Quid pro Quo: Friendly Information Exchange between Rivals

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We show that information exchange via disclosure is possible in equilibrium even if only one party benefits from the information *ex post*. The incentive to disclose results either from an expectation of disclosure being reciprocated – the *quid pro quo* motive – or from the possibility of learning from the rival's failure to act in response to a disclosure – the screening motive. Alternating and gradual disclosures are generally indispensable for information exchange and the number of disclosure rounds grows without bound if the agents' initial information becomes sufficiently diffuse – in that sense, the less informed agents are the more they talk. Patient individuals can achieve efficiency by means of continuous alternating disclosures of limited amounts of information. This provides a rationale for protracted dialogues.

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1 Introduction

Can two decision makers each of whom have critical information concerning which of a number of possible actions is the correct one share that information when both have a preference for acting on it alone? This question naturally arises, for example, in R&D races/joint ventures, or when multiple government agencies collect information intended to avert a terrorist attack or when separate researchers work on a common problem, as happened in the pursuit of a proof of Fermat's Last Theorem. While there may be a common benefit to making the correct decision, e.g. when a new technology

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is developed, a terrorist is arrested in the planning stage of an attack or a chosen proof strategy yields results, the desire to be the principal beneficiary of an invention or to receive primary credit may stand in the way of information sharing.

Suppose, for example, that each of two rival intelligence agencies conducted an independent investigation of a crime and came up with a list of multiple suspects. If they knew that combining their information would reveal who is responsible, would they voluntarily share their information, even when both are motivated to be the first to identify the true culprit? If they did, in what manner would/should the information exchange take place? This paper provides some new insights into these questions by delineating the key factors that incentivize information exchange in such environments. The main findings are: (1) disclosures are made in anticipation of obtaining information in return; (2) due to the risk from disclosing too much information, information sharing is necessarily gradual, requiring multiple rounds of alternating disclosures; (3) the necessary number of disclosure rounds to guarantee that the truth will eventually be discovered grows without bound as initial information becomes more diffuse; and, (4) irrespective of the initial information, as long as the payoff from taking the correct action is at least double the disutility from the rival taking the correct action, there always exists an equilibrium in which information exchange continues until the truth is discovered.

In this paper we investigate the case where monetary incentives are not available and instead individuals are motivated by concerns for the future. Then there are two possible reasons for providing information, the *quid-pro-quo* reason, that arises from the expectation that information will be disclosed by the other party if (and only if) information is first disclosed to that party, and the *screening* reason, that relies on the fact that information may be gleaned from others not acting on information provided. The *quid-pro-quo* reason is familiar from many dynamic environments in which in equilibrium individuals forgo short-term gain in the interest of future payoffs, and in particular is related to incremental exchange, incremental public goods provision and turn taking. The screening reason, as the name suggests, is reminiscent of dynamic screening settings where, for example, a seller extracts information about a buyer's valuation for an object by tempting the buyer with a sequence of price offers. We show that the combination of *quid-pro-quo* and *screening* motivations generates intertemporal incentives that may counter the detrimental effects of the desire for primacy.

Our focus is on exchange of information via disclosure. We represent (payoff-relevant) information as a subset of some (payoff) state space. Initially, two agents independently and privately learn a finite set, their "possibility set", to which the true state belongs. To avoid degeneracy, we assume that it is common knowledge that combining their information is useful in the sense that it reveals the state of the world without error, i.e., the true state of the world is the unique common element of both agents' possibility sets.

Each period, agents make one of two kinds of choices; either they take an action or they make a disclosure. For simplicity, we identify the space of actions with the state space. Each agent's objective is to take the action that corresponds to (is optimal in) the true state of the world. Having the other agent take the correct action is less desirable than no action being taken but not as damaging as taking a wrong action. If and when the correct action is taken, the game ends.

Agents disclose information by revealing states in their possibility sets. They need not disclose fully but must be truthful. Thus, a disclosure decision amounts to picking a subset of the undisclosed elements in one's possibility set. To highlight the role of disclosure, we shut down all other avenues for communication. For this reason we assume that agents have a uniform prior over the state space and lack a common language for the undisclosed elements of the state space, so that the only property of a disclosed set that matters is its size, not the identity of its elements.

With each disclosure an agent risks revealing the true state and thereby giving the other agent an opportunity to identify the true state and act on it. For any agent to disclose, therefore, there should be a prospect for him to be able to identify the true state in the future, for instance, because the other agent is expected to disclose in return. However, this *quid-pro-quo* reason is not enough to initiate information exchange because there may only be a finite number of disclosures and the last disclosure cannot be motivated by this reason. The aforementioned screening reason comes to the rescue here: If the one to disclose last disclosed all but one element in his possibility set, he retains the prospect of identifying the undisclosed element as the true state should the other agent not end the game after the last disclosure.

This reasoning illuminates some key equilibrium features: (1) each disclosure must be motivated by a future prospect of obtaining enough information in return; (2) once started, the agents take turns in disclosing information until the true state of the world is identified by one of the agents; and, (3) since disclosing too much information at once is too risky, communication necessarily takes the form of prolonged dialogues during which both agents become increasingly informed.

We study equilibrium behavior as the time delay between choices vanishes. For the case where both agents start out with relatively accurate information, i.e., their possibility sets contain no more than three elements, we fully characterize the set of symmetric Markov equilibria. For the general case, in which the agents' possibility sets may contain any number of states, we proceed in two steps. We first construct a "focal" symmetric Markov equilibrium that exhibits a maximum *quid-pro-quo* flavor when agents are similarly informed at the outset, as measured by the cardinality of their possibility sets. When they start equally informed, in particular, both agents initially randomize over disclosing one or no state until there is a first disclosure after which agents start alternating in disclosing pairs of states until one of them identifies the true state. Then, we show that the underlying basic principle generalizes to cases where the agents are too unequally informed at the outset for any disclosure to take place in the focal equilibrium: in particular, we construct non-symmetric equilibria in which the agent starting with a larger possibility set discloses proportionally more information in alternating rounds until the true state is identified. The number of disclosure rounds grows without bound as the agents become less informed at the outset. Nonetheless, the equilibria converge to efficieny as the agents become infinitely patient, because the disclosure continues without delay until the true state is identified.

Following the seminal papers by Grossman (1981) and Milgrom (1981) on disclosure and Crawford and Sobel (1982) on cheap talk, an extensive literature has developed on communication by costless messages. In models with one round of communication a privately informed sender sends a message to a receiver who then takes an action that affects both players' payoffs. The disclosure strand of this literature, which includes Milgrom and Roberts (1986), Shin (1994), Seidmann and Winter (1997) and Glazer and Rubinstein (2004), permits senders to withhold information but does not allow them to lie. The cheap-talk literature, with contributions by Green and Stokey (1980, 2007), Austen-Smith (1990), Baliga and Morris (2002), Matthews (1989), Morgan and Stocken (2003), Ottaviani and Sørensen (2006), Blume, Board and Kawamura (2007), places no constraints on sender messages.

Multi-round communication in sender-receiver settings has been studied by Forges (1990a), Amitai (1996), Aumann and Hart (2003), Krishna and Morgan (2001, 2004), Goltsman, Hörner, Pavlov and Squintani (2009), Forges and Koessler (2008), Esö and Fong (2010) and Golosov, Skreta, Tsyvinski, and Wilson (2014). There has been work on mediated communication by Myerson (1982) and Forges (1986) and recently in the Crawford-Sobel environment by Goltsman, Hörner, Pavlov and Squintani (2009). Another line of papers characterize the set of equilibrium outcomes obtainable when static games of three or more players are augmented by unmediated communication, as in Forges (1990b), Barany (1992), Ben-Porath (2003), and Gerardi (2004). A general message from this literature is that with three or more players one can find communication protocols for which the set of Nash equilibrium outcomes coincides with the set of equilibrium outcomes that can be achieved by mediation.

Single-round communication between multiple, privately informed players has been studied by Fried (1984), Farrell and Gibbons (1989), Matthews and Postlewaite (1989), Okuno-Fujiwara, *et al.* (1990), Park (2002), Goltsman and Pavlov (2014), among others. The present paper contributes to a small but growing literature on multi-round information exchange between privately informed parties with conflicting interests. Stein (2008) examines an environment in which competing players engage in continued exchange of newly developed ideas driven by the fact that future ideas can only be discovered if current ideas are shared. Rosenberg, Solan, and Vieille (2013) study repeated games with incomplete information and show that two players facing completely unrelated decision problems can engage in mutually beneficial information exchange. Hörner and Skrzypacz (2012) study the acquisition and gradual sale of information when there is no outside enforcement.

In Stein (2008) payoffs are complementary; in Rosenberg, *et al.* (2013) they are independent. In contrast, the current paper, like Hörner and Skrzypacz (2012), considers environments where payoffs are negatively correlated. Unlike in Rosenberg, *et al.* (2013) in the present paper each player's information is useful for both players.

As a result, although we share with Rosenberg, *et al.* (2013) the insight that each disclosure is motivated by the anticipation of receiving information in return, owing to the aforementioned screening reason we obtain full disclosure in finite time. In Hörner and Skrzypacz (2012) only one party (seller) has information valuable for the other (buyer) and thus, although gradual disclosure in multiple rounds increases the total price by enhancing buyer's trust, it is not necessary for trade. In contrast, gradualism is generally indispensable for any information exchange in our setting because the risk of losing from each disclosure needs to be kept small enough to be offset by future prospects of winning from returned information, in order for the process to be viable.

The need for protracted information exchange that arises in our environment resembles incremental contributions studied in the public goods literature, where they help overcome the free-riding problem stemming from a lack of commitment technology (Admati and Perry (1991), Marx and Matthews (2000)). Unlike in the public goods environment where an ability to commit to reciprocate contributions with a specific contribution of one's own would remove the need for incremental contributions, in our setting a similar ability to commitment to reciprocate disclosures with a disclosure of a pre-specified size would have no such effect. Such a commitment ability would not remove gradualism in our model as it is needed to rein in the immediate risk of losing from each and every disclosure. Moreover, the learning component which is essential in our setting and gives rise to the screening motive is absent in the public goods setting. Compte and Jehiel (2004) identify an alternative source of gradualism in public goods and bargaining environments, namely the fear of raising one's opponent's termination option value too much by large concessions, which acts as a lower bound of equilibrium payoff. This aspect is not present in our setting.

