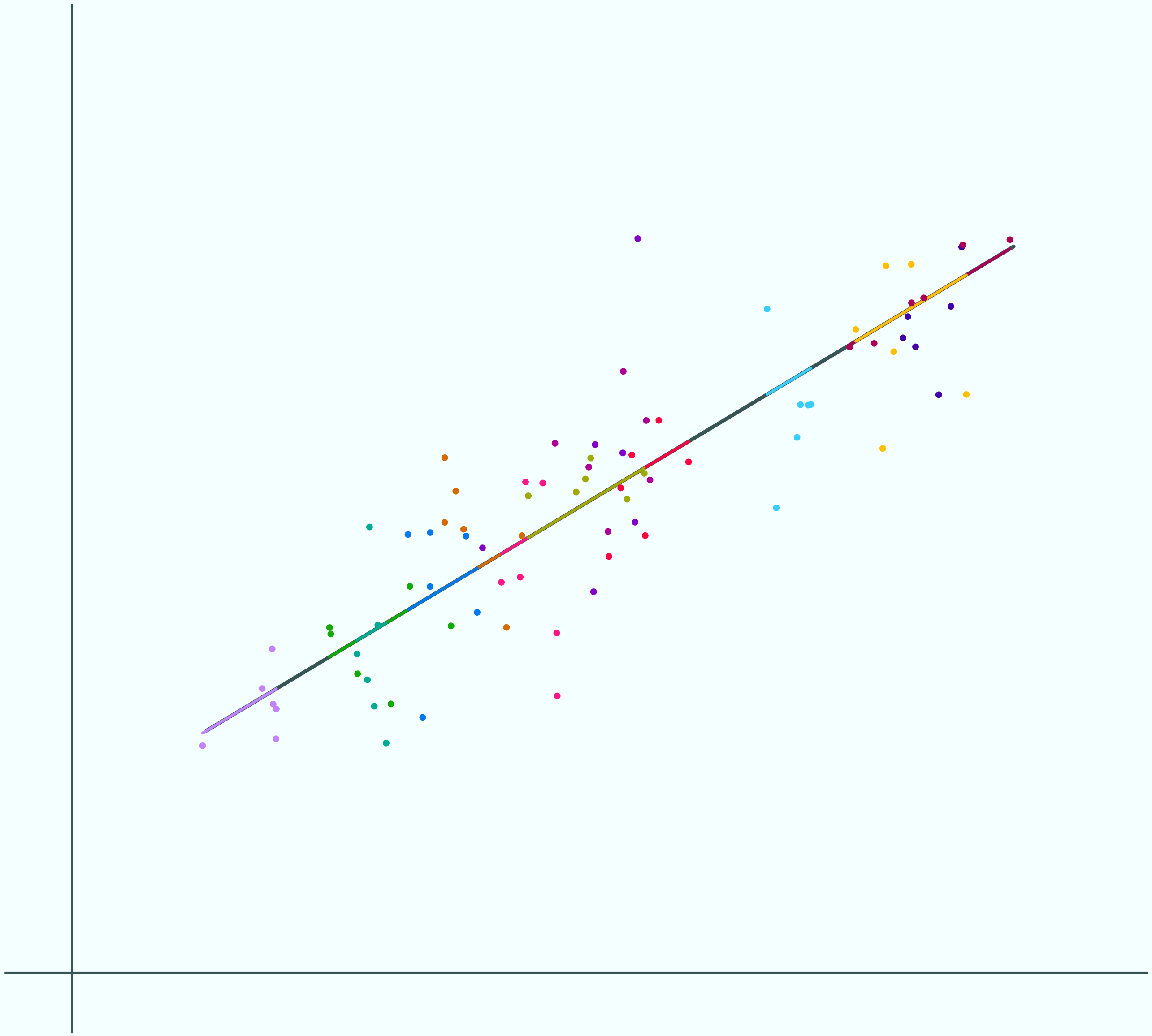












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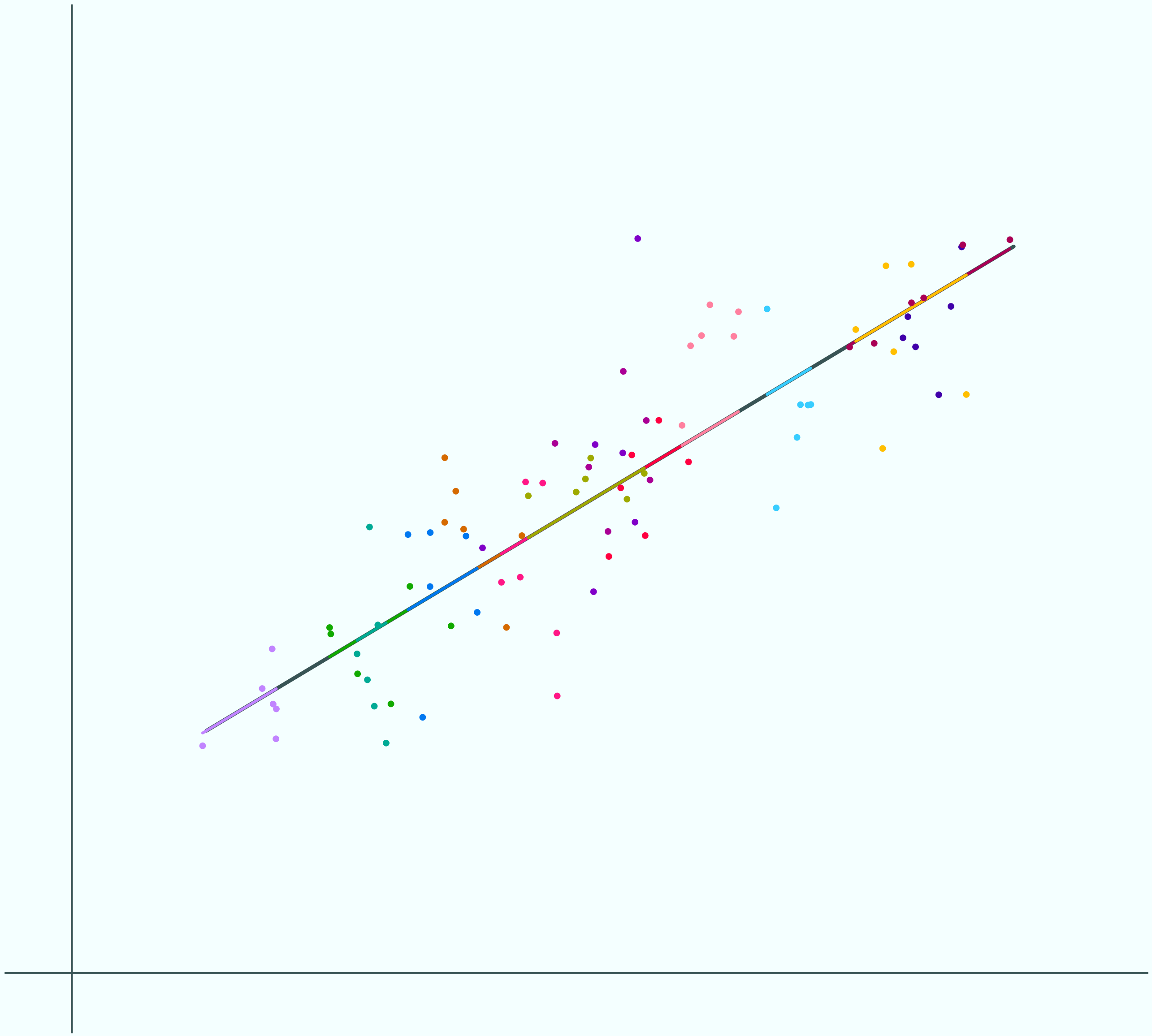

Interpreting the value of ρ

Theoretical limits for ρ

- Looking at the formula for ρ, we can see that in theory the smallest it can be is 0 and the largest it can be is 1

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- and more clustering for observations on people within families than pupils within schools, for example

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Incorporating the clustering

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- And to do that we need to return to the technicalities of the model

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- Level 2 residuals for different groups are uncorrelated
- Level 1 residuals for different observations are uncorrelated
- Level 2 and level 1 residuals are uncorrelated
- Residuals and covariates are uncorrelated

V, the correlation matrix

The correlation matrix gives the correlation between every pair of level 1 units in our dataset after controlling for the explanatory variables

Single level model

	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
1	1	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
2	0	1	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
3	0	0	1	0	0	0	0	0	0	0	0	0	0	$\cdots$
4	0	0	0	1	0	0	0	0	0	0	0	0	0	$\cdots$
5	0	0	0	0	1	0	0	0	0	0	0	0	0	$\cdots$
6	0	0	0	0	0	1	0	0	0	0	0	0	0	$\cdots$
7	0	0	0	0	0	0	1	0	0	0	0	0	0	$\cdots$
8	0	0	0	0	0	0	0	1	0	0	0	0	0	$\cdots$
9	0	0	0	0	0	0	0	0	1	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

V for random intercepts model

L2		1	1	1	1	2	2	3	3	3	3	$\cdots$
	L1	1	2	3	4	1	2	1	2	3	4	$\cdots$
1	1	1	ρ	ρ	ρ	0	0	0	0	0	0	$\cdots$
1	2	ρ	1	ρ	ρ	0	0	0	0	0	0	$\cdots$
1	3	ρ	ρ	1	ρ	0	0	0	0	0	0	$\cdots$
1	4	ρ	ρ	ρ	1	0	0	0	0	0	0	$\cdots$
2	1	0	0	0	0	1	ρ	0	0	0	0	$\cdots$
2	2	0	0	0	0	ρ	1	0	0	0	0	$\cdots$
3	1	0	0	0	0	0	0	1	ρ	ρ	ρ	$\cdots$
3	2	0	0	0	0	0	0	ρ	1	ρ	ρ	$\cdots$
3	3	0	0	0	0	0	0	ρ	ρ	1	ρ	$\cdots$
3	4	0	0	0	0	0	0	ρ	ρ	ρ	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- The correlation matrix is identical to the matrix for the variance components model

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1	4	ρ	ρ	ρ	1	0	0	0	0	0	0	$\cdots$
2	1	0	0	0	0	1	ρ	0	0	0	0	$\cdots$
2	2	0	0	0	0	ρ	1	0	0	0	0	$\cdots$
3	1	0	0	0	0	0	0	1	ρ	ρ	ρ	$\cdots$
3	2	0	0	0	0	0	0	ρ	1	ρ	ρ	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- The correlation matrix is identical to the matrix for the variance components model
- As expected, observations within the same group are correlated but observations from different groups are uncorrelated

More on correlation matrices

See also the audio presentation on our website at

http://www.cmm.bristol.ac.uk/learning-training/videos/index.shtml#correlation
(which gives details of how we derive the entries of these correlation matrices)

