Abstract

We investigate Brownian motion and Laplacians in various geometries, their interplay with quantum spin/particle systems, with particular emphasis on phase transitions, long range order, Bose-Einstein condensation. We also plan to investigate related problems of asymptotics and scaling of random processes with path-wise self-interaction, exploit the power of stochastic representations of quantum systems. Special emphasis will be put on the interdisciplinary (mathematics/physics) aspects of the project. Senior participants at all nodes have remarkable record in these investigations. Joining their expertise will lead to new and efficient approaches to these fundamental and difficult problems.

Context and aim of the programme

Context and background:
The interplay of Brownian motion, its geometry and its applications to other parts of mathematics and sciences has a rich and long history. Frank Spitzer showed in a seminal paper (1962) how the heat flowing from a region in Euclidean space into its complement is governed by the Newtonian capacity. Spitzer's analysis relies on the probabilistic – Brownian motion – representation of the solution of the heat equation. Around the same time Mark Kac wrote his milestone paper "Can one hear the shape of a drum?" (1966). Here the object of investigation was the asymptotic behaviour of the heat kernel trace for a planar region with a polygonal boundary. Again the analysis relies on the probabilistic representation of the heat kernel this time in terms of a Brownian bridge. Further on in time, during the 1970-s the importance of stochastic representations in quantum statistical physics and quantum field theory was realized: random walk and diffusion representations of quantum correlations led to much better understanding of fundamental problems, such as long range order and phase transitions in these systems. These computations also pointed out the close link between the deep problems of quantum statistical physics and those of interacting random walks – in particular variants of self-avoiding walks. The research planned within the frame of the present network concentrates on three intertwined subjects. The senior participants all made ground-breaking contributions and continue to produce major results in these areas.

(1) Spectral theory of Laplacians in fractal geometries:
We study spectral properties of Laplace-type operators under a variety of boundary conditions, and restricted geometries, including fractal boundaries such as the von Koch snow flake. These investigations lead to refined understanding of long and short time asymptotics of heat transfer through the boundary. In the crushed ice model a collection of ice balls is arranged in cuboidal geometry. The number of ice balls in each layer increases, while their size decreases near the bottom of the cube. Depending on these rates of increase and decrease this self-similar arrangement exhibits rich behaviour (surface area dominated versus capacity dominated regimes). However, these phenomena are far from being mathematically fully understood yet. We plan to complement the already existing results by fully exploiting the relations with Brownian motion and its asymptotic analysis.
A related problem is the geometry of the regions left empty by a Brownian motion in a compact domain. In recent work the expectation and large deviation probabilities of the total heat content and the largest in-radius of these regions was computed in dimensions three and more, in the large time limit. Precise description was obtained of what the Brownian motion looks like in the vicinity of the empty regions. The problem is completely open and much harder in two-dimensions. The geometry of the empty regions appears to exhibit strong space-time correlations. There are deep connections with two-dimensional interlacement percolation and with branching Brownian motion, which are also hot topics in probability theory. Our aim is to elucidate these connections and extend the results to the two-dimensional cases.

(2) Stochastic representations in quantum statistical physics and their use:
Since the mid-nineties various efficient stochastic representations of quantum spin and particle systems were introduced, based on Feynman-Kac and/or Trotter formulae. These representations helped to understand the behaviour of quantum systems in the thermodynamic limit: providing various rigorous bounds on the long range order parameters for interacting boson gas and spin systems, but there is much more potential in these representations.
Thanks to these representations, many interesting problems of condensed matter physics can be given a probabilistic formulation. Probabilists have already started working on models of quantum mechanical origin. One objective of the present proposal is to encourage probabilists to look into some deep questions originating from physics, and to transfer the outcomes back to condensed matter theory. We plan to further investigate the still open questions related to mathematically rigorous proof of existence of long range order at low positive

temperature in various quantum spin systems.
We investigate an *interacting* Bose gas on the Bethe-lattice. Topological simplicity of the underlying tree structure facilitates understanding the effect of a repulsive interaction on long range order. We propose a new type of stochastic representation of the spin-1/2 quantum Heisenberg ferromagnet on lattices. Applying this approach to the quantum Heisenberg ferromagnet on Bethe-lattice we arrive at an analytic problem related to Brownian motion on Lie-groups. Finally, the problem of establishing the transition between phases without/with long range order is formulated as a non-linear bifurcation problem in an infinite dimensional space. Similar methods applied to the quantum Ising model with transversal external field leads to a similar but somewhat simpler problem which is still widely open problem.