The next section describes the model and equilibrium concept. Section 3 lays down some fundamental insights common to all equilibria. Section 4 defines Markov equilibrium and characterizes the set of Markov equilibria when the agents are well informed at the outset. Extending the basic underlying principle, Section 5 establishes that efficient information exchange may arise in general environments in the form of alternating and gradual disclosures. Section 6 contains some concluding remarks, followed by technical proofs in Appendix.

2 Model

There is a finite set Ω of (payoff) states. Two agents, 1 and 2, are interested in identifying the *true state.* At the beginning each agent $i \in \{1,2\}$ privately learns a subset of the state space, denoted by $S_i \subset \Omega$ and referred to as his *possibility set*, that contains the true state. For both i, $\#(S_i) = \nu_i > 1$ and $\#(S_1 \cap S_2) = 1$. Thus, agents can jointly but not individually identify the true state. Define

$$\mathcal{S}(\nu_1, \nu_2) := \{(R_1, R_2) \subset 2^{\Omega} \times 2^{\Omega} | \#(R_i) = \nu_i \text{ and } \#(R_1 \cap R_2) = 1\}.$$

The assumption that the two players' possibility sets have exactly one element in common, facilitates our analysis greatly but is not critical. The analysis continues to hold as long as the probability of there being a single common element is sufficiently high, as we will explain in Section 6.

The set $\mathcal{S}(\nu_1, \nu_2)$ is the set of "states of the world" in the usual sense, as it determines both which payoff state is the the true state and what information players have. We will use "state" throughout to refer to payoff states. The game begins with nature drawing a pair (S_1, S_2) from a uniform distribution on $\mathcal{S}(\nu_1, \nu_2)$. The lone element of $S_1 \cap S_2$ becomes the true state, which will be denoted by ω^*. The manner in which possibility sets and the true state are determined is assumed to be common knowledge between the two agents, as is the remainder of the description of the game below.

Notice that according to this formulation all elements of Ω play a symmetric role in the determination of players' initial information and in the selection of the true state. Thus *a priori* the names of states do not matter; this will enable us later to restrict attention to players using strategies that treat states identically as long as they have not been distinguished by the history of play.

After learning their possibility sets privately, the two players play a potentially infinite-horizon game as described below. In each period $t = 1, 2, \cdots$, as long as the game has not ended by then, each agent i has the option to make a "move," which is either a "disclosure" of a nonempty subset of S_i (of elements that have not been disclosed already), or an"action," where each player's set of possible actions coincides with the state space, Ω. Alternatively, either player may opt to "do nothing." The two players' moves are simultaneous in each period. The game ends when either player takes an action that is ω^*.

Formally, the set of possible choices in period 1 is $C_i = 2^{S_i} \cup S_i$ for player $i = 1, 2$, where $D \in 2^{S_i} \setminus \{\emptyset\}$ denotes disclosing a non-empty subset D of S_i, $\omega \in S_i$ denotes taking the action $\omega \in S_i$, and $\emptyset$ doing nothing.[1] To avoid confusion between disclosing $\{\omega\}$ and taking the action ω, we denote the latter as $\langle\omega\rangle$ in the sequel. Also, for ease of terminology in exposition, doing nothing is considered a choice but not a move. The outcome of period 1, denoted by c^1, records the choices taken by the two players in period 1, that is, $\mathsf{c}^1 = (c_1, c_2) \in C_1 \times C_2$.

Recursively, conditional on the game not having ended, a public history at the beginning of period $t = 1, 2 \cdots$, denoted by $h^t = (\mathsf{c}^1, \cdots, \mathsf{c}^{t-1})$, records how the game has been played prior to period t. For completeness, define $h^1 = \emptyset$. Player i's private history $h_i^t = (S_i, h^t)$ combines the public history with player i's private information about his possibility set S_i. Let $\mathbb{D}_j^t$ denote the set of all elements disclosed by player $j \in \{1, 2\}$ according to h^t, and $\mathbb{A}_j^t$ the set of actions taken by j. Then, given any private history $h_i^t = (S_i, h^t)$, player i's information set is given by

$$I(h_i^t) := \left\{(R_1, R_2, h^t) | R_i = S_i,\ R_{-i} \supset (\mathbb{D}_{-i}^t \cup \mathbb{A}_{-i}^t),\ \#(R_{-i}) = \nu_{-i},\ \#(R_1 \cap R_2) = 1\right\}.$$

Any information set of player i is a maximal set of histories that player i cannot distinguish by what he has learned during the course of the game. A history h_i^t is an *extension* of h_i^τ if the two coincide prior to period τ; it is a *simple extension* if no move took place from period τ onward.

The set of possible choices for player i in period t with a private history $h_i^t = (S_i, h^t)$ is $C(h_i^t) = 2^{S(h_i^t)} \cup S_i$ where $S(h_i^t) = S_i \setminus (\mathbb{D}_i^t \cup \mathbb{A}_i^t)$ is the subset of S_i that consists of the elements that i has not yet disclosed or taken as an action. The outcome of period t is $\mathsf{c}^t = (c_1, c_2) \in C(h_1^t) \times C(h_2^t)$.

If player i alone takes an action in period t and that action is ω^*, then his payoff is $\alpha > 0$ in that period while his opponent, denoted by $-i$, receives a payoff $\beta < 0$; if both players take action ω^* in the same period, each receives the payoff $\frac{\alpha+\beta}{2}$; if player i takes action $\omega \neq \omega^*$, then i's payoff equals $\gamma < 0$ and $-i$'s payoff is zero[2]. In any

[1] The restriction of player i's set of actions to S_i, rather than allowing the entire state space, is for convenience and without loss of generality because taking an action outside of S_i is strictly dominated due to assumption (1) below.

[2] Although set at 0 for expositional ease, this payoff is unimportant for our result because no player would take an action $\omega \neq \omega^*$ in equilibrium due to the assumption (1) below.

period t in which no action is taken players receive a payoff of zero. Players have a common discount factor, $\delta \in (0,1)$, and maximize the expected presented discounted sum of per-period payoffs.

Taking the correct action ω^* is socially desirable, $\alpha + \beta > 0$, even if costly to the player who is not the one taking it, $\beta < 0$. Taking an incorrect action , $\omega \neq \omega^*$, is worse than being preempted, $\gamma < \beta$, and so much so that a player would reject an equal probability chance of taking the correct or an incorrect action, $\alpha + \gamma < 0$. Throughout of the paper, we assume the stronger condition

$$\frac{2\alpha + \gamma}{2} \; < \; \beta \; < \; 0 \; < \; \alpha + 2\beta, \tag{1}$$

except in Section 3 where the last inequality can be weakened to $0 \; < \; \alpha+\beta$. The first inequality ensures that a player would reject an equal probability chance of taking the correct or an incorrect action, even if it guaranteed him identifying the correct action in the immediate following period; as a consequence, no player would want to try to preempt his opponent by taking an action when his posterior is uniform over a non-singleton set of states. The last inequality implies that each player i prefers that the true state becomes known provided that his chance of taking the correct action before player $-i$ is at least one-third.[3]

We now define a player's strategy by specifying a choice for every possible private history.[4] Specifically, a *planned choice* of player i in period t with private history h_i^t is $\sigma_i(h_i^t) \in \Delta C(h_i^t)$ where ΔX is the set of all probability distributions over the set X; it is a *pure* planned choice if $\sigma_i(h_i^t)$ assigns probability one to a single element of $C(h_i^t)$. A *strategy* of player i, denoted by σ_i, is a collection of planned choices, one for each and every possible history (of any length). Given a strategy σ_i and a history h_i^t, a "continuation strategy" of player i is σ_i restricted to h_i^t and all possible extensions of it. Note that, given S_i, only those histories are possible according to which player i neither discloses elements outside of S_i nor takes them as actions, which we take for granted.

We will assume, in the spirit of Crawford and Haller (1990), that there is no

[3]It is used to ensure that there are nontrivial equilibria for general (ν_1, ν_2).

[4]Note that each player's possibility set S_i can be perceived as his private type and thus, each player's strategy may be described as type-contingent choice for every possible public history. The current approach is equivalent to this.

common labeling of the elements of Ω. As a result, from player $-i$'s perspective, player i's behavior treats elements of Ω identically as long as they are not distinguished by the history of play. Here, ω and ω' are undistinguished by history h_i^t if $\omega \in S_i \Leftrightarrow \omega' \in S_i$; $\omega \in D_j^\tau \Leftrightarrow \omega' \in D_j^\tau$ for all $\tau < t$ and $j = 1, 2$ where D_j^τ denotes the set of states disclosed by player j in period τ; and $\omega, \omega' \notin \mathbb{A}_j^t,\ j = 1, 2$. Formally, adopting the perspective of player $-i$, this means that for each player i his strategy σ_i is invariant under permutations of the elements of the state space. Denote a permutation of the state space Ω by π and the set of all such permutations by Π. For every private history h_i^t let $\pi(h_i^t)$ stand for the private history obtained by renaming the elements of h_i^t according to the permutation π; and for every choice $c \in C(h_i^t)$ let $\pi(c)$ stand for the choice obtained by renaming the elements of c according to π. Then, we have *no common labeling* (NCL) if

$$\sigma_i(h_i^t)(c) = \sigma_i(\pi(h_i^t))(\pi(c)) \quad \forall c \in C(h_i^t)$$

for all $h_i^t = (S_i, h^t)$ and $\pi \in \Pi$. We take it for granted throughout the paper that all strategies satisfy this property.

A perfect Bayesian equilibrium is a strategy-belief pair (σ, μ) consisting of a strategy profile $\sigma = (\sigma_1, \sigma_2)$ and a belief system μ that assigns a belief to every information set, with the property that strategies are sequentially rational given beliefs and beliefs are derived from Bayes' rule where possible. We strengthen the requirement on beliefs in the manner of Fudenberg and Tirole (1991, p. 331-3):

Definition 1 *A strategy-belief pair (σ, μ) forms a perfect Bayesian equilibrium (PBE) if at every possible private history h_i^t, (i) σ_i is a best response of player i given σ_{-i} and μ, and (ii) for all possible extensions of h_i^t, the belief assigned by μ is obtained from σ by Bayes' rule based on $\mu(I(h_i^t))$, where possible.*

Under the NCL assumption, whenever player i makes a disclosure player $-i$ updates his posterior belief about the true state by dismissing the disclosed states and concentrating beliefs on the remaining states, unless he finds one of the disclosed states, say ω, in his possibility set S_{-i}; in the latter case, $\omega = \omega^*$ and it is clearly optimal for player $-i$ to take action ω in the next period (as will be formalized shortly). A straightforward consequence of the NCL assumption is therefore that each player continues to assign the same probability of being the true state to each of the elements in his possibility set that are undistinguished by history. When player i considers a disclosure after history h_i^t (as ω^* has not been identified by then), all that matters strategically is how many elements to disclose, not their identities, since all elements of $S(h_i^t)$ are undistinguished by history.