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Reminder

For a single level model, the residual for an observation is an estimate for e_i

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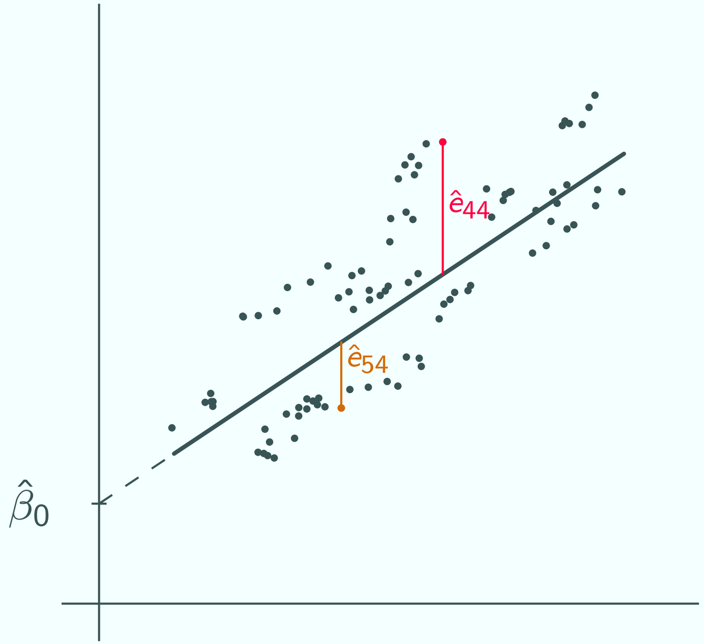


Residuals

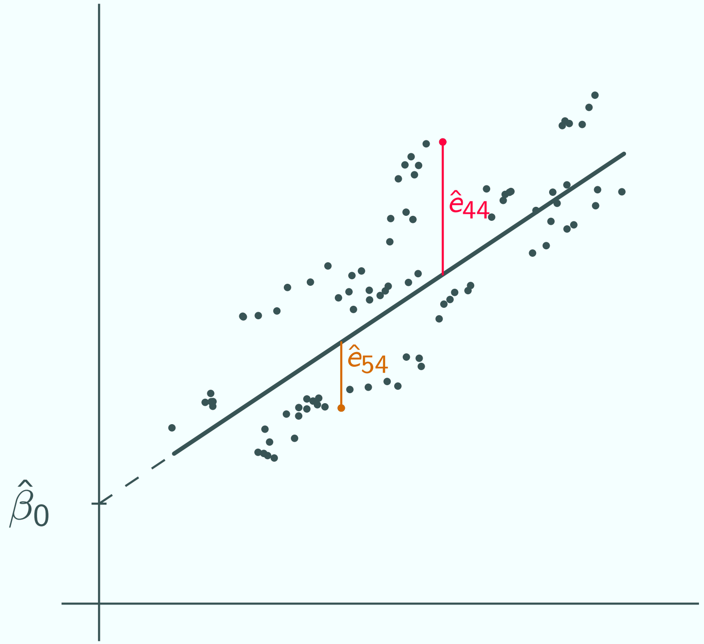
- Having looked at the correlation between residuals, let's look at the residuals themselves in more detail.
- Residuals are estimates for the random part
  - For a single level model, we often say 'estimates for the error term'

Reminder

For a single level model, the residual for an observation is an estimate for e_i

$$y_i = \beta_0 + \beta_1 x_i + e_i$$

If we write $\hat{y}_i = \beta_0 + \beta_1 x_i$ then $\hat{e}_i = y_i - \hat{y}_i$:
the observed value − the value predicted by the regression line



Why are we interested in the residuals?

Often we're not, but they can be useful in some cases:

Diagnostics

- We can plot the residuals to check their Normality
- This is part of checking how well the model fits

Rankings

Interest in a unit

Prediction/ visualisation

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Interest in a unit

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Prediction/ visualisation

- The level 2 residuals are needed to make predictions for individuals in a particular level 2 unit
- We need them to graph the group lines

Why are we interested in the residuals?

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- 'Was our Normality assumption justified?'

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Prediction/ visualisation

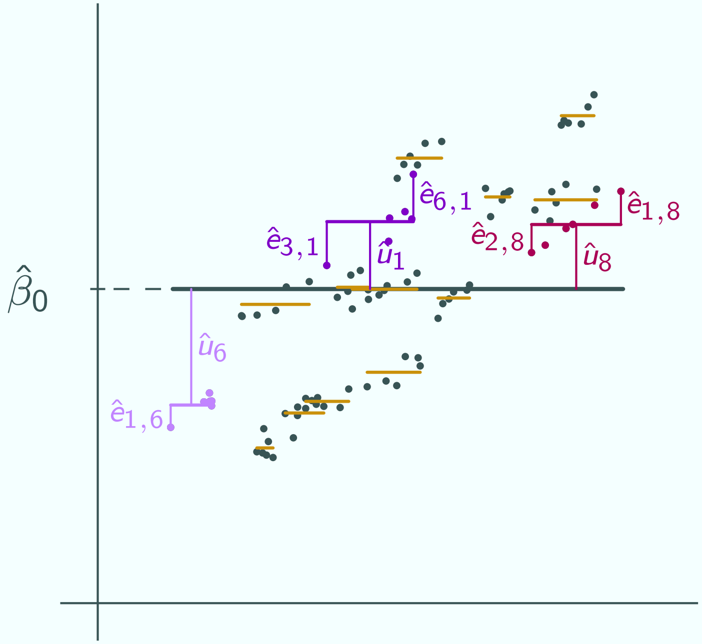
- 'What is the expected weight of a salmon from Fish Farm 28?'
- 'What does our model look like?'

Multilevel residuals

Variance components model

$$y_{ij} = \beta_0 + u_j + e_{ij}$$

Recall that now that we have 2 random terms, we have 2 kinds of residual:



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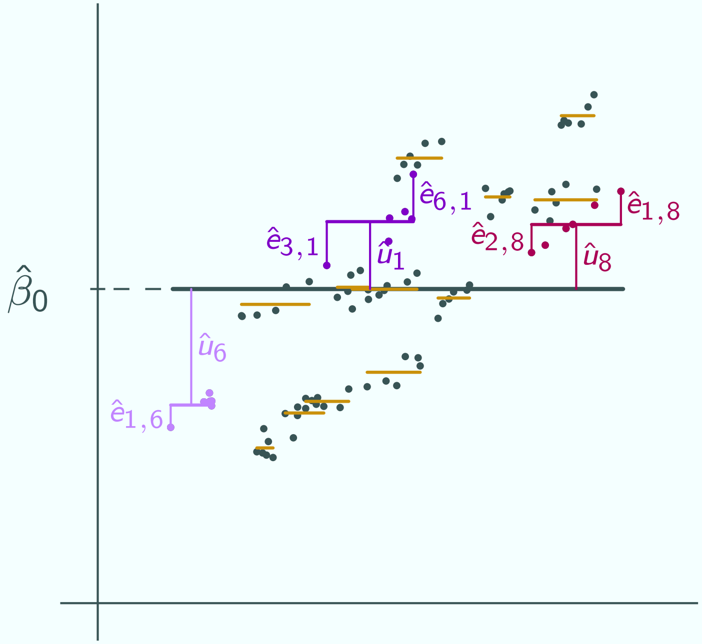

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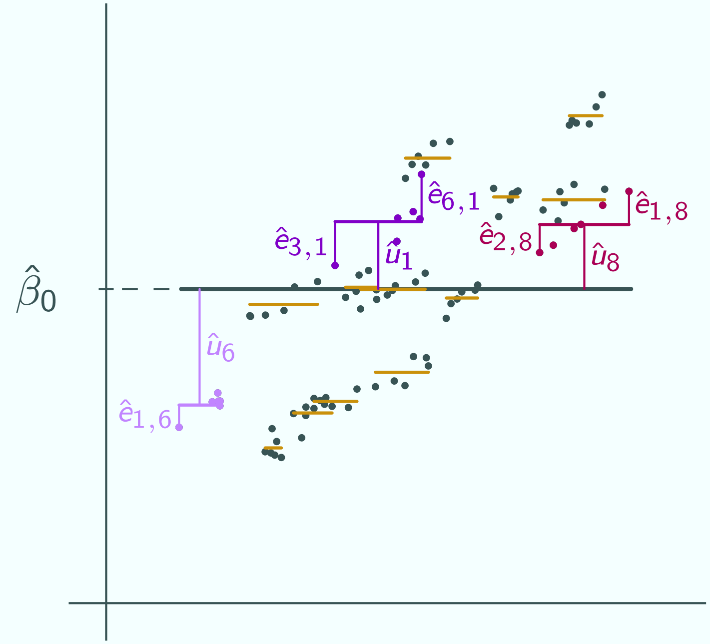

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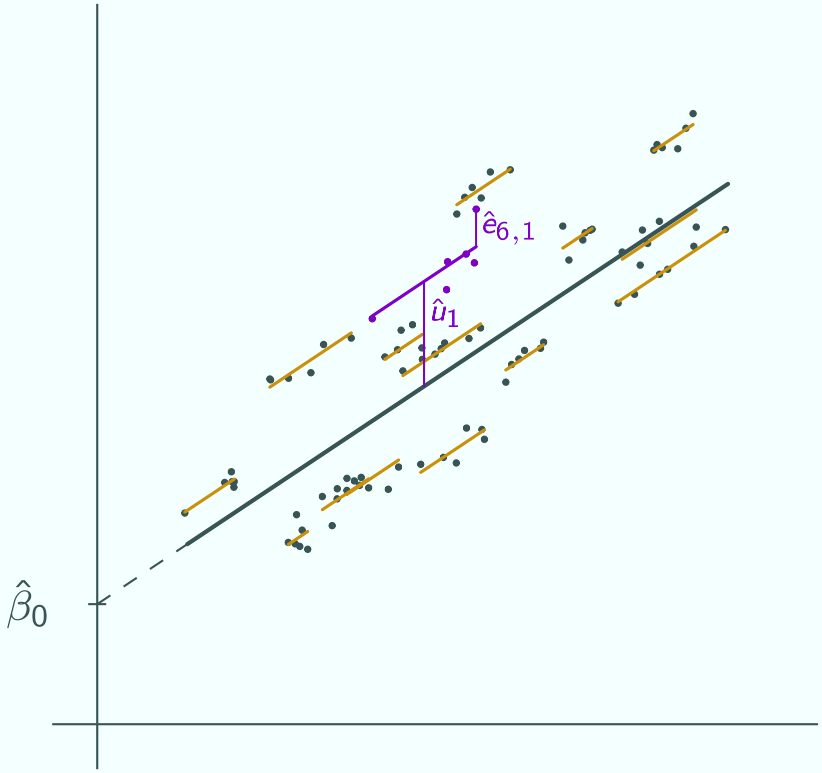

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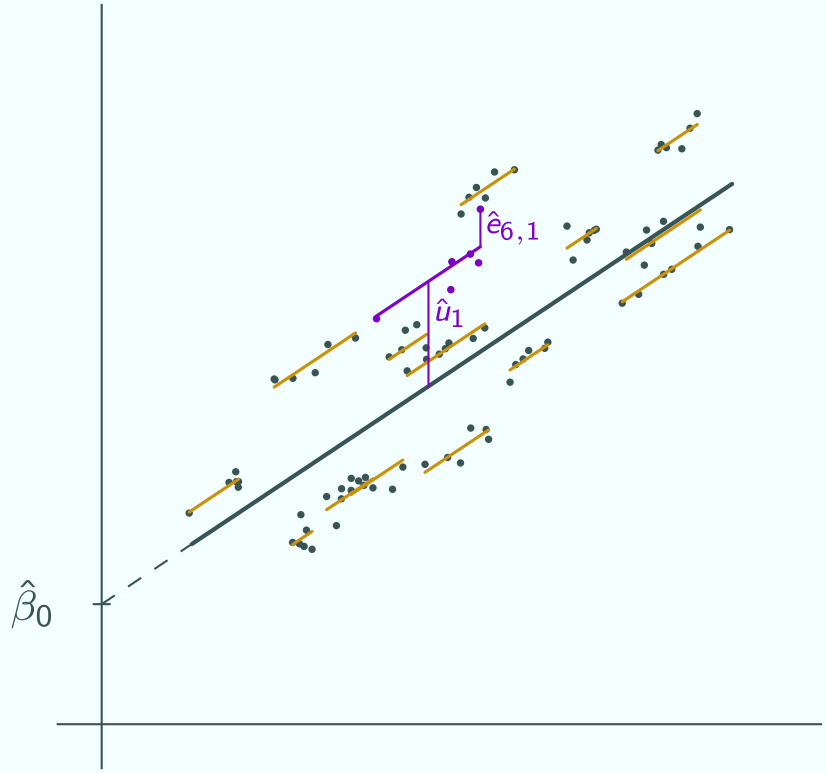