(3) Self-interacting random walks and diffusions:
We study the asymptotic behaviour of self-interacting, in particular self-repelling or self-avoiding random walks. One of the most prominent open problems in probability theory is to prove super-diffusivity for the self-avoiding standard random walk on $\boldsymbol{Z}^d$, $d=2,3$, i.e. to prove that the mean end-to-end distance of an n-step walk is larger than of order *.* The so-called *elastic* models in $d=1$ are also interesting and challenging. The model has two parameters, one that corresponds to (inverse) temperature and another one which controls the polynomial decay of interaction between positions, as function of separation distance. The model is expected to show a rich phase diagram, showing similarity to the standard self-avoiding walk problem: it is expected to be ballistic in one parameter range, sub-ballistic but super-diffusive in another range, and diffusive in a third one. Only a small part of these conjectures is proved. In particular nothing is rigorously known for the most interesting range where sub-ballistic and super-diffusive behaviour is expected. Convincing, but mathematically non-rigorous, scaling arguments for the scaling exponents are available. It appears that this model is simpler than the standard self-avoiding walk in two and three dimensions, but still very challenging. We plan to develop a rigorous a renormalization argument to prove super-diffusivity.

Description of the institutions involved

The five nodes (Bristol, Leiden, Warwick, Zürich universities and Technion Haifa) are worldwide leading research centres in the mathematical and physical sciences. The leading senior scientists, formally representing their institutions and coordinating the research in each and across the nodes have a track record with substantial lists of publications in leading journals, plenary talks at prestigious international meetings, and supervision of postgraduate research as well as organizational expertise. We list the scientific strength of the various nodes. Bristol: spectral analysis and variational calculus (van den Berg), probability and statistical physics (Tóth); Leiden: large deviations and variational calculus (den Hollander); Warwick: random loop representations (Ueltschi); Zürich: random walks, large deviations (Bolthausen); Haifa: mathematical physics, large deviations (Ioffe). Besides the scientists named above, at each node there are a considerable number of heavy weight senior researchers and younger colleagues who will take part in the project. More details about the node institutions are available in the section “network partners” of this application.

Significance of the Network

The proposed work entails the understanding of one of the most challenging problems in mathematical physics: the mathematical mechanism of phase transitions and long range order in quantum statistical systems such as Heisenberg ferromagnet and the interacting Bose gas. Dissemination is via leading international journals, lectures at international conferences, and a dedicated web-site. The successful outcome of the proposed project depends crucially on the participation and interaction between the institutions (2 UK, 3 Overseas). Even though there are some established collaborations already (van den Berg-Bolthausen-den Hollander, Bolthausen-Ioffe, Ioffe-Toth) most of these interactions will be newly constituted. A further important feature of the proposal is the multidisciplinary character of the research. While the models where formulated by physicists, substantial progress in the understanding of the physics will require tools from a whole range of mathematical branches in particular, stochastic analysis, combinatorics, spectral theory, calculus of variations and tools from mathematical physics.

Arrangements for the Network

We plan an opening and a closing workshop, with approximately 20-25 active participants. (Estimated budget: 20 kGBP each.) We plan yearly 15 short (up to one week) and 2 longer (one month) visits between the various nodes of the network. (Estimated budget: 21 kGBP per year.) These individual visits will facilitate scientific interaction and progress of joint research. We plan yearly one mini-workshop at various nodes, with 5-8 participants, concentrating on one particular topics of the project. (Budget included in planned short visits.) The visiting participant will give seminar and/or colloquium lectures at the visited department, presenting the actual state of her/his research related to the project activity. Special attention will be put on the activity and mobility of the young participants, postgraduate students and postdoctoral researchers involved in the project.