The role of the NCL assumption is to emphasize the hard-information nature of our model: player i cannot indirectly communicate information about the elements in S_i; all that player $-i$ learns about S_i from a disclosure D_i by player i is that $D_i \subset S_i$. This also implies that at the disclosure stage player i's only relevant decision concerns how many (further) elements of S_i to disclose. In the sequel, therefore, we represent a disclosure move by the number of the elements to disclose (irrespective of their identities), i.e., we can write the set of available choices following private history h_i^t as $C(h_i^t) = \{1, \cdots, \#(S(h_i^t))\} \cup S_i \cup \{\emptyset\}$.

3 General properties of PBE

In this section we establish some core properties that pertain to all equilibria. The overall picture that emerges from these results is that a players will take an action if and only if he is certain of it being correct, either because the other player has revealed the true state or failed to respond to a revelation of all but one state; a player will generally not reveal all of the remaining elements of his possibility set at once; equilibrium continuation payoffs are bounded from below by zero; and, as a result, there is a limit on the size of disclosed sets. Therefore, disclosure must be gradual, involve reciprocation and the length of time required to guarantee finding the true state grows without bound, as the initial uncertainty, represented by the size of the initial possibility sets, increases.

Recall that $S(h_i^t) = S_i \setminus (\mathbb{D}_i^t \cup \mathbb{A}_i^t)$ is the subset of S_i consisting of the elements that player i has not yet disclosed or taken as an action according to h_i^t. For brevity, we use S_i^t as a shorthand for $S(h_i^t)$ and refer to it as player i's *remaining possibility set* when no confusion arises. The following two classes of histories are of special interest.

$$\begin{aligned} H_i^*(\omega) &:= \left\{ h_i^t \,|\, \{\omega\} = S_i \cap \mathbb{D}_{-i}^t \right\} \\ H_i^\circ(\omega) &:= \left\{ h_i^t \,|\, \{\omega\} = S_i^t \text{ and player } i \text{ disclosed no element in period } t-1 \right\} \end{aligned}$$

The class $H_i^*(\omega)$ consists of all private histories of player i in which he has identified ω to be the true state, $\omega = \omega^*$, because his opponent has disclosed it as being also in his own possibility set. The class $H_i^\circ(\omega)$ consists of all private histories of player i in which he has disclosed every state in his initial possibility set with the exception of the state ω, and even though his opponent had a chance to make a move, the game has not ended; note that from this player i can infer that ω must be the true state. The next two lemmas state that player i always takes an action ω that has been identified as the true state in one of these two manners. Furthermore, player i never takes an action otherwise. We define

$$H_i^* := \cup_{\omega \in S_i} H_i^*(\omega) \quad \text{and} \quad H_i^\circ := \cup_{\omega \in S_i} H_i^\circ(\omega).$$

Lemma 1 *In any PBE, $\sigma_i(h_i^t)(\langle\omega\rangle) = 1$ for all $h_i^t \in H_i^*(\omega)$; and $\sigma_i(h_i^t)(\langle\omega\rangle) = 1$ for all $h_i^t \in H_i^\circ(\omega)$ on the equilibrium path.*[5]

Proof: It is obvious that $\sigma_i(h_i^t)(\langle\omega\rangle) = 1$ for all $h_i^t \in H_i^*(\omega)$. This implies that if $h_i^t \in H_i^\circ(\omega)$ is along the equilibrium path then $\omega = \omega^*$, since otherwise ω^* must have been disclosed by player i and thus, the other player must have ended the game by taking action ω^*. Hence, $\sigma_i(h_i^t)(\langle\omega\rangle) = 1$ if $h_i^t \in H_i^\circ(\omega)$ is on the equilibrium path. □

Lemma 2 *In any PBE, on the equilibrium path,*

(a) if $\#(S_i^t) \geq 2$ and $h_i^t \notin H_i^*$, then $\sigma_i(h_i^t)(\langle\omega\rangle) = 0$ $\forall \omega \in S_i$;

(b) if $\#(S_i^t) = 1$ and $h_i^t \notin H_i^* \cup H_i^\circ$, then $\sigma_i(h_i^t)(\langle\omega\rangle) = 0$ $\forall \omega \in S_i$.

Proof: Being on the equilibrium path, ω^* has not been disclosed by player i before period $t-1$, for if it had been, then the game would have ended according to Lemma 1. By the NCL property, therefore, any of the elements that have not been disclosed by player i before period $t-1$ is equally likely to be ω^*. Let n be the number of such elements. Note that $n \geq 2$ because $\#(S_i^t) \geq 2$ for (a), and $\#(S_i^t) = 1$ and $h_i^t \notin H_i^\circ$ for (b). Thus, player i's payoff in the continuation from taking any of these elements

[5] The qualification "on the equilibrium path" is needed here because if in period $t-1$, in which player $-i$ had a chance to respond, player $-i$ made an unexpected move, PBE permits player i to believe in period t that she did disclose ω^* at some earlier point in time, without player $-i$ having responded optimally by taking the action ω^*. In that case player i would believe in period t that $\omega \neq \omega^*$. We will later put further restrictions on beliefs, which rule out this possibility.

as an action in period t is bounded from above by what he obtains if he wins in the next period when the action taken is not ω^*, i.e., $\frac{(n-1)(\gamma+\delta\alpha)+\alpha}{n} < \frac{(n-1)\gamma+n\alpha}{n} \leq \frac{\gamma+2\alpha}{2}$, which is less than the lower bound, β, of the payoff from doing nothing forever due to (1). Since the payoff from taking any other element of S_i as an action is bounded above by $\gamma + \delta\alpha < \beta$, we conclude that taking any element in period t as an action is suboptimal for player i. $\square$

The next lemma addresses a situation for a player i in period t who has disclosed all but one element of his possibility set and does not know whether the sole remaining element corresponds to the true state or not. It says that unless the other player is in the same situation, player i makes no move and instead waits to see whether the other player ends the game by taking action ω^*, anticipating that if player $-i$ does not end the game, he will be in a position to end it himself by making the optimal choice (Lemma 1).

Lemma 3 *In any PBE, on the equilibrium path, if* $\#(S_i^t) = 1 < \#(S_{-i}^t)$ *and* $h_i^t \notin H_i^* \cup H_i^\circ$ *then* $\sigma_i(h_i^t)(\emptyset) = 1$.

Proof: If $h_{-i}^t \in H_{-i}^*$, player $-i$ will take action ω^* in period t by Lemma 1. In this contingency, it is trivially optimal for player i to do anything other than taking some action ω (i.e., either disclose his remaining element or do nothing). In the other contingency that $h_{-i}^t \notin H_{-i}^*$, which has positive probability, player $-i$ will not take any action ω in period t by Lemma 2 (a). If player $-i$ does not end the game in period t, therefore, player i will correctly infer that his remaining element must be ω^*.

Given that $\#(S_{-i}^t) \geq 2$, if player i discloses the lone remaining element in his possibility set in period t, in the contingency that $h_{-i}^t \notin H_{-i}^*$, both players will take action ω^* in period $t+1$. If player i does not disclose his lone remaining element in period t, in the same contingency player i is certain to take action ω^*, but player $-i$ will take action ω^* only with probability strictly less than one. This proves that $\sigma_i(h_i^t)(\emptyset) = 1$. $\square$

The next result states that a player does not disclose all the remaining elements in his possibility set as long as his opponent has two or more elements undisclosed and therefore it remains uncertain whether his opponent has identified ω^*.

Lemma 4 *In any PBE, on the equilibrium path, $\sigma_i(h_i^t)(S_i^t) = 0$ if $\#(S_{-i}^t) \geq 2$.*

Proof: In the contingency that player $-i$ knows ω^*, disclosing nothing is trivially optimal for player i. In the alternative contingency, which has a positive probability when $\#(S_{-i}^t) \geq 2$, let $d \in \{0, 1, \cdots\}$ denote the number of elements that player $-i$ dicloses in period t. By "doing nothing forever unless player i knows w^* for sure", player i would get an expected payoff strictly above $\delta \frac{d\alpha + (\#(S_{-i}^t) - d)\beta}{\#(S_{-i}^t)}$ because player $-i$ would never take an action by Lemma 2 unless history evolved so that it is in H_{-i}°. If player i discloses S_i^t in period t, on the other hand, player $-i$ would take action ω^* in period $t+1$ with probability one, and thus, player i's expected payoff would be $\delta \frac{d(\alpha+\beta)/2 + (\#(S_{-i}^t) - d)\beta}{\#(S_{-i}^t)} \leq \delta \frac{d\alpha + (\#(S_{-i}^t) - d)\beta}{\#(S_{-i}^t)}$. Hence, disclosing S_i^t in period t is dominated by "doing nothing forever unless player i knows w^* for sure". $\square$

The last result established that it is generally not optimal for a player to disclose his entire remaining possibility set. Additional constraints on the size of disclosed sets follow from the fact that equilibrium continuation payoffs are bounded from below by zero, as shown in the next lemma. As a result, player i will avoid making large disclosures, which would result in a high probability of the opponent discovering the true state, taking an action and leaving player i with a negative payoff, as shown in Theorem 1 below.

Lemma 5 *In any PBE, after any private history h_i^t with $\#(S_j^t) \geq 2$ for $j = 1, 2$, player i's expected payoff conditional on $h_{-i}^t \notin H_{-i}^*$, is no less than zero.*

Note that for this observation, a player's expectation is taken conditional on information that is not available to him. The observation is of interest because the event that player $-i$ has already discovered ω^* is irrelevant for player i's disclosure decision.