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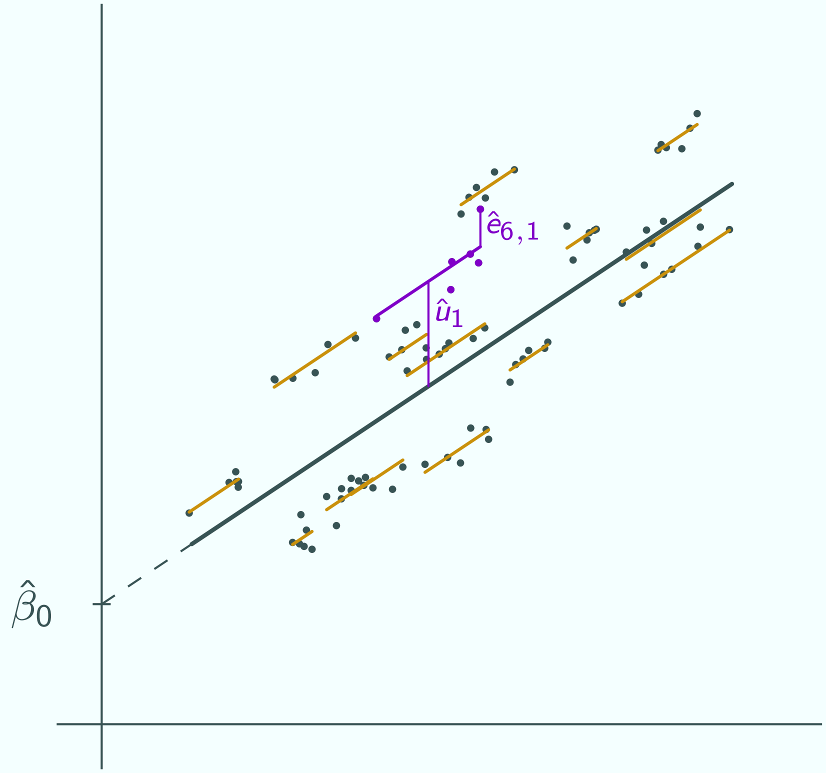

Calculation of residuals

Raw mean residual

- Recall $r_{ij} = y_{ij} - \hat{y}_{ij}$ and

Shrinkage factor

Level 2 residual

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- $\hat{e}_{ij} = y_{ij} - \hat{y}_{ij} - \hat{u}_j = r_{ij} - \hat{u}_j$

Why do we shrink? A thought experiment

The situation

- Suppose we have exam results for 6 pupils each from a number of schools

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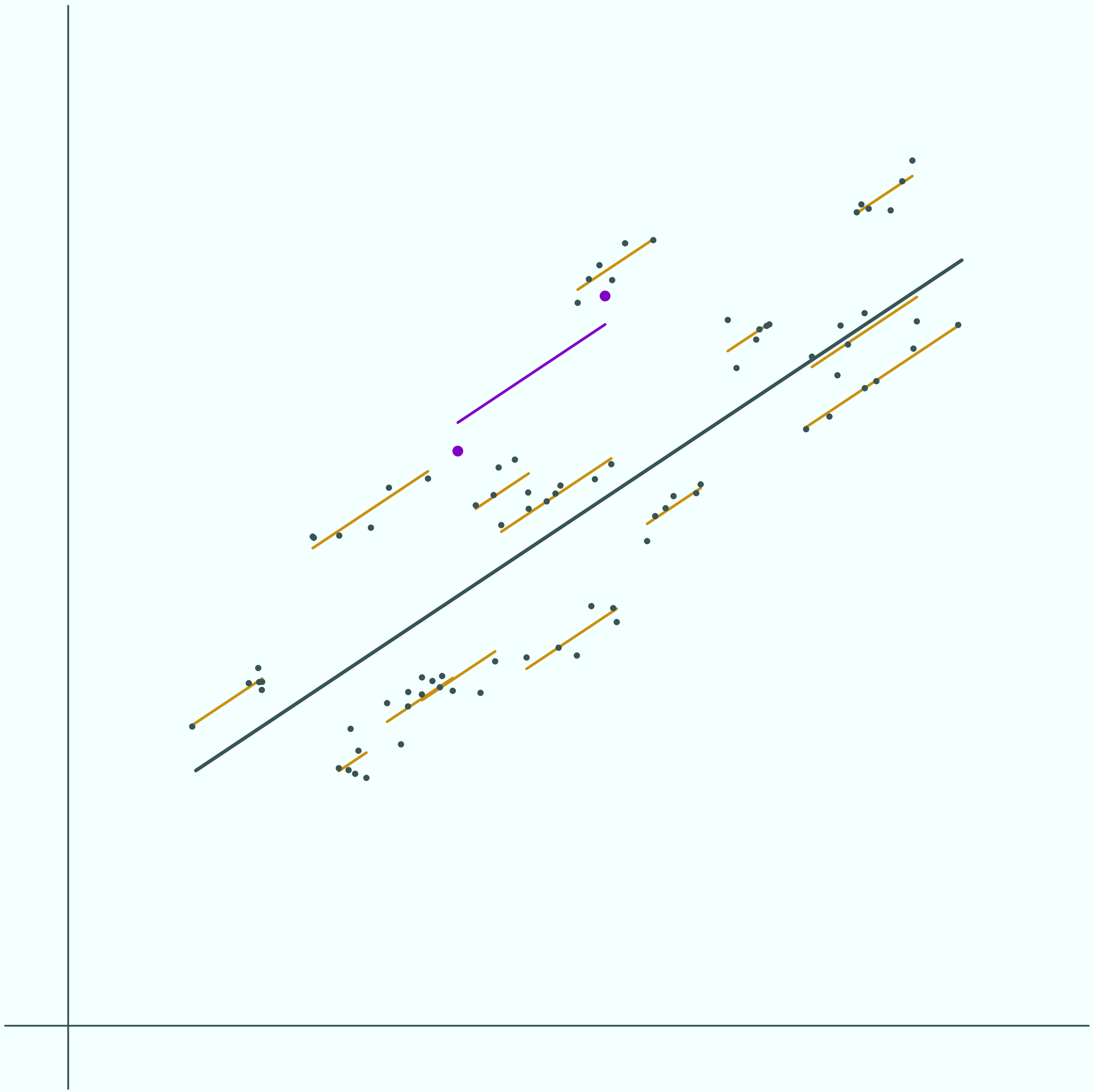
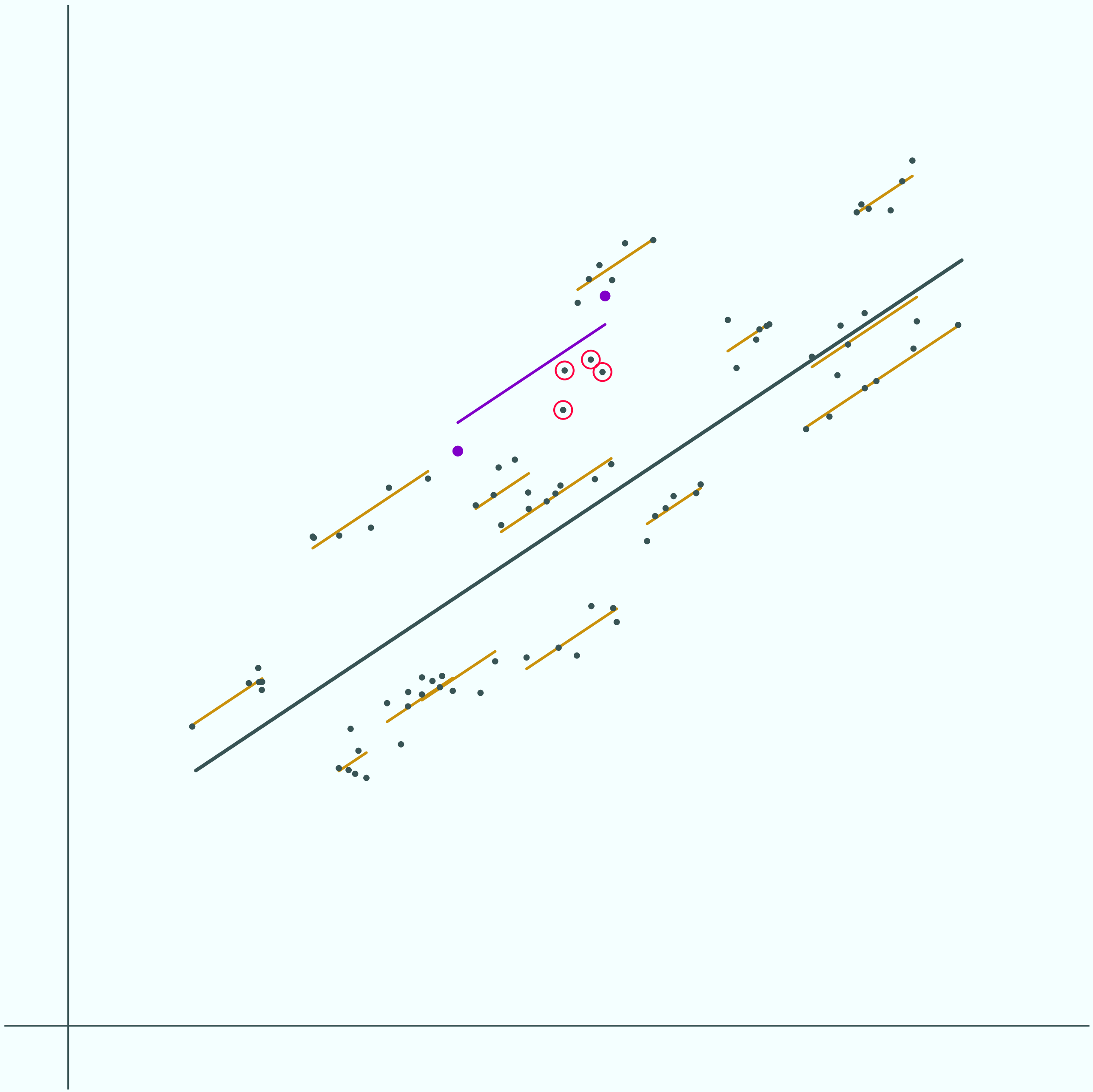
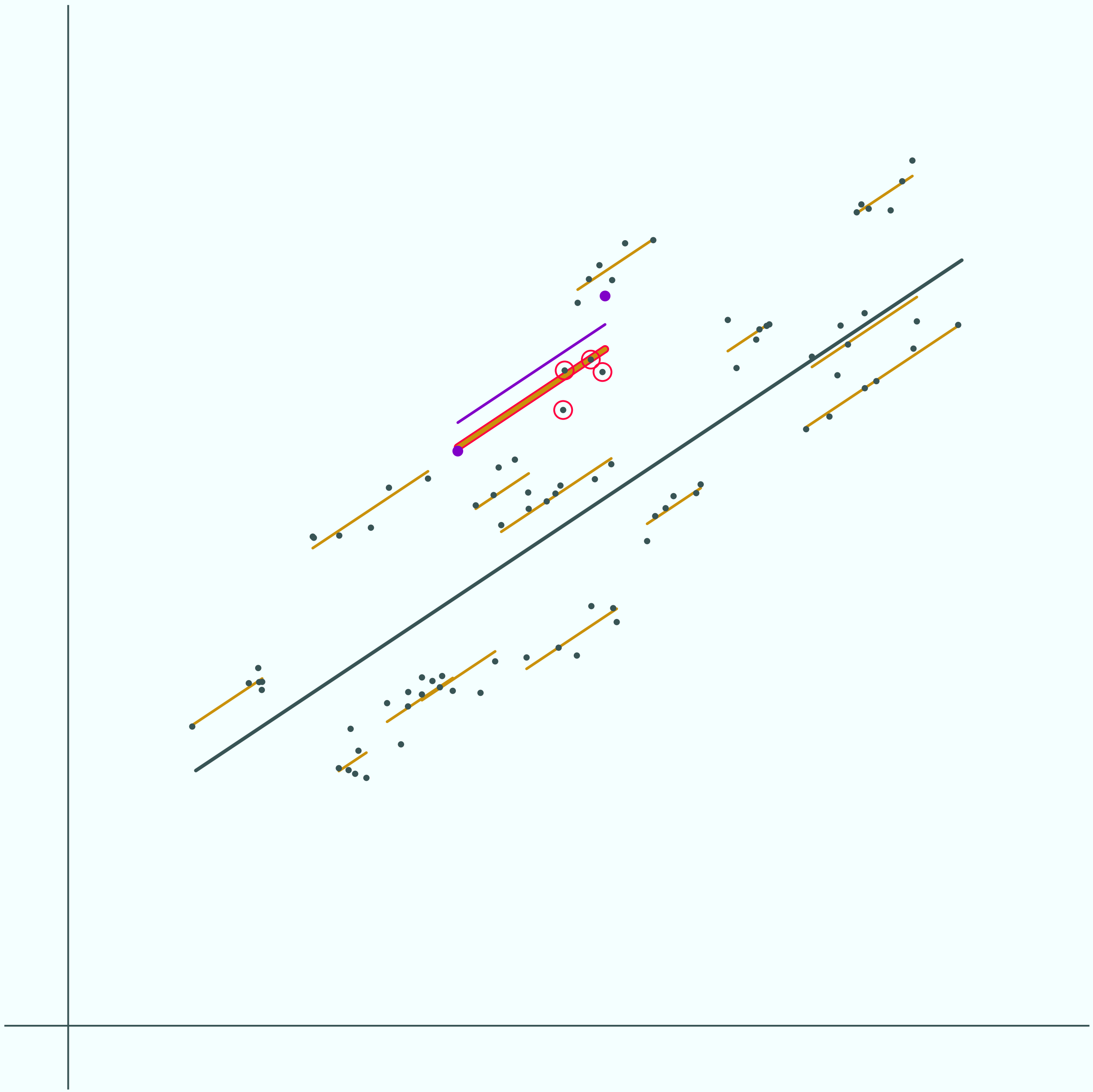
- It depends how "typical" those 2 pupils are of the full set of 6.
- Since we're picking 2 pupils from 6, it's quite likely that we might pick 2 untypical pupils
- Then the school line drawn using those 2 will be quite far from the school line using all 6 pupils- as happens in this example







