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Multilevel covariance component models

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SUMMARY

A straightforward extension to the multilevel linear model with nested covariance components is described. This allows the specification and efficient estimation of a very general mixed linear model with both crossed and nested covariance components.

Some key words: Covariance component; Multilevel model; Mixed effects model; Variance component.

Goldstein (1986) describes the analysis of the multilevel mixed effects linear model with random coefficients, where the variance and covariance components have a nested structure across levels. The purpose of the present note is to show how a simple extension to the formulae in that paper can accommodate cross-classifications of the components within any level of the nesting, thus enabling quite general covariance component models to be specified and efficient parameter estimates obtained. For simplicity the 3-level model is used, with the extension to 4 or more levels being straightforward.

We write the random part of the 3-level model as

$$e = X_1\beta_1 + X_2\beta_2 + X_3\beta_3, \tag{1}$$

where X_i is the design matrix for level i and β_i is a vector of random variables at level i with $E(\beta_i) = 0$ and $\text{cov}(\beta_i) = \Omega_i$.

Appendix 2 of Goldstein (1986) shows how the inverse of the matrix $V_3 = E(ee^{\mathrm{T}})$ can be derived as a function of these design and covariance matrices, where at level i the elements of β_i have a single multivariate distribution over the units at that level. In many applications, however, the units at level i are themselves structured by a cross-classification. For example, in a 2-level educational model, students, i.e. level 1, may be cross-classified by the school they attend and the neighbourhood they live in. Thus the basic level-2 unit is the school/neighbourhood combination, and we may wish to describe the variation between these level-2 units as a sum of the variation or covariation between schools and that between neighbourhoods.

In (1) suppose now that there are c factors classifying the level-2 units. We can rewrite the second term in the sum on the right-hand side of (1) for the kth level-3 unit as

$$\textstyle\sum_j U_{kj}\beta_{k2j}, \tag{2}$$

where U_{kj} is the design matrix for the jth factor of the kth unit of level 3; U_{kj} is of order $n_k \times pq_{kj}$, where n_k is the number of level-1 units belonging to the kth level-3 unit, p is the number of explanatory variables defining the random variation at level 2, q_{kj} is the number of design variables for the jth factor and β_{k2j} is the $pq_{kj} \times 1$ vector of random variables defined over the q_{kj} levels of the jth factor.

We have $\text{cov}(\beta_{k2j}) = I \otimes \Omega_{2,j}$ and assume $\text{cov}(\beta_{k2j}, \beta_{k2m}) = 0$ $(j \neq m)$, where $\Omega_{2,j}$ is the $p \times p$ covariance matrix of the p random coefficients for factor j.

The contribution of the level-2 random terms to the kth block of V_3 can now be written as $\Sigma_j U_{kj}(1 \otimes \Omega_{2,j})U_{kj}^{\mathrm{T}}$, which reduces to the form given in Appendix 2 of Goldstein (1986) when $c = 1$. A similar result is obtained for classifications of the level-1 and level-3 units. Using the general results in Appendix 2 we calculate V_3^{-1}, which in general involves the inversion of matrices of order pq_{kj}; hence we obtain maximum likelihood and iterative generalized least-squares estimates for the parameters of the models discussed by Goldstein (1986). The computational procedures are more efficient than those which do not recognize explicitly a nested component of the data structure.

By allowing the random coefficients of a multilevel linear model to have a cross-classified structure, a very general class of covariance component models has been obtained, which has existing models as special cases. Thus, when $p = 1$ in a 2-level model with a simple random term at level 1 then we obtain the usual c-way variance components or mixed model. When the p random coefficients are in fact the p variates of a multivariate distribution (Goldstein, 1986, § 6.3), then we obtain the usual c-way covariance components or mixed model.

We also note that the 'factors' classifying the units may, for high-order designs, include interaction effects as well as main effects, and that not all the 'cells' of a design need to be present.

Reference

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