Proof: Following h_i^t, consider letting player i adopt the strategy $\tilde{\sigma}_i$ of never disclosing any elements and taking an action if and only if that action is revealed to be ω^*. At those private histories of player $-i$ that are consistent with player i using strategy $\tilde{\sigma}_i$, it is never optimal for player $-i$ to take an action unless player $-i$ has disclosed all but one of the elements in his possibility set and player i had an opportunity to respond. Therefore, given player i's strategy $\tilde{\sigma}_i$, for any sequentially optimal strategy of player $-i$, either neither player will ever take an action, and therefore both players receive a payoff of zero, or player $-i$ makes disclosures before taking an action. Each time player $-i$ makes a disclosure there is a chance that he discloses ω^*, in which case player i receives a positive payoff. Only in the event that player $-i$ has disclosed all but one element, and player i had a chance to respond, will player $-i$ take an action. At the moment player $-i$ makes the final disclosure that leaves him with one undisclosed element, player i's expected payoff conditional on player $-i$ disclosing all but one of his remaining K elements is $\delta\frac{K-1}{K}\alpha + \delta^2\frac{1}{K}\beta > 0$. Therefore unless player $-i$ has already identified ω^*, player i adopting strategy $\tilde{\sigma}_i$ following private history h_i^t always results in a nonnegative payoff for player i. $\square$

We now use the observation that players' equilibrium continuation payoffs are bounded from below by zero to show that disclosure sizes are bounded and, as a consequence, that ensuring discovery of the truth requires that the number of disclosure rounds grows without bound when players' initial information is made sufficiently imprecise.

Theorem 1 *For any integer M, if ν_1 and ν_2 are sufficiently large, disclosure goes on for M or more rounds with positive probability in any PBE in which the true state is revealed with certainty.*

Proof: Consider any private history $h_i^t \notin H_i^*$ of player i after which $n_j \geq 2$, $j = 1, 2$, elements remain in players' possibility sets. With positive probability $h_{-i}^t \notin H_{-i}^*$; otherwise player i's disclosure decision is irrelevant. Consider player i's problem of how many elements, K_i, to disclose in period t following h_i^t. There are two possibilities for player i to consider: one is that player $-i$ contemporaneously discloses ω^*; conditional on that event, the unique optimal choice of K_i would be zero. Hence player i only possibly discloses $K_i > 0$ elements in consideration of the possibility of being in the second case, in which player $-i$ does not disclose the true element in period t. In that case player i's payoff from disclosing K_i elements is bounded from above by

$$\frac{K_i}{n_i}\beta + \left(1 - \frac{K_i}{n_i}\right)\alpha.$$

Since in the (positive probability) event that player i's disclosure decision is relevant, i.e. $h^t_{-i} \notin H^*_{-i}$, player i can guarantee a payoff no less than zero by Lemma 5, in order for player i to be willing to disclose K_i elements, it has to be the case that

$$\frac{K_i}{n_i}\beta + \left(1 - \frac{K_i}{n_i}\right)\alpha \geq 0. \tag{2}$$

Let N be the smallest (integer) value of n such that $\alpha + n\beta < 0$. Then the condition in equation (2) amounts to

$$\frac{\frac{K_i}{n_i}}{1 - \frac{K_i}{n_i}} < N \quad \Longleftrightarrow \quad \frac{K_i}{n_i} < \frac{N}{N+1}.$$

Therefore neither player will ever disclose a fraction $\frac{N}{N+1}$ or more of the elements of his remaining possibility set. Hence, after M disclosure rounds, player i's remaining possibility set contains at least $\left(\frac{1}{N+1}\right)^M \nu_i$ elements, provided $\left(\frac{1}{N+1}\right)^M \nu_i \geq 1$, which can be ensured by choosing ν_i sufficiently large. Choose ν so that $\left(\frac{1}{N+1}\right)^M \nu \geq 2$. Then for any $\nu_j \geq \nu$, $j = 1, 2$, as long as neither player discloses the true element in any of the first M disclosure rounds neither player will take an action by Lemma 2. Hence for any $\nu_j \geq \nu$, $j = 1, 2$, with positive probability there are at least M disclosure rounds. $\square$

4 The set of Markov equilibria when $\#(S_1)$ and $\#(S_2)$ are small

In this section we introduce Markov strategies and Markov equilibria. We then fully characterize the set of Markov equilibria for patient players with the best possible information short of knowing the truth (i.e. both players' possibility sets contain two elements); as a byproduct, we get to see the screening motive for disclosure in operation in equilibrium, under conditions where the *quid-pro-quo* motive is absent. Finally, we turn to the case of players being maximally well informed consistent with them having a *quid-pro-quo* motive for disclosure (i.e. at least one player's possibility set contains three elements). For this case and again with patient players, we fully characterize the set of symmetric Markov equilibria. This gives us a preview of the focal equilibria we study in the general case, an understanding of how efficiency is achieved in those equilibria, and an illustration of the operation of equilibrium disclosures that are driven by the *quid-pro-quo* motive.

4.1 Markov Strategies

We begin by introducing Markov strategies and Markov equilibria. This requires an appropriate state space. As shown in the previous section, the key variables that govern agents' decisions are how many elements each player has disclosed and whether or not a player's private history satisfies $h_i^t \in H_i^* \cup H_i^\circ$. This inspires our definition of a *Markov state* of a private history h_i^t as a tuple

$$\big(\#(S(h_i^t)),\, \nu_{-i} - \#(\mathbb{D}_{-i}^t \cup \mathbb{A}_{-i}^t),\, \mathbf{1}(h_i^t)\big) \in \mathbb{N} \times \mathbb{N} \times \{0,1\}$$

where $\mathbf{1}(h_i^t)$ is an indicator function such that

$$\mathbf{1}(h_i^t) = 1 \ \text{ if } \ h_i^t \in H_i^* \cup H_i^\circ \quad \text{and} \quad \mathbf{1}(h_i^t) = 0 \ \text{ if } \ h_i^t \notin H_i^* \cup H_i^\circ.$$

A strategy σ_i is a *Markov strategy* if $\sigma_i(h_i^t)$ depends only on the Markov state of h_i^t for every h_i^t.

Definition 2 *A PBE $(\sigma_1, \sigma_2, \mu)$ is a Markov equilibrium if σ_1 and σ_2 are Markov strategies. It is symmetric if $\sigma_1(h_1^t) = \sigma_2(h_2^\tau)$ whenever h_1^t and h_2^τ have identical Markov states.*

Along the equilibrium-path of any PBE, a player's posterior belief about which state is the true state ω^* is the prior concentrated on the remaining possibility set. A key feature of Markov equilibrium is that this principle is maintained for off-equilibrium contingencies as well.[6]

In the remainder of this section we characterize the set of Markov equilibria for patient and well-informed players, indicated by the cardinalities of their initial possibility sets being small. We consider two subcases below, one that permits a full

[6] A PBE does not imply this principle on off-equilibrium contingencies. For instance, if player i disclosed some elements in period $t-2$ and player $-i$ did nothing even if he was supposed to disclose some elements in case D_i^{t-2} did not include ω^* according to the equilibrium, then off-equilibrium belief may prescribe that player i assigns positive probability that D_i^{t-2} included ω^* (but player $-i$ did nothing in period $t-1$). Therefore, for off-equilibrium history h^t, a PBE prescribes a belief that assigns to each element of S_i^t the same probability of being ω^*, and possibly a positive probability to other elements of S_i as well, again the same probability to elements that were disclosed at the same time (due to NCL).

characterization of the set of Markov equilibria, and another for which we characterize the set of all symmetric Markov equilibria. The analysis foreshadows that for arbitrary initial possibility sets, which will be the subject of the next section.

4.2 The set of Markov equilibria when $\#(S_1) = \#(S_2) = 2$

We begin by studying the case in which both players are as well informed as they can be without knowing the true state.

Suppose that one of the two players discloses one of the two elements they started out with. Clearly, if the disclosed element is ω^*, then the discloser loses the game in the next period because the opponent will end the game then (Lemma 1). If the disclosed element is not ω^*, however, the opponent will not end the game in the next period (Lemma 2), and this very fact reveals to the discloser that what he retained must be ω^* and therefore, he will win the game in the subsequent period (Lemma 1). Hence, the discloser faces equal chances of losing in period 2 and winning in period 3, and his expected payoff is

$$\delta \frac{\beta + \delta\alpha}{2} > 0 \tag{3}$$

where the inequality holds when $\delta < 1$ is large enough.

Therefore, if one player adopts a Markov strategy of doing nothing after all histories whose Markov states coincide with the initial state, it is uniquely optimal for the other player to disclose one element if δ is large enough because neither taking an action nor disclosing both elements is optimal by Lemmas 2 and 4, respectively, and doing nothing in period 1 at best postpones disclosing one element and obtaining the payoff of (3) to a later period.

Conversely, if one player adopts a Markov strategy that involves disclosing one element in period 1 (and in all simple extensions of h^1; we omit analogous qualifications hereafter), the other player gets an expected payoff of $\delta\frac{\alpha+\delta\beta}{2}$ by doing nothing because then he wins the game in the next period if the disclosed item is ω^* but otherwise loses in the following period, as explained above. In fact, this is uniquely optimal for him because disclosing both elements is not optimal by Lemma 4, and his payoff from disclosing one element as well is lower at $\delta(\frac{3+\delta}{4})(\frac{\alpha+\beta}{2})$: this is because if both disclose one element, each player wins, loses, and ties in the next period with probability 1/4 each; and when ω^* is disclosed by neither, which happens with the remaining probability of 1/4, neither takes an action in the next period by Lemma 2 (b) and then both take action ω^* in the subsequent period by Lemma 1.

Thus, assuming that $\delta < 1$ is sufficiently large, we have identified one class of Markov equilibria: One player, say i, does nothing and the other player, $-i$, discloses one element in period 1; player i takes action ω^* in the next period if this was the element disclosed by player $-i$, while he mixes between doing nothing and disclosing some element of S_i if player $-i$ disclosed a state ω other than ω^* (by Lemmas 2 and 4), followed by player $-i$ taking action ω^* in the subsequent period. The exact manner in which player i mixes (including degenerate mixtures) between doing nothing and disclosing some element when ω^* was not disclosed by player $-i$ does not matter because they have identical payoff-relevant consequences. Note that these exhaust all Markov equilibria in which either player discloses one element with certainty or does nothing with certainty in the initial period.