Thought experiment cont.

Drawing the group line

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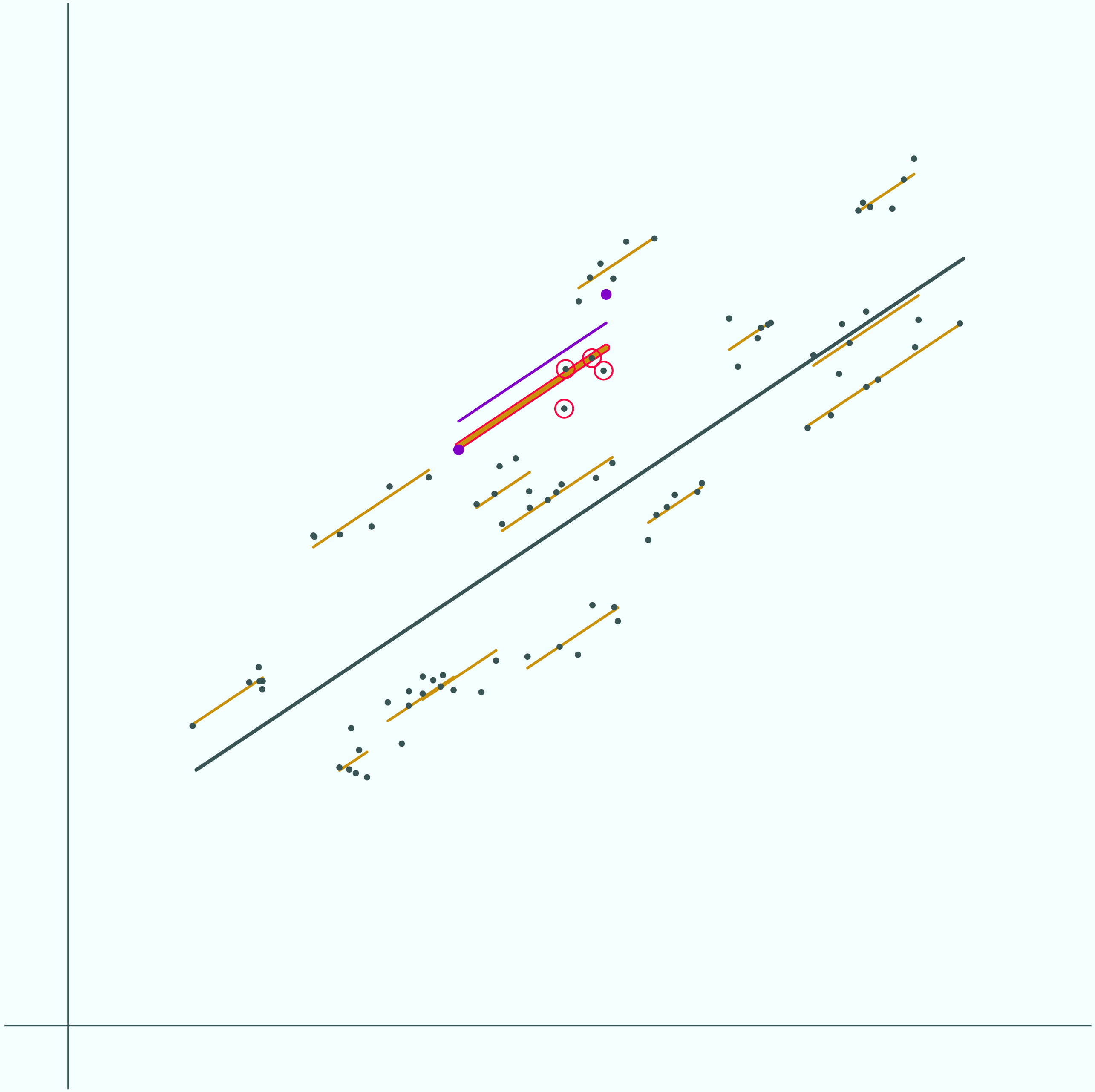
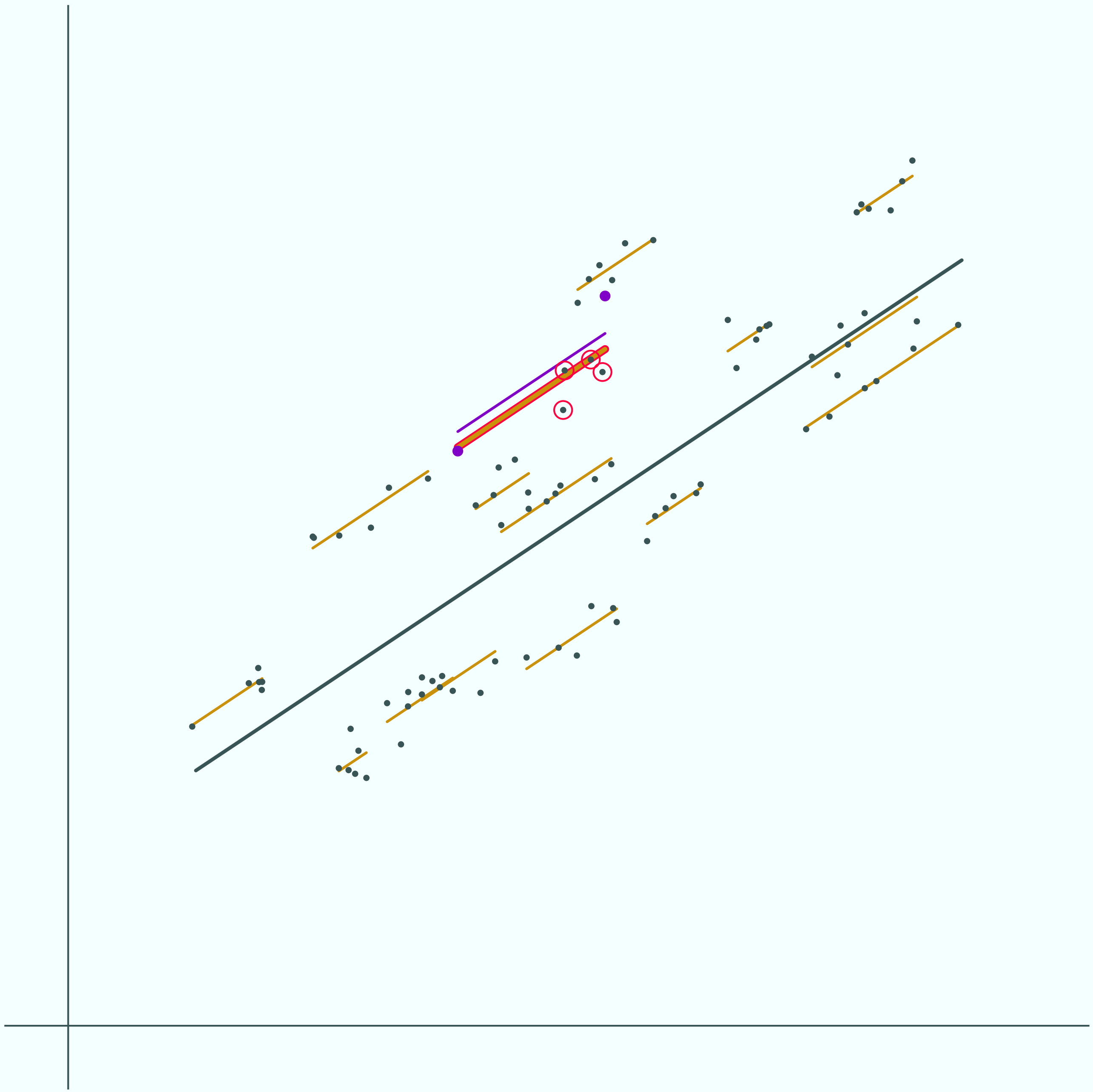
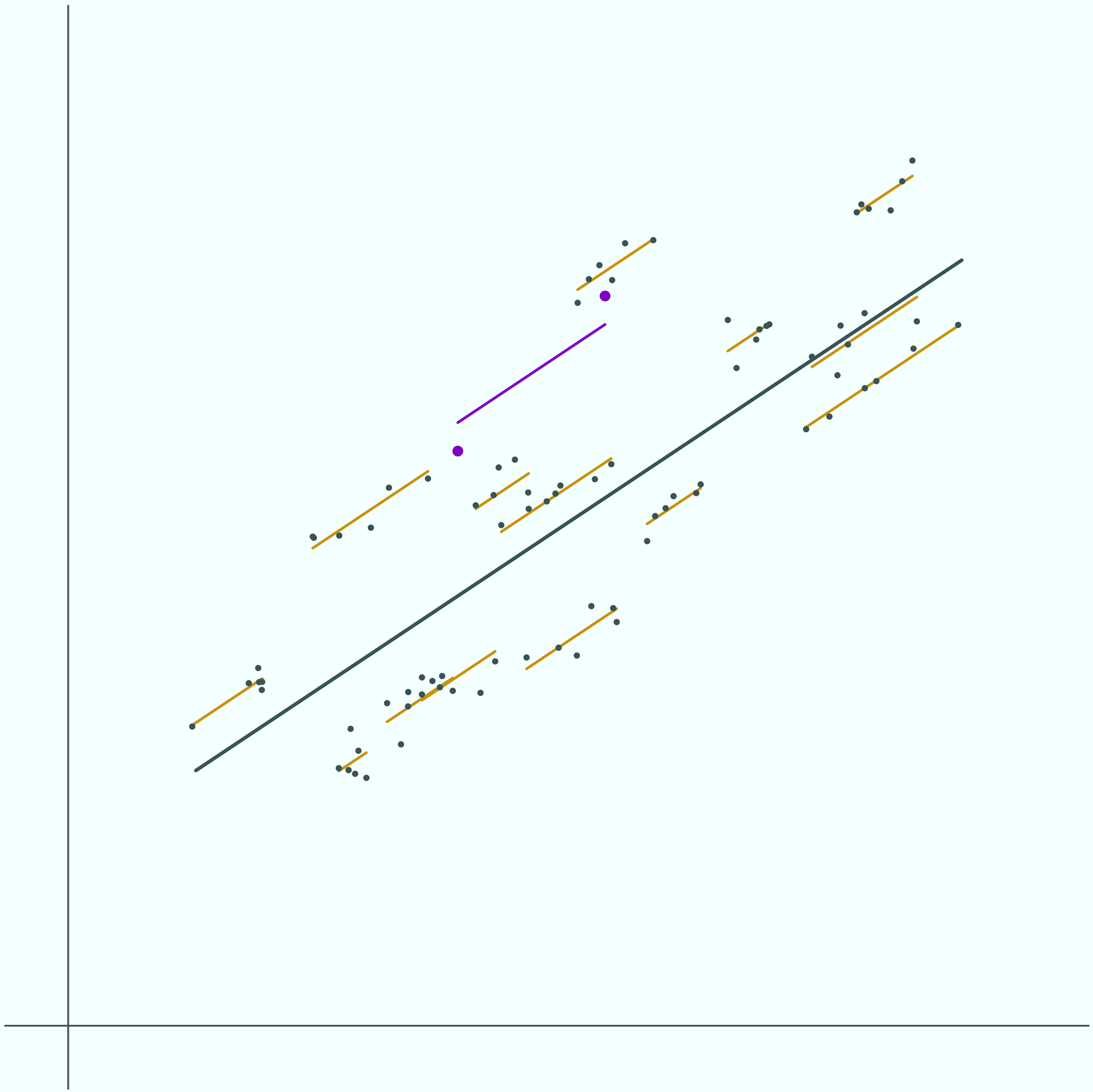
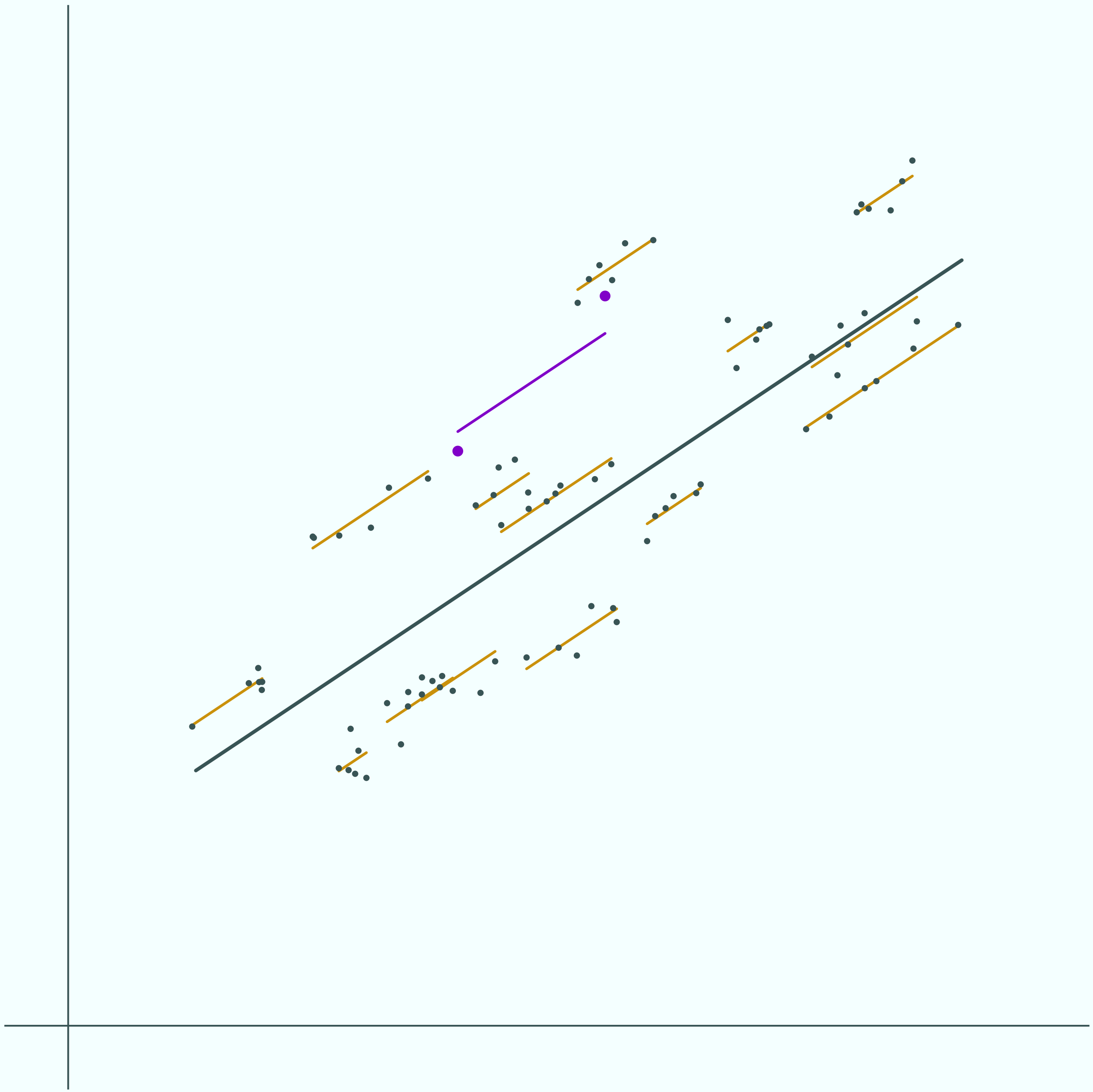
- So we can improve our positioning of the line by shrinking it in towards the overall average









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How does this generalise?

- We are in the same situation with our dataset as a whole
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- The position of the other school lines gives us information about the likely position of the pupils not included in the dataset

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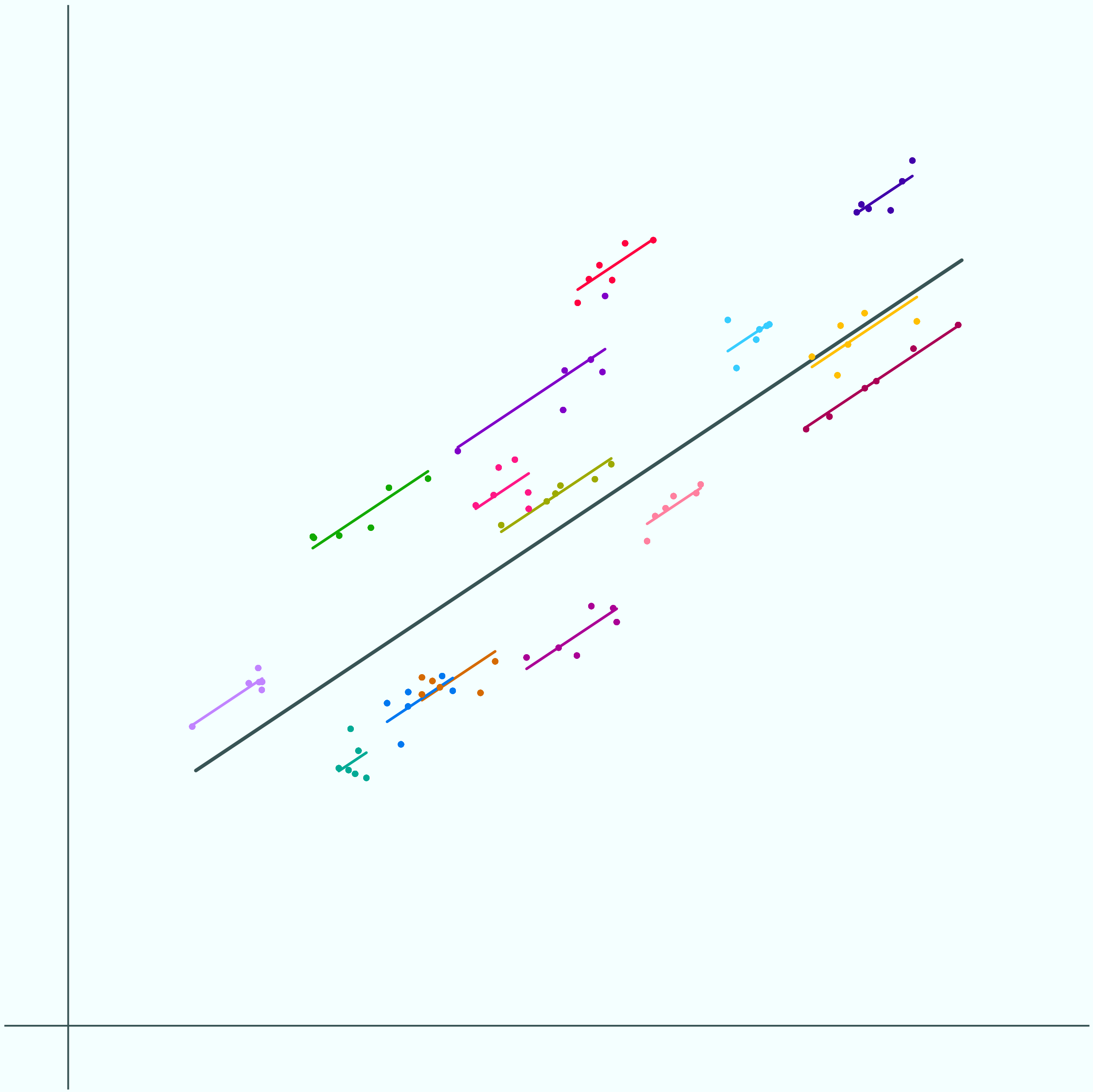
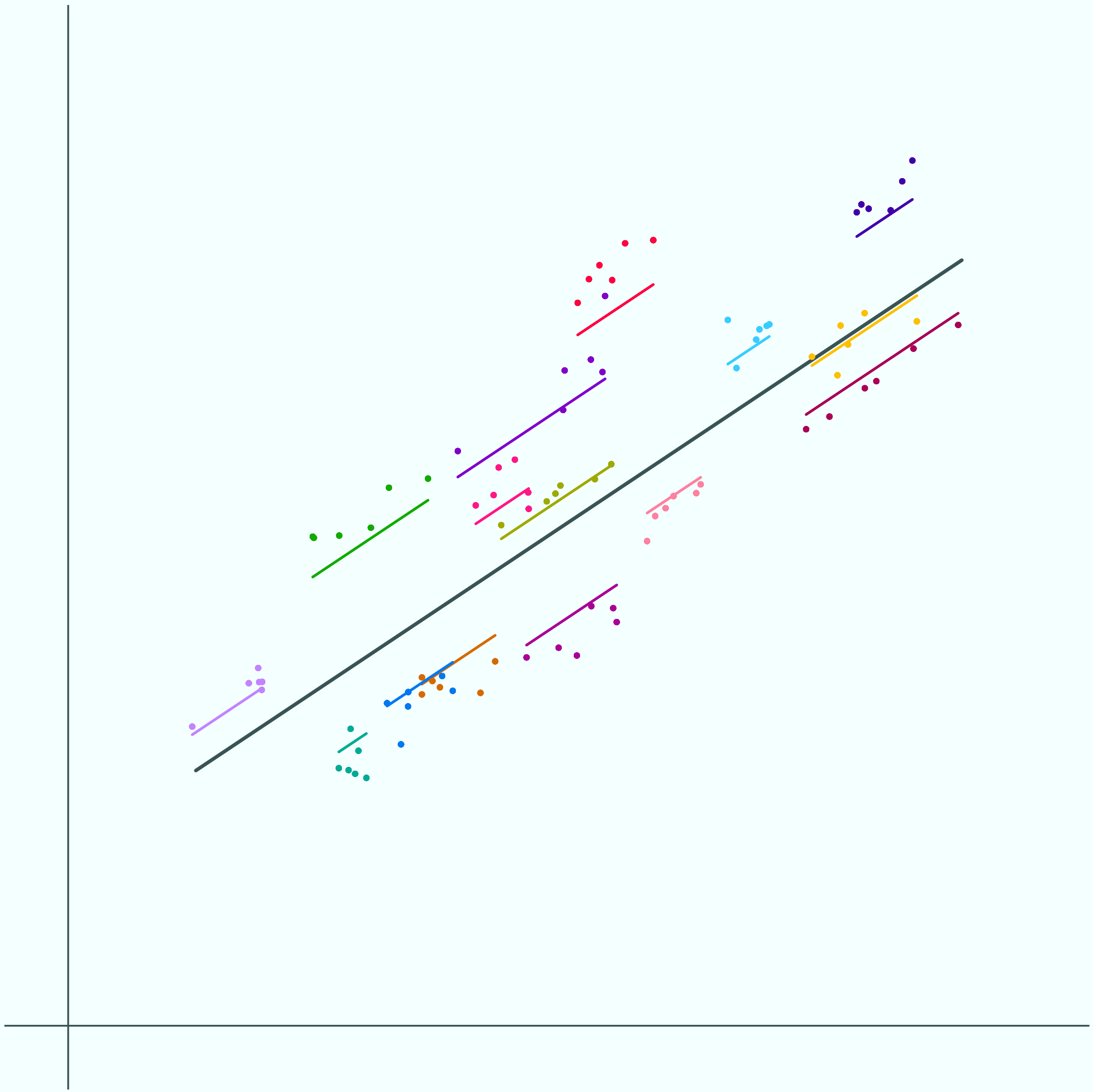
- Again, we get a better estimate of the group line by shrinking it in towards the overall average





Points to note about shrinkage

When do we shrink?

Always!

- We always shrink the residuals because we always have a sample from each level 2 unit

How much do we shrink by?

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- The amount we shrink by depends on the absolute number of level 1 units, not the proportion of the total for that level 2 unit

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  - even if we have 499 out of the 500 pupils attending a school
  - even if we have all the pupils attending all the schools in the UK
- Even if our dataset contains the whole population, we regard that as a sample from the superpopulation and shrink

How much do we shrink by?

Recall that:

- We don't shrink by a fixed amount
  - If we have 500 pupils from a school we shrink less than if we have 7
- The amount we shrink by depends on the absolute number of level 1 units, not the proportion of the total for that level 2 unit
- We can also see that the amount of shrinkage depends on the variances σ_u^2 and σ_e^2

Points to note about shrinkage

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- $\hat{\sigma}_u^2$ is the estimated variance of the level 2 units in the population not in our sample

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Model visualisation

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We focus on the second use

Visualising the model

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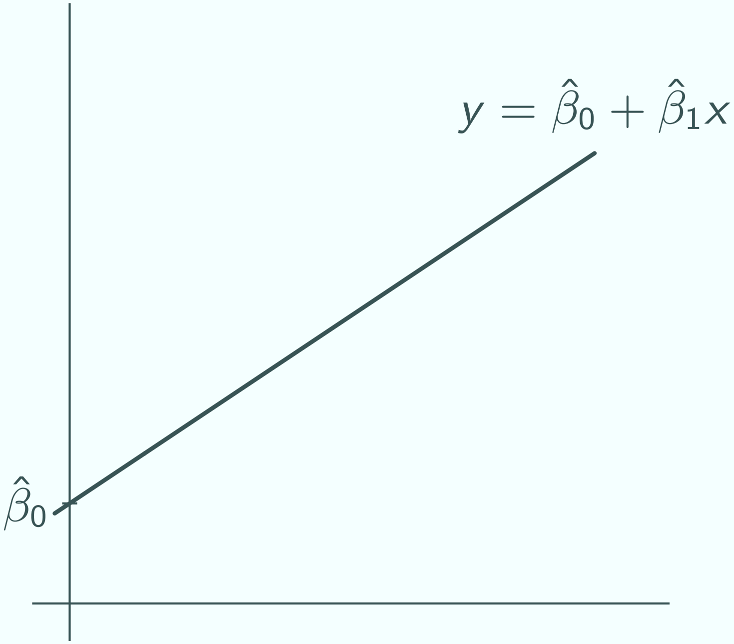
- We plot $y = \hat{\beta}_0 + \hat{\beta}_1 x$ to get our graph



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Visualising a single level model

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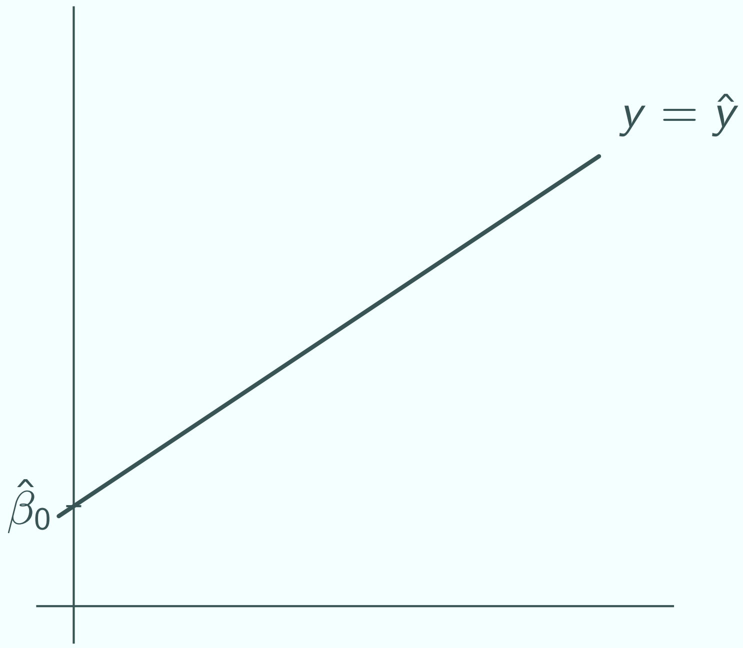

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- We plot $y = \hat{\beta}_0 + \hat{\beta}_1 x$ to get our graph
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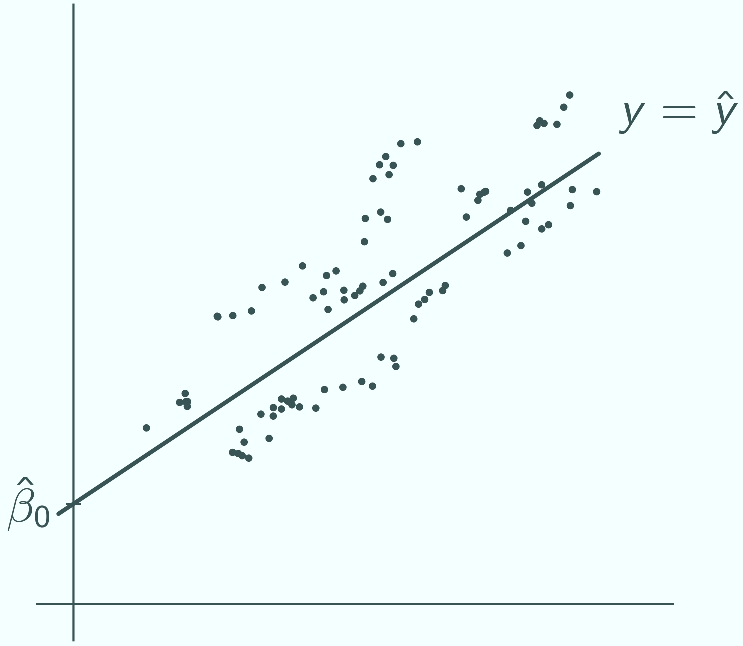
- We can add on the actual data points



Visualising the random intercepts model

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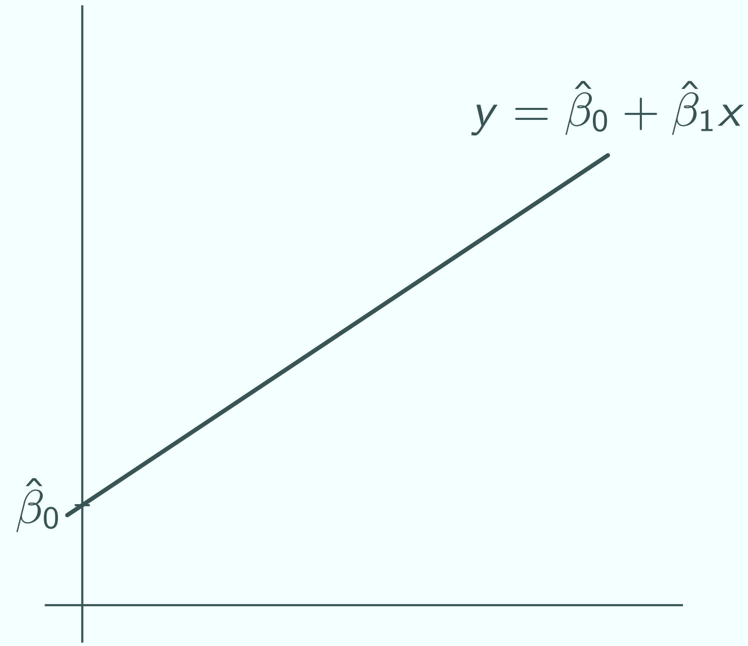