Since in period 1 neither player takes an action (Lemma 2) or discloses all elements of his possibility set (Lemma 4), the remaining possibility for a Markov equilibrium is that both players mix between disclosing one element and doing nothing in the initial period. Let $p_i \in (0, 1)$ denote the probability with which player i discloses one element. Then, the payoff of player $-i$ from doing nothing is

$$p_i \delta \frac{\alpha + \delta\beta}{2} + (1 - p_i)\delta V_{-i}$$

where V_{-i} is the equilibrium payoff of player $-i$, because if player i discloses one element player $-i$ wins in the next period and loses in the following period with equal probabilities as explained above. If player $-i$ discloses one element, instead, his payoff would be

$$p_i \delta \frac{\alpha + \beta}{2} \left(\frac{3 + \delta}{4}\right) + (1 - p_i)\delta \frac{\beta + \delta\alpha}{2}.$$

Here, the first term captures what happens when both disclose one element each as explained earlier, and the second term what happens when he is the sole discloser, and therefore obtains the payoff in (3). Since both payoffs must be the same and equal to the equilibrium payoff, V_{-i}, routine calculations[7] establish that there is a

[7]The equation is expressed as a quadratic function of p_i being equal to 0, where the function is negative at $p_i = 0$ and positive at $p_i = 1$ due to (1).

unique equilibrium value $p_i \in (0,1)$. This value equals

$$p_2^* = \frac{\alpha+\beta+\delta(7\alpha-\beta)-\sqrt{(\alpha+\beta+\delta(7\alpha-\beta))^2-16\delta(3\alpha-\beta)(\delta\alpha+\beta)}}{\delta(6\alpha-2\beta)} \quad (4)$$

$$\to \frac{8\alpha-\sqrt{64\alpha^2-16(3\alpha-\beta)(\alpha+\beta)}}{(6\alpha-2\beta)} \quad \text{as} \quad \delta\to 1. \quad (5)$$

Note that the equilibrium value of p_{-i} is the same by symmetry. Hence, for both players the payoff in this symmetric equilibrium for the game in which both players start with two elements in their possibility sets equals

$$V_{2,2} := \delta\frac{\beta+\delta\alpha}{2}+p_2^*\delta\Big(\frac{(3-3\delta)\alpha-(1-\delta)\beta}{8}\Big) \to \frac{\alpha+\beta}{2} \text{ as } \delta\to 1. \quad (6)$$

The next result summarizes the findings when both start with two elements each, preceded by a remark about a convention we adopt for off-equilibrium specifications.

Remark 1: For each class of Markov equilibria identified above, there is a continuum of equilibria that only differ in the manner in which a player mixes between doing nothing and disclosing some element when the opponent first disclosed one element that is different from ω^*. These differences are superficial because they are payoff-irrelevant as explained earlier. In other games where the players start with more than two elements, the same superficial differences arise when an agent disclosed all but one element in the previous period and the other agent has not identified ω^*. For expositional convenience, we treat Markov equilibria that differ only in this superficial sense as one and the same in the remainder of the paper.

Proposition 1 *When $\#(S_1)=\#(S_2)=2$, there exists a $\underline{\delta}\in(0,1)$ such that for all $\delta\in(\underline{\delta},1)$ the set of Markov equilibria takes the following form:*

1. *There is a unique symmetric equilibrium: both players disclose one element of S_i with probability p_2^*, as given in (4), in all periods without previous moves. If both players disclose one element each in some period, player i takes an action $\omega\in S_i$ in the next period if and only if it was disclosed by the other player, which ends the game; if both of the disclosed elements are not in the other player's possibility set, then both players take an action equal to the sole remaining element in their respective possibility set in the subsequent period,*

ending the game. If only one player, say i, disclosed one element in some period without previous moves, player $-i$ takes an action $\omega \in S_{-i}$ in the next period if and only if that was the state disclosed by player i, ending the game; if the element disclosed by player i is not in player $-i$'s possibility set, and therefore player $-i$ does not take an action, then player i takes an action equal to the sole remaining element in the subsequent period, ending the game. The payoff in this equilibrium is $V_{2,2}$, as given in (6).

2. *There are two other equilibria: in each, one player discloses one element while the other discloses none in all periods without previous moves; and, once there is a disclosure, players continue as in the symmetric equilibrium.*

Proof: See Appendix. □

4.3 Symmetric Markov equilibria when $\max\{\#(S_1), \#(S_2)\} = 3$

In this section we characterize the set of symmetric Markov equilibria for the case of sufficiently patient players whose initial possibility sets contain three or fewer elements. The reader only interested in the construction of Markov equilibria in the general case can skip this section without loss of continuity.

We begin by exploring possible continuations in which one player has fewer than three undisclosed elements, and then use our findings to characterize symmetric Markov equilibria when both players start with three elements.

Consider a continuation in which one player, say 1, retains all three elements and the other player, 2, has two elements remaining undisclosed. Any move in period 1 of the continuation must be a disclosure move. We start with the possibility that some player discloses with certainty in period 1.

First, consider the possibility that player 2 (with two elements) discloses one element for sure in period 1 (he never discloses both elements by Lemma 4). Then, it is uniquely optimal for player 1 to do nothing in period 1. To see this, note that his expected payoff from doing so is $\delta\frac{\alpha+\delta\beta}{2}$ as he faces equal chances of winning in the next period (if the disclosed element is ω^*) and losing in the subsequent period (otherwise, because then his opponent would take an action corresponding to his sole remaining element and end the game), while that from disclosing one element himself is smaller since his payoff would be reduced if the element he disclosed turns out to be ω^* (otherwise, his payoff would be the same as when he disclosed none), and that from disclosing two is easily calculated to be smaller still because he discloses ω^* with a higher probability.

Conversely, conditional on player 1 doing nothing in period 1, it is easily verified that disclosing one element is uniquely optimal for player 2 in period 1 because it generates a payoff of $\delta\frac{\beta+\delta\alpha}{2}$ while by doing nothing he at best delays disclosing one element and obtaining $\delta\frac{\beta+\delta\alpha}{2}$ to a later period.

Thus, we have identified a symmetric Markov equilibrium, in which player 2 discloses one element for sure and player 1 does nothing in period 1. The equilibrium payoffs of players 1 and 2, respectively, are $\delta(\alpha+\delta\beta)/2$ and $\delta(\beta+\delta\alpha)/2$.

Next, consider the possibility that player 1 (with three elements) discloses one or two elements for sure in period 1. Then, it is straightforwardly verified that doing nothing is optimal for player 2 in period 1, because disclosing one element himself halves the chance of winning in period 2 (they would tie, instead) that overshadows any possible future benefits.[8] Given that player 2 does nothing in period 1, if player 1 discloses one element, he loses in the next period if it turns out to be ω^* (probability 1/3); else, the two players start a continuation game with two elements each, which is shown to have a unique symmetric equilibrium with payoff $V_{2,2}$ in (6) in Proposition 1. Hence, player 1's expected payoff from disclosing one element in period 1 is $\delta(\beta+2\delta V_{2,2})/3$. If player 1 discloses two elements instead, he loses in the next period if either of them is ω^* (probability 2/3); else, player 2 does not end the game in the next period and from this fact player 1 infers that his sole remaining element must be ω^* and thus wins the game in the subsequent period. Hence, player 1's expected payoff from disclosing two elements in period 1 is $\delta(2\beta+\delta\alpha)/3$. A routine calculation comparing the two expected payoffs verifies that

$$\begin{aligned}&\delta(2\beta+\delta\alpha)/3-\delta(\beta+2\delta V_{2,2})/3\\ &= \frac{\delta(1-\delta)\big[-\alpha+7\beta+\delta(\alpha+\beta)+\sqrt{(\alpha+\beta+7\delta\alpha-\delta\beta)^2-16\delta(3\alpha-\beta)(\delta\alpha+\beta)}\,\big]}{24}\end{aligned}$$

[8]If player 1 discloses one element, player 2's expected payoff from doing nothing is $\delta(\alpha+2V_{2,2})/3$ which exceeds that from disclosing one element, $\delta(\alpha+(\alpha+\beta)/2+2\beta+\delta(\alpha+\beta))/6$, and from disclosing two is never optimal (Lemma 4). The calculation is analogous for the case that player discloses two elements.

which is positive for large enough $\delta < 1$ as the value in the square bracket converges to $4(\alpha+\beta)$ as $\delta \to 1$. For large enough $\delta < 1$, therefore, since doing nothing is worse, disclosing two elements is optimal for player 1, conditional on player 2 adopting a Markov strategy of doing nothing in period 1.

This establishes another symmetric Markov equilibrium, in which player 1 discloses two elements and player 2 does nothing in period 1. The equilibrium payoffs of players 1 and 2, respectively, are $\delta(2\beta + \delta\alpha)/3$ and $\delta(2\alpha + \delta\beta)/3$.

This covers all symmetric Markov equilibria in which one player starts with three and the other with two elements in his possibility set and at least one player takes a pure planned choice in period 1. There may also exist equilibria in which both players adopt mixed planned choices in period 1. They are described in the next lemma (proved in Appendix), which summarizes the discussion so far.

Lemma 6 *When $\#(S_1) = 3$ and $\#(S_2) = 2$ there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$ there are two symmetric Markov equilibria in pure strategies along the equilibrium-path, as described in (a) and (b) below:*

(a) $\sigma_1(h^1)(2) = 1$ and $\sigma_2(h^1)(\emptyset) = 1$, and the associated equilibrium payoffs of players 1 and 2 are $\delta(2\beta + \delta\alpha)/3$ and $\delta(2\alpha + \delta\beta)/3$, respectively.

(b) $\sigma_1(h^1)(\emptyset) = 1$ and $\sigma_2(h^1)(1) = 1$, and the associated equilibrium payoffs of players 1 and 2 are $\delta(\alpha + \delta\beta)/2$ and $\delta(\beta + \delta\alpha)/2$, respectively.

For some parameter values, there are two other symmetric Markov equilibria, as described below:

(c) Both players mix between disclosing one element and doing nothing in any period without previous moves. In any such equilibrium, player 1's payoff converges to $(\alpha + 2\beta)/3$ and player 2's payoff to $(\alpha + \beta)/2$ as $\delta \to 1$.

(d) Player 1 mixes between disclosing two elements and doing nothing and player 2 mixes between disclosing one element and doing nothing in any period without previous moves. Such an equilibrium is possible only if $\alpha + 5\beta > 0$ and player 1's payoff converges to $(\alpha+2\beta)/3$ and player 2's payoff to $(\alpha+\beta)/2$ as $\delta \to 1$.

In all equilibria above, when both players have two or fewer elements undisclosed, the unique symmetric equilibrium described in Proposition 1 prevails.

Now consider the game in which both players start with three elements, i.e.,

$\#(S_1) = \#(S_2) = 3$. For each continuation[9] in which one player has three and the other two elements in his remaining possibility set, fix one of the symmetric Markov equilibria identified in Lemma 6, and let $V_{2,3}$ and $V_{3,2}$ denote the equilibrium payoff of the player with 2 and 3 elements, respectively. Note that a player with three elements has the option of disclosing two elements in period 1. He loses in the next period if he turns out to have disclosed ω^* (probability 2/3) and the other player ends the game by taking action ω^*. Otherwise, he infers that his sole remaining element is ω^* and wins in the subsequent period. Hence,

$$V_{3,2} \;\geq\; \delta\frac{2\beta+\delta\alpha}{3}, \tag{7}$$

which is positive for sufficiently large $\delta < 1$ by (1).

Consider the possibility that both players disclose one element for sure in period 1, i.e., $\sigma_i(h_i^1)(1) = 1$ for $i \in \{1, 2\}$, in a symmetric Markov equilibrium when both players start with three elements. Then, player i would get a payoff of $\delta(\alpha + 2V_{3,2})/3$ by disclosing none because then he would win in the next period if ω^* is disclosed by his opponent (probability 1/3) and otherwise would get a continuation payoff of $V_{3,2}$ in the next period; he would get $\delta(5(\alpha + \beta)/2 + 4V_{2,2})/9$ by disclosing one element because then he would win, lose, and tie in the next period if ω^* is disclosed only by his opponent, only by himself, and by both, respectively, and otherwise they would enter a continuation with two elements each; and, he would get $\delta(\alpha+4\beta+(\alpha+\beta)+2\delta\alpha)/9$ by disclosing two elements because then he would win, lose and tie in the next period with probabilities 1/9, 4/9, and 2/9 (when ω^* is disclosed by either player), respectively, and otherwise he would win in the subsequent period by taking his sole remaining element as an action. A trivial calculation shows that, for large enough δ, the payoff is highest from disclosing none, lower from disclosing one, and lowest from disclosing two. Therefore, $\sigma_i(h_i^1)(1) = 1$ is impossible in a symmetric Markov equilibrium.

By similar reasoning, it is easily verified that neither $\sigma_i(h_i^1)(2) = 1$ nor $\sigma_i(h_i^1)(\{1, 2\}) =$

[9] Continuations are not games in their own right. Note, however, that unless either player has discovered the true state, they face the same situation as at the beginning of a proper game. Furthermore, if a player has discovered the true state his decision problem is trivial, and if his opponent has discovered the true state any strategy that is optimal conditional on his opponent not yet having discovered the true state remains optimal. Therefore a strategy is part of a Markov equilibrium in a game starting with some pair of possibility sets if and only if it is an optimal Markov strategy at the Markov state of player i with the same pair of possibility sets and where player i has not yet discovered the truth.

1 for both $i \in \{1,2\}$, is possible in a symmetric Markov equilibrium. Therefore, we consider the possibility of mixing over disclosing some (either one or two) elements and none in the initial period of a symmetric Markov equilibrium below.

If players in a putative equilibrium mixed between disclosing one and none with probability $p \in (0,1)$ and $1-p$, respectively, then the equilibrium payoff, $V_{3,3}$, would satisfy

$$
\begin{aligned}
V_{3,3} &= p\delta\Big(\frac{\alpha + 2V_{3,2}}{3}\Big) + (1-p)\delta V_{3,3} && (8)\\
&= p\delta\Big(\frac{5}{9}\frac{(\alpha+\beta)}{2} + \frac{4}{9}V_{2,2}\Big) + (1-p)\delta\Big(\frac{\beta + 2V_{2,3}}{3}\Big) && (9)\\
&\geq p\delta\Big(\frac{\alpha+4\beta}{9} + \frac{2}{9}\frac{(\alpha+\beta)}{2} + \frac{2\delta\alpha}{9}\Big) + (1-p)\delta\frac{2\beta+\delta\alpha}{3} && (10)
\end{aligned}
$$

where the RHS of (8), (9) and (10) are the expected payoffs from disclosing none, one and two elements, respectively. Rearranging the equations (8) and (9), we get

$$
V_{3,3} = \frac{p\delta}{1-\delta+p\delta}\Big(\frac{\alpha+2V_{3,2}}{3}\Big) \quad \text{and} \tag{11}
$$

$$
p\Big(\frac{5\alpha - \beta + 8V_{2,2} - 12V_{2,3}}{18}\Big) + \frac{\beta + 2V_{2,3}}{3} = \frac{p}{1-\delta+p\delta}\Big(\frac{\alpha+2V_{3,2}}{3}\Big). \tag{12}
$$

From the values of $V_{2,3}$ in Lemma 6, note that when $p = 0$ the RHS of (12) is 0 while the LHS is positive provided δ is large enough; when $p = 1$, the LHS is $(5\alpha + 5\beta + 8V_{2,2})/18$ and converges to $(\alpha+\beta)/2$ as $\delta \to 1$ by (6), while the RHS is $(\alpha + 2V_{3,2})/3$, which converges to a value no lower than $(5\alpha+4\beta)/9$ as $\delta \to 1$ by (7). Hence, there exists $p \in (0,1)$ that solves (12) for sufficiently large $\delta < 1$. The solution is unique because the LHS of (12) is linear in p while the RHS is concave in p. Let p^* denote this solution. To establish the putative equilibrium, it remains to verify that inequality (10) holds at p^*.

Note that the first term of (9) converges to $(\alpha+\beta)/2$ while the first term of (10) converges to $(4\alpha+5\beta)/9$ as $\delta \to 1$. Therefore, the inequality (10) is satisfied for sufficiently large δ if $\lim_{\delta\to 1} V_{2,3} > (\alpha+\beta)/2$ because then the second term of (9) exceeds that of (10) for large enough δ. This is indeed the case when the equilibrium from Lemma 6 (a) governs the continuation.

In contrast, straightforward calculations show that (10) is violated for large enough δ when the equilibrium from Lemma 6 (b) governs the continuation. Specifically,

$V_{2,3} = \delta(\beta + \delta a)/2 \to (\alpha + \beta)/2$ as $\delta \to 1$, and $V_{3,2} = \delta(\alpha + \delta\beta)/2$ in this case. Then, the value of p that equates (9) and (10), denoted as $\hat{p}$, is calculated as

$$p\Big(\frac{\alpha + 2\delta\alpha + \beta + 8V_{2,2} - 12V_{2,3}}{18}\Big) = \frac{\beta + \delta\alpha - 2V_{2,3}}{3} \tag{13}$$

$$\Rightarrow \quad \hat{p} \; = \; \frac{-6(1-\delta)(\beta + \delta\alpha)}{2(\alpha - \beta)(1-\delta)\delta - \beta(1-2\delta) - \alpha(1 + 6\delta - 6\delta^2)}. \tag{14}$$

Subtracting the LHS of (12) from the RHS and evaluating at $p = \hat{p}$, we find that the difference converges to

$$\frac{(\alpha - \beta)(5\alpha + 4\beta)}{3(7\alpha + 5\beta)} \; > \; 0 \tag{15}$$

as $\delta \to 1$. Since the LHS of (12) is linear in p while the RHS is concave in p, this means that $p^* < \hat{p}$. Because the first term of (9) is strictly larger than that of (10) for large enough δ as shown earlier, whereas (9) and (10) are equated at $\hat{p}$, it follows that (9) falls short of (10) at p^*, violating the inequality. This means that there does not exist a symmetric Markov equilibrium in which the two players mix between disclosing one and none if the equilibrium from Lemma 6 (b) governs the continuation. By extending this argument, we show in the appendix that the same holds if the equilibria from Lemma 6 (c) or (d) govern the continuation. This completes the characterization of Markov equilibria in which players mix between disclosing one element and none in period 1.

Next, consider putative equilibria in which players mix between disclosing two and none with probability $q \in (0,1)$ and $1-q$, respectively, in the initial period. Then, the equilibrium payoff, $V_{3,3}$, has to satisfy

$$V_{3,3} \;\; = \;\; q\delta\frac{2\alpha + \delta\beta}{3} + (1-q)\delta V_{3,3} \tag{16}$$

$$= \;\; q\delta\frac{(8+\delta)}{9}\frac{(\alpha + \beta)}{2} + (1-q)\delta\frac{2\beta + \delta\alpha}{3} \tag{17}$$

$$\geq \;\; q\delta\Big(\frac{4\alpha + \beta}{9} + \frac{2}{9}\frac{(\alpha + \beta)}{2} + \frac{2\delta\beta}{9}\Big) + (1-q)\delta\frac{\beta + 2\delta V_{2,3}}{3}. \tag{18}$$

Rearranging (16) and (17),

$$V_{3,3} \; = \; \frac{\delta q}{(1 - \delta + \delta q)}\Big(\frac{2\alpha + \beta\delta}{3}\Big) \; = \; q\delta\Big(\frac{(8-5\delta)\alpha - (4-\delta)\beta}{18}\Big) + \delta\frac{2\beta + \delta\alpha}{3}. \tag{19}$$

As before, there is a unique solution $q^* \in (0,1)$. To check if the inequality (18) holds at p^*, find the value $\hat{q}$ that equates (17) and (18):

$$\hat{q}\Big(\frac{(-2-5\delta)\alpha - (8+3\delta)\beta}{18} + \frac{\beta + 2\delta V_{2,3}}{3}\Big) = \frac{-\delta\alpha - \beta + 2\delta V_{2,3}}{3}. \tag{20}$$

If $V_{2,3} = \delta(\beta + \delta\alpha)/2$, the LHS of (19) exceeds the RHS when evaluated at $\hat{q}$ (the difference converges to $\frac{11\alpha^2 - \alpha\beta - 10\beta^2}{39\alpha + 33\beta}$ as $\delta \to 1$), which implies that $q^* < \hat{q}$. This means that the inequality (18) is satisfied at q^* when $V_{2,3} = \delta(\beta + \delta\alpha)/2$, i.e., when the equilibrium from Lemma 6 (b) governs the continuation. We show in Appendix that the same also holds when the equilibrium from Lemma 6 (c) or (d) governs the continuation. But, if the equilibrium from Lemma 6 (a) governs the continuation so that $V_{2,3} = \delta(2\alpha + \delta\beta)/3$, the inequality (18) fails for large enough δ because $p^* \to 0$ as $\delta \to 1$ and the second term of (18) exceeds and is bounded away from that of (17).

Finally, consider the equilibrium possibility that players mix among disclosing one, two and none with probability p, q, and $1-p-q$, respectively. Then, the equilibrium payoff, $V_{3,3}$, has to satisfy

$$\begin{aligned} V_{3,3} &= p\delta\Big(\frac{5}{9}\frac{\alpha+\beta}{2} + \frac{4}{9}V_{2,2}\Big) + q\delta\Big(\frac{4\alpha+\beta}{9} + \frac{2}{9}\frac{(\alpha+\beta)}{2} + \frac{2\delta\beta}{9}\Big) + (1-p-q)\delta\Big(\frac{\beta}{3} + \frac{2}{3}V_{2,3}\Big) \\ &= p\delta\Big(\frac{\alpha+4\beta}{9} + \frac{2}{9}\frac{(\alpha+\beta)}{2} + \frac{2\delta\alpha}{9}\Big) + q\delta\frac{(8+\delta)}{9}\frac{(\alpha+\beta)}{2} + (1-p-q)\delta\frac{2\beta+\delta\alpha}{3} \qquad (21) \\ &= p\delta\Big(\frac{\alpha}{3} + \frac{2}{3}V_{3,2}\Big) + q\delta\frac{2\alpha+\delta\beta}{3} + (1-p-q)\delta V_{3,3} \qquad (22) \end{aligned}$$

To compare (21) and (22) term by term, note that $\frac{\alpha+4\beta}{9} + \frac{2}{9}\frac{(\alpha+\beta)}{2} + \frac{2\delta\alpha}{9} < \frac{\alpha}{3} + \frac{2}{3}V_{3,2}$ from (7), and that $\frac{(8+\delta)}{9}\frac{(\alpha+\beta)}{2} < \frac{2\alpha+\delta\beta}{3}$. In addition, note that

$$V_{3,3} = \frac{1}{1-\delta(1-p-q)}\Big(p\delta\Big(\frac{\alpha}{3}+\frac{2}{3}V_{3,2}\Big)+q\delta\frac{2\alpha+\delta\beta}{3}\Big) \;\to\; \frac{p}{p+q}\Big(\frac{\alpha}{3}+\frac{2}{3}V_{3,2}\Big)+\frac{q}{p+q}\frac{2\alpha+\delta\beta}{3}$$

as $\delta \to 1$ and $\frac{\alpha}{3} + \frac{2}{3}V_{3,2} > \frac{2\beta+\delta\alpha}{3}$ and $\frac{2\alpha+\delta\beta}{3} > \frac{2\beta+\delta\alpha}{3}$. Hence, (22) exceeds (21) for sufficiently large $\delta < 1$, violating the equality. Therefore, it is not possible that both players mix among disclosing one, two, and none in a symmetric Markov equilibrium. The next result summarizes these findings for the case that both players start with three elements each, and is proved in Appendix.

Proposition 2 *When $\#(S_1) = \#(S_2) = 3$, there exists a $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$ the set of symmetric Markov equilibria consists of*

(a) an equilibrium in which both players disclose one element with probability $p^* \in (0,1)$ and none otherwise, where p^* is the unique solution to (12), and the equilibrium from Lemma 6 (a) governs the continuation after only one player disclosed one element; and

(b) a class of equilibria in which both players disclose two elements with probability $q^* \in (0,1)$ and none otherwise, where q^* is the unique solution to (19), and an equilibrium from parts (b)–(d) of Lemma 6 governs the continuation after only one player disclosed one element.

5 Markov equilibria for any $\#(S_1)$ and $\#(S_2)$

We saw in the previous section that even when the initial possibility sets are small and we restrict attention to symmetric equilibria, there remain multiple Markov equilibria with different implications for behavior and payoffs. With larger initial possibility sets, if anything, multiplicity will be only more severe. For this reason, in the general case, we consider Markov equilibria with a view toward the following desiderata: a natural *quid-pro-quo* pattern of information exchange, symmetry, existence in a broad class of games, and efficiency (in the limit as $\delta \to 1$).

The first result in this section demonstrates that it is possible to jointly satisfy the first three desiderata while partially satisfying the fourth. We construct a symmetric Markov equilibrium with a natural *quid-pro-quo* pattern of information exchange: players eventually establish a routine of alternating disclosure of two elements each, after every step leaving a gap of one between the remaining possibility sets. To guarantee that with this equilibrium the true state ω^* will eventually be identified requires that the sizes of the initial possibility sets do not differ by too much: if this is the case, then initially the player with the larger possibility set bridges the gap by disclosing the difference plus one, but if the gap is too large doing so is excessively costly and there is no disclosure. Existence can be guaranteed for any pair of initial possibility sets provided players are sufficiently patient. Efficiency, in the limit as the discount factor converges to one, is achieved whenever the initial possibility sets differ in size, but not by too much. The *quid-pro-quo equilibrium* may fail to be efficient for two reasons. The first possibility is that the sizes of the initial possibility sets are too different. The second possibility is that the sizes are identical; this induces a war of attrition prior to the first information disclosure that is the result of a natural tradeoff between wanting to get the information exchange started and being put at a disadvantage by being the first to disclose information.

For the second and main result of this section, we use the core insights from the construction of the *quid-pro-quo equilibrium* to construct a Markov equilibrium that is efficient in the limit as the discount factor converges to one, regardless of the relative sizes of the initial possibility sets. This *efficient equilibrium* sacrifices symmetry in order to avoid inefficient wars of attrition that arise when players in the *quid-pro-quo equilibrium* compete to delay initial disclosures that are required to break symmetry. In the *efficient equilibrium* the true state ω^* is always eventually identified; in the case where the difference in the sizes of the initial possibility sets is large, which causes problems for efficiency with the alternative equilibrium construction, this is achieved by not trying to have players bridge the gap in one step but having them exchange information at rates that gradually narrow the gap.

5.1 The *quid-pro-quo* equilibrium

We begin by constructing a Markov strategy, called the *quid-pro-quo equilibrium* and denoted by σ^*, that constitutes a symmetric Markov equilibrium for any $\#(S_1)$ and $\#(S_2)$, provided players are sufficiently patient. The basic pattern of behavior in this equilibrium can be best understood in the case that $\#(S_1) = \#(S_2)$: players initially mix between disclosing one element and none and after the first disclosure begin to alternate, disclosing two elements at a time until ω^* is identified. In the case with $\#(S_1) \neq \#(S_2)$ players either start alternating immediately, if $|\#(S_1) - \#(S_2)| = 1$; or, if the difference between $\#(S_1)$ and $\#(S_2)$ is not too large, the player with the larger set discloses the difference plus one, after which alternation starts; or, if the difference is large, the player with the smaller possibility set discloses all but one element, unless doing so yields a negative payoff, in which case there is no disclosure.

In order to describe σ^* it proves useful to recursively define the payoff $\phi(n)$ that a player receives when n elements remain in his opponent's possibility set, $n-1$ remain in his set, and starting from his opponent the players alternate disclosing two elements until action ω^* is taken by one player either because it was disclosed by the other player or because he disclosed all but one element and yet the other player did not end the game thereafter. Similarly, we define the payoff $\psi(n)$ that a player receives when n elements remain in his set, $n-1$ elements remain in his opponent's set, and starting with himself players alternate disclosing two elements until action

ω^* is taken by one player as explained above. Thus,

$$\phi(3) := \delta\left(\frac{2}{3}\alpha + \frac{\delta}{3}\beta\right), \quad \psi(3) := \delta\left(\frac{2}{3}\beta + \frac{\delta}{3}\alpha\right);$$
$$\phi(n) := \delta\left(\frac{2}{n}\alpha + \frac{n-2}{n}\psi(n-1)\right), \quad \psi(n) := \delta\left(\frac{2}{n}\beta + \frac{n-2}{n}\phi(n-1)\right), \quad n \geq 4.$$

For expositional ease, we define $\phi(2) = \delta\alpha$ and $\psi(2) = \delta\beta$.[10]

In the sequel an $n \times k$ game corresponds to the case with $\#(S_1) = n \geq 2$ and $\#(S_2) = k \geq 2$. For players in the $n \times n$ game to be willing to mix between disclosing one element and none in period 1 as described above, the payoff from doing so, denoted by $V_{n,n}$, must satisfy:

$$\begin{aligned} V_{n,n} &= p_n\delta\frac{(2n-1)(\alpha+\beta)/2 + (n-1)^2 V_{n-1,n-1}}{n^2} + (1-p_n)\delta\frac{\beta + (n-1)\phi(n)}{n} \\ &= p_n\delta\frac{\alpha + (n-1)\psi(n)}{n} + (1-p_n)\delta V_{n,n} \end{aligned} \tag{23}$$

where p_n is the probability with which each player discloses one element in the initial period, $V_{n-1,n-1}$ is the continuation payoff when $\#(S_1) = \#(S_2) = n-1 \geq 2$, and the first and second line in equation (23) refer to the payoff from disclosing one and none respectively. Given that the values $V_{n,n}$ have been obtained for $n = 2$ and 3 in the previous sections, the values $V_{n,n}$ for higher n can be calculated by solving the simultaneous equation system (23) for $V_{n,n}$ and p_n recursively for $n = 4, 5, \cdots$. We show in the appendix that there is a unique legitimate solution to (23) in the sense that $p_n \in (0,1)$, provided that $\delta < 1$ is sufficiently large. The solution value of p_n thus obtained, denoted by p_n^*, converges to 0 as δ tends to 1.

Using the functions ϕ, ψ and p_n^*, we now describe the strategy σ^* as below:

$$\sigma_i^*(h_i^t)(\langle\omega\rangle) = 1 \quad \text{if} \quad h_i^t \in H_i^*(\omega) \cup H_i^\circ(\omega);$$

[10] The payoff $\phi(2) = \delta\alpha$ can be thought of as applying to the situation in which as part of the alternating pattern of disclosures of two elements each, starting with $n \geq 4$ and the initial sizes of possibilitiy sets differing by one, a player has disclosed all but one of the elements in his possibility set and his opponent has failed to act in response. The player then anticipates taking the correct action himself in the next period. The payoff $\psi(2) = \delta\beta$ has a similar interpretation.

and for all other $h_i^t \in H_i$, denoting $n = \#(S_i^t)$ and $k = \#(S_{-i}^t)$,

(i)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $k = 1$ (variations possible as in Remark 1 above)	
(ii)	$\sigma_i^*(h_i^t)(1) = p_n^*$ $\sigma_i^*(h_i^t)(\emptyset) = 1 - p_n^*$	if $n = k \geq 2$	
(iii)	$\sigma_i^*(h_i^t)(2) = 1$	if $n = k + 1 > 2$	
(iv)	$\sigma_i^*(h_i^t)(n-1) = 1$	if $n > k = 2$	and $(n-1)\beta + \phi(2) > 0$
(v)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $n > k = 2$	and $(n-1)\beta + \phi(2) \leq 0$
(vi)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $n = 2 < k$	and $(k-1)\beta + \phi(2) > 0$
(vii)	$\sigma_i^*(h_i^t)(1) = 1$	if $n = 2 < k$	and $(k-1)\beta + \phi(2) \leq 0$
$(viii)$	$\sigma_i^*(h_i^t)(n-k+1) = 1$	if $n > k > 2$	and $(n-k+1)\beta + (k-1)\phi(k) > 0$
(ix)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $n > k > 2$	and $(n-k+1)\beta + (k-1)\phi(k) \leq 0$
(x)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $k > n > 2$	and $(k-n+1)\beta + (n-1)\phi(n) > 0$
(xi)	$\sigma_i^*(h_i^t)(n-1) = 1$	if $k > n > 2$	and $(k-n+1)\beta + (n-1)\phi(n) \leq 0$ and $(n-1)\beta + \phi(2) > 0$
(xii)	$\sigma_i^*(h_i^t)(\emptyset) = 1$	if $k > n > 2$	and $(k-n+1)\beta + \delta(n-1)\phi(n) \leq 0$ and $(n-1)\beta + \phi(2) \leq 0$.

The general principles that guide the formulation of this strategy are that if players have identically sized possibility sets, they randomize over when to disclose one element, (ii); and, otherwise the player with the larger possibility set discloses just enough elements to reverse the order of informedness, provided the size differential is not so large as to make such a disclosure (followed by two-by-two alternation) unprofitable, (iii), (iv) and (viii). If the size differential is too large to make the minimal order-reversing disclosure profitable, then the player with the larger possibility set stays put, (v), (ix). The player with the smaller possibility set stays put when it is profitable for the player with the larger possibility set to make a minimal order-reversing disclosure, (vi) and (x). Otherwise, the player with the smaller possibility set discloses all but one element, provided doing so is profitable, (vii) and (xi). If the size difference is too large to make a minimal order-reversing disclosure by the large size player unprofitable and it is not profitable for the small-size player to disclose all but one element, then the small-size player stays put, (xii).

We briefly summarize the key elements of verifying that if both players adopt this strategy we have an equilibrium (the details are in the Appendix): When the strategy σ^* prescribes that the player with the larger possibility set discloses some number of elements, then this number is equal to the size difference plus one. To disclose fewer elements, and following the strategy thereafter, would be suboptimal because following such a disclosure the strategy σ^* prescribes that the same player discloses again, either because players now have equal sized possibility sets and engage in a war of attrition to break the symmetry, or because the order of the possibility set sizes has remained the same; in either case because of discounting the player with the larger possibility set prefers disclosing a given number of elements all at once rather than disclosing that same number in multiple installments. To disclose more would be suboptimal because it would result in giving away too much information too quickly (see Lemma 7 in Appendix for the details).

A player who according to the strategy σ^* is designated to disclose when having the smaller possibility set is meant to disclose all but one of the elements of that set, and the expected payoff from doing so is positive. To disclose less, and thereafter continuing to follow σ^*, would be suboptimal because the same player would be called upon to disclose again, since the size gap would have increased, lessening the incentive of the other player to make the minimal order-reversing disclosure, and as before because of discounting it is preferable to disclose all but one element all at once rather than disclosing that same number in multiple installments. Hence, if the player with the larger possibility set does not disclose, it is optimal for the player with the smaller possibility set to disclose all but one as long as the payoff from disclosing some number of elements if positive.

Having verified the optimality of disclosures stipulated by σ^*, we now turn to the optimality of non-disclosure: There are the following four cases in which a player i is supposed not to disclose:

(1) Player i has the larger possibility set and player $-i$ does not disclose either. In this case, if instead player i disclosed such a small number of elements that he remains in this case, then the direct payoff impact of the initial disclosure is negative and there are no other consequences. If he discloses more, but short of reversing the order, then σ^* prescribes that thereafter he makes the minimal order-reversing disclosure. If this were profitable, then by discounting he would have been better off making the minimal order-reversing disclosure at the outset, but a defining characteristic of the present case is that making the minimal order reversing disclosure is not profitable. Disclosing even more is not profitable because of the negative payoff impact of parting with more information too soon (as detailed in Lemma 7).

(2) Player i has the larger possibility set and player $-i$ discloses all but one of his elements. In this case the immediate payoff consequence of disclosure is sufficiently negative to make disclosure unattractive.

(3) Player i has the smaller possibility set and player $-i$ does not disclose. If instead player i disclosed a sufficiently small number then this would take us back to the same case. If player i disclosed a number of elements large enough that σ^* then would prescribe to disclose all but one, and if doing so were profitable, then by discounting it would be even more attractive to disclose all but one immediately, and doing so is unprofitable under the conditions of the case in question.

(4) Player i has the smaller possibility set and player $-i$ makes the minimal order-reversing disclosure. Suppose a deviation of player i disclosing as well were profitable. Then player i would do even better by postponing that disclosure until the next period, because the resulting state would be the same and he could first take advantage of player $-i$'s disclosure, assuming players are patient enough.

The following result summarizes our discussion and is proved in the appendix.

Proposition 3 *For any $N \geq 2$, there is a $\delta_N \in (0, 1)$ such that for all $\delta \in (\delta_N, 1)$ and all $S_i, i = 1, 2$ such that $\max\{\#(S_1), \#(S_2)\} \leq N$, the strategy σ^* constitutes a symmetric Markov equilibrium. There exists an integer $\lambda \geq 2$ such that σ^* is efficient in the limit as $\delta \to 1$ provided $0 < |\#(S_1) - \#(S_2)| \leq \lambda$.*

5.2 The efficient equilibrium

The focal symmetric *quid-pro-quo* equilibrium, σ^*, achieves efficiency whenever information exchange takes place, except when both players start with the same *odd* number of elements. In the latter case, inefficiency stems from a delay in the war of attrition stage needed in early periods to break symmetry. The inefficiency does not disappear even when $\delta \to 1$ if players start with an odd number of elements (think of three, for example), because the first-mover disadvantage persists given that the probability with which the first-mover will win is calculated to be a ratio of integers with an odd number as the denominator that does not exceed one half (and thus is bounded away from one half).

The *quid-pro-quo* equilibrium payoff of a $n \times n$ game is routinely calculated to converge, as $\delta \to 1$, to

$$\frac{\alpha+\beta}{2} \text{ if } n \text{ is even;} \quad \frac{(n^2-1)\alpha+(n^2+1)\beta}{2n^2} \text{ if } n \text{ is odd.}$$

Note that the inefficiency disappears asymptotically as $n \to \infty$.

In contrast, if one sacrifices symmetry, efficiency can be achieved for any (n, k) in the limit as $\delta \to 1$. Without the symmetry requirement, we can avoid costly wars of attraction. The other potential inefficiency that arose with σ^* in Proposition 3 (not tied to symmetry) was that sometimes the gap in the sizes of the initial possibility sets was too large to bridge in one step. This can be circumvented by having players close the gap gradually while maintaining the general principle that they alternate disclosing a relatively small fraction of remaining elements. For example, in a 5×11 game, counting his 11 elements as 5 *units* of two elements and one residual, player 2 first discloses three elements (leaving one less *units* undisclosed than his opponent's elements), then player 1 discloses 2 elements, followed by 2 disclosing 2 units (4 elements), and so on, until ω^* is identified. This give us our final major result, which is stated below and proved in the appendix.

Theorem 2 *For every $n \times k$ game there is a $\underline{\delta} \in (0, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a Markov equilibrium in which information disclosure takes place in every round until ω^* is identified. Consequently, the game ends in no more than $n + k + 1$ rounds and efficiency is achieved in the limit as $\delta \to 1$.*

For any (n, k), in the *efficient equilibrium* constructed for Theorem 2, disclosure starts without delay and continues until ω^* is identified, which takes place within a finite number periods. At the same time, our earlier result (Theorem 1) implies that the maximum number of information exchanges that may take place in any such equilibrium increases without bound as n and k increase. In summary, if players are patient, or equivalently disclosures can be made at a rapid rate, information exchange can be made efficient even though it requires protracted rounds of alternating information provision.

6 Concluding Remarks

Our interest in this paper has been in understanding the interaction between rivals who compete to be the first to learn the truth, but depend on each other's information to be able to do so. We found that in such situations information exchange is possible even if only one party gains *ex post*. It necessarily takes multiple rounds, with the number of rounds growing without bound as each side is made less informed. The desire not to be the first to reveal information can be a source of inefficiency. Nevertheless, with patient players there are equilibria in which efficient information exchange is achieved.

A key driver of our analysis is the screening motive that sometimes makes players willing to disclose all but one of the elements of their possibility sets in the hope to infer that it is the true state from a failure of the other player ending the game. This argument makes use of our assumption that the two players' possibility sets have exactly one element in common, but this is not critical. The analysis continues to hold as long as the probability of there being a single common element is sufficiently high.

In the symmetric equilibrium σ^* specified in Section 5, for example, the player's equilibrium strategy is uniquely optimal for every Markov state except when the two players' remaining possibility sets are of