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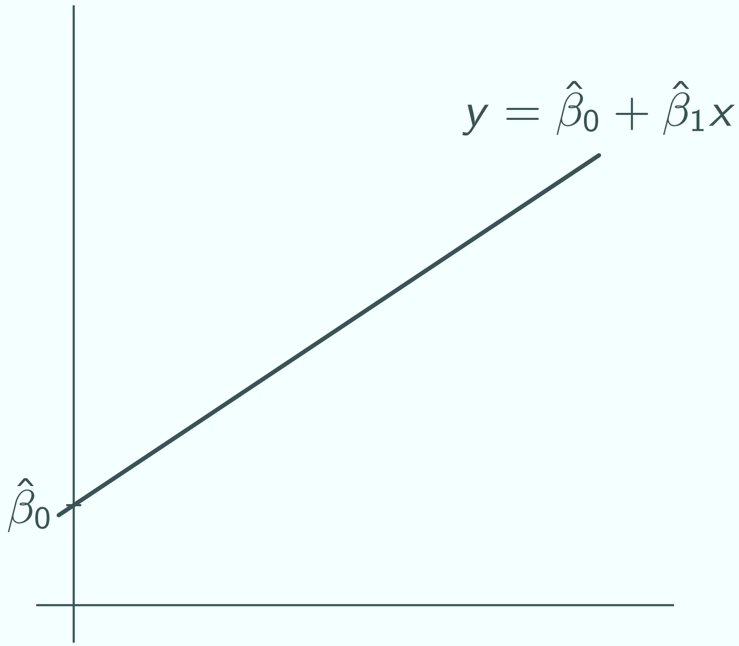

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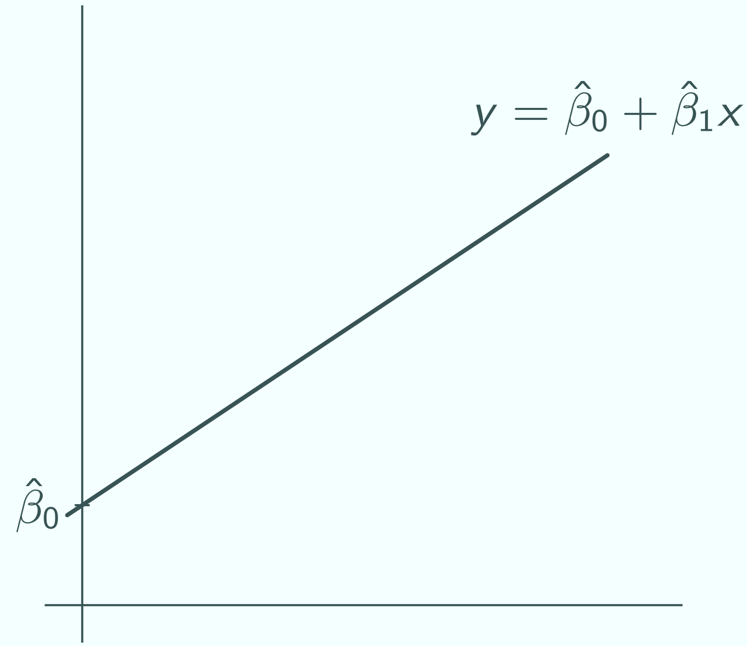

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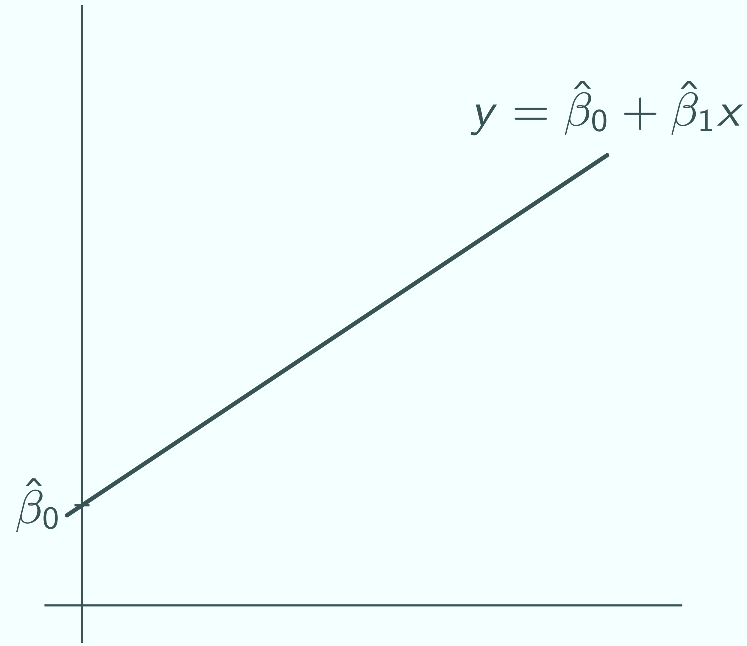

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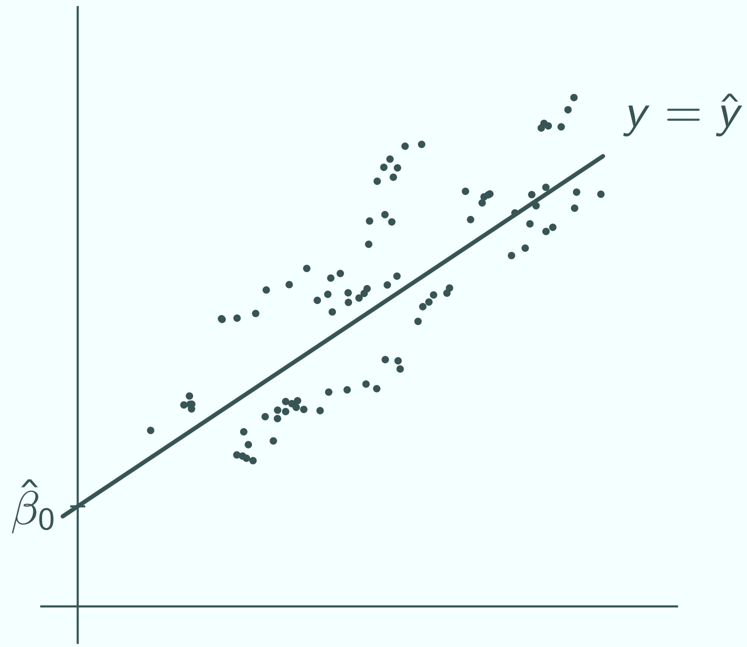
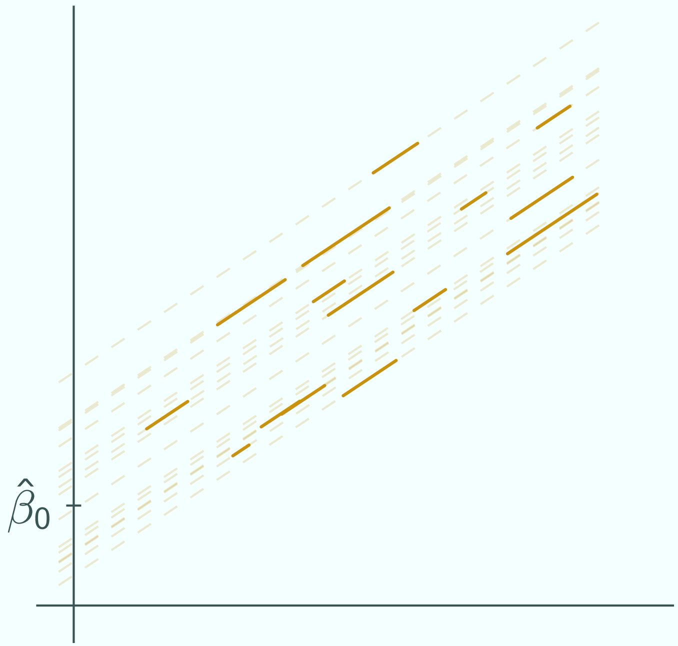


- The value of $\hat{y}_{ij}$ does not depend on the group j in this case, only the explanatory variables
- So this prediction only produces one line
- This is what we would predict if we didn't know which group a data point belonged to

Visualising the random intercepts model

Group lines

- Adding in the group residual u_j gives the group lines



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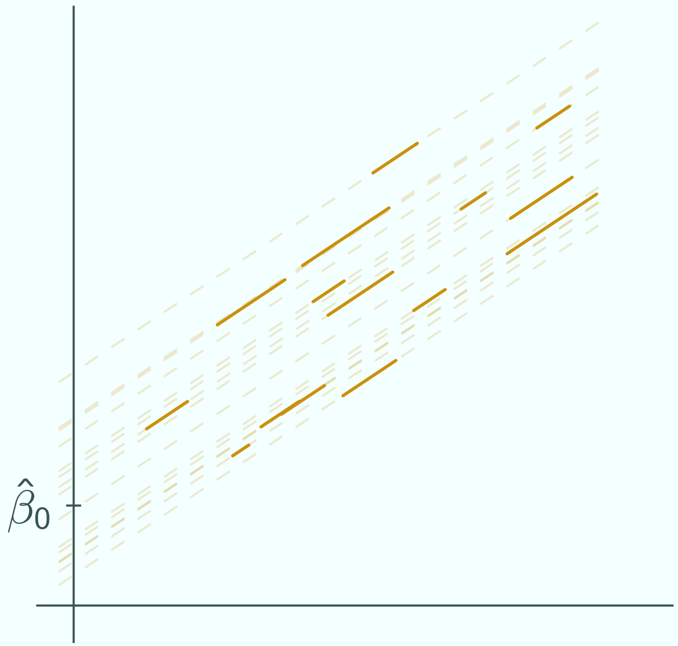

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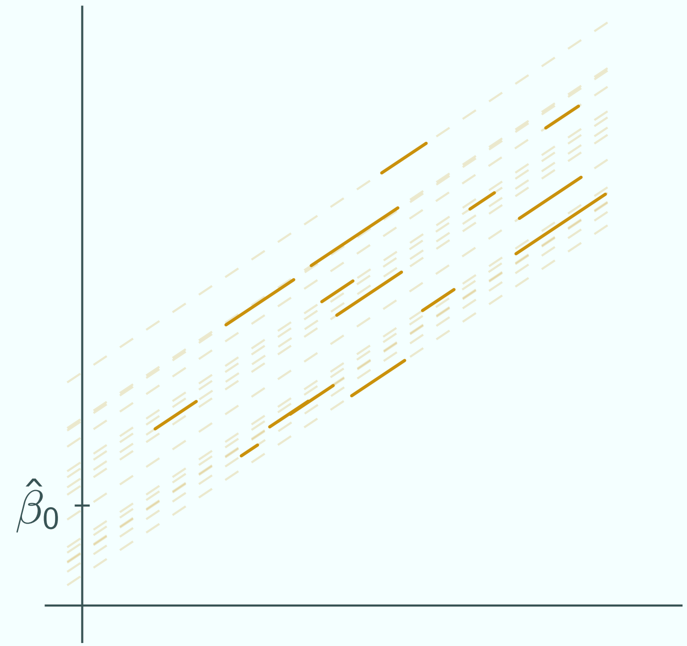

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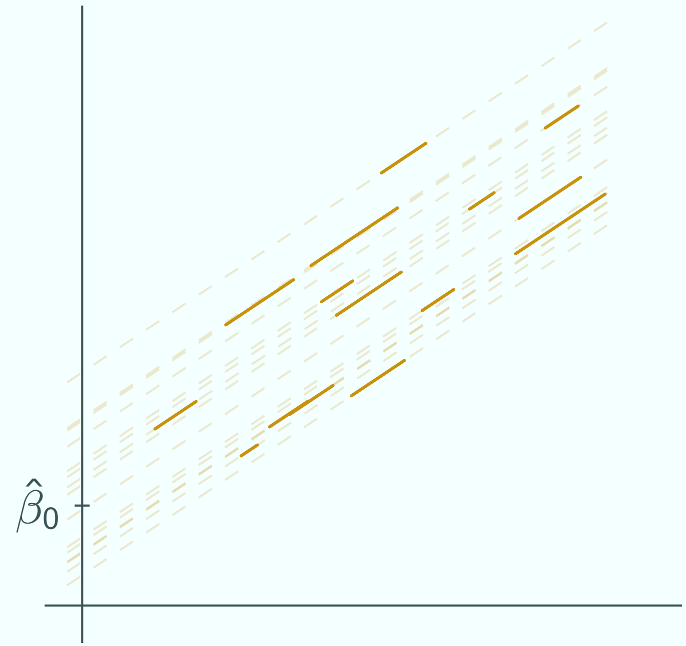

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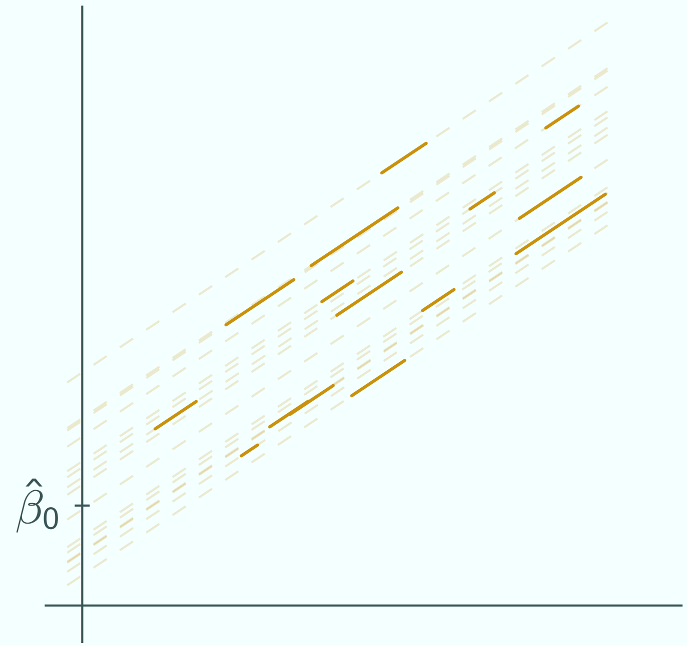

- So there is a different line for each group
- The group line is what we would predict if we did know which group a data point belonged to

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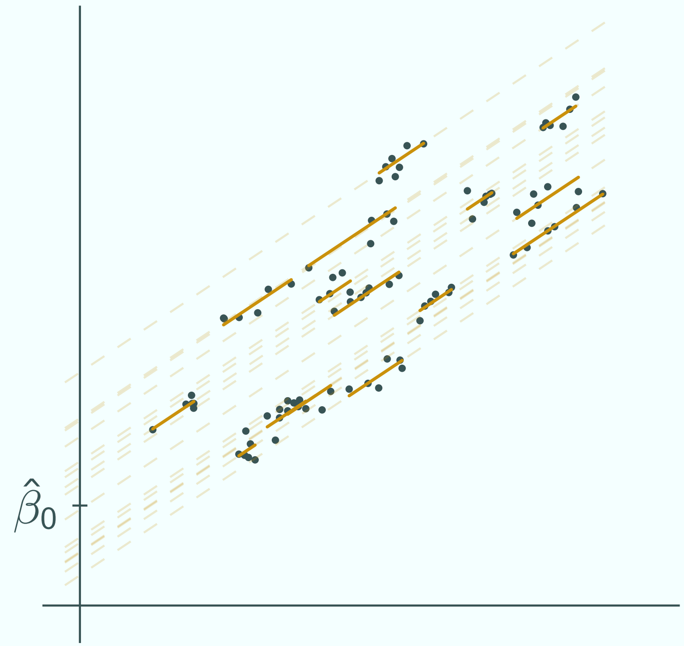

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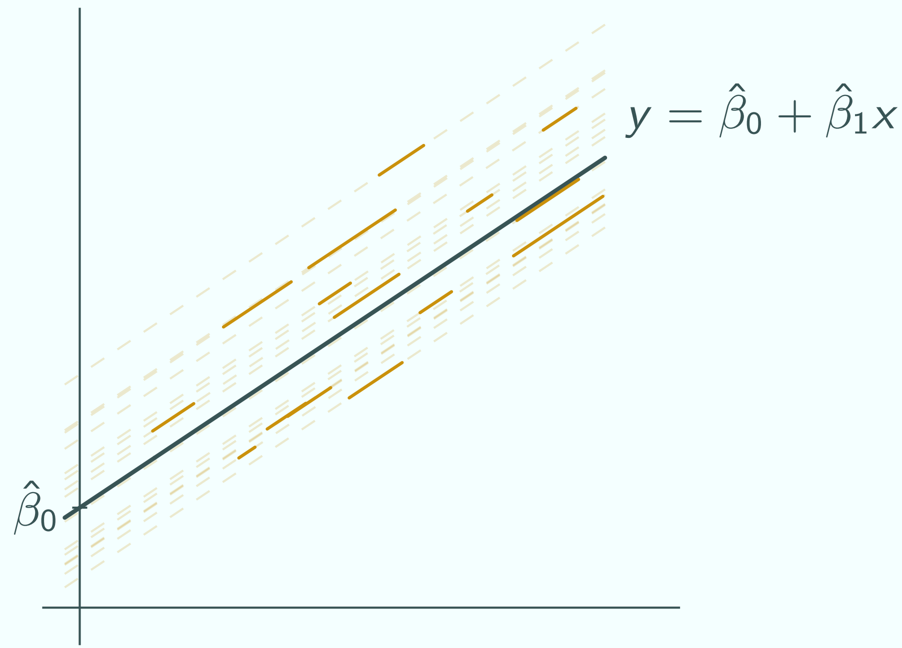
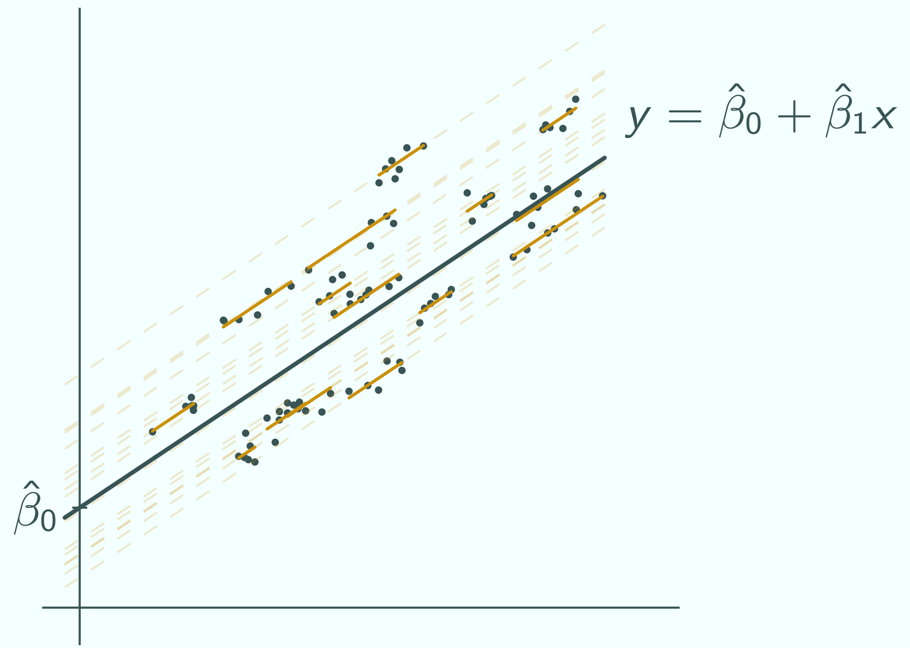


Complete model

- We can combine the predictions from the fixed and random part in one graph to get a complete visualisation of the model

Visualisng the random intercepts model



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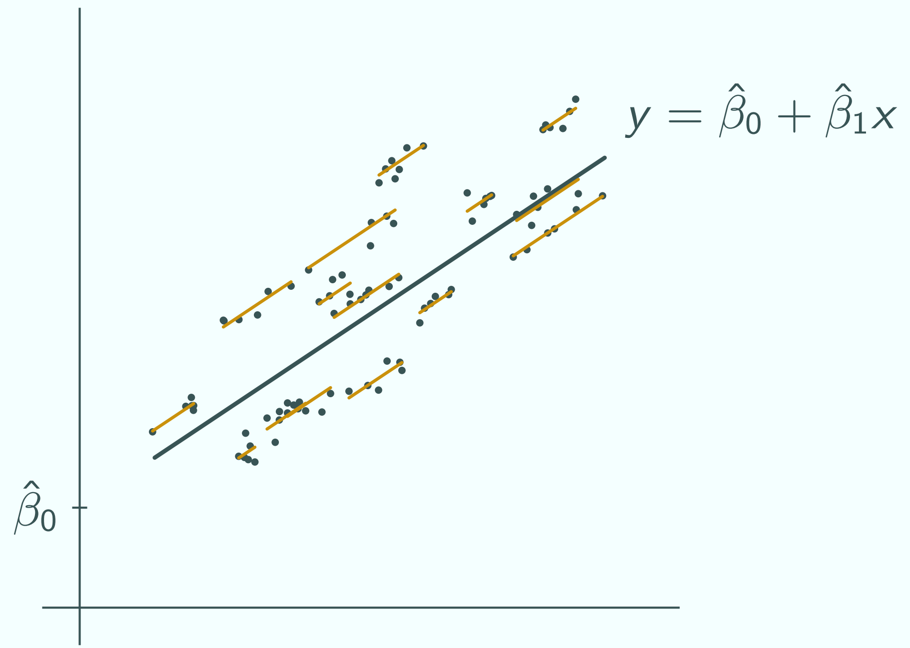

Complete model

- We can combine the predictions from the fixed and random part in one graph to get a complete visualisation of the model
- and we can add in the actual data points for comparison
- Usually we only plot predictions for the range of values we have in our dataset

Exercises: Session 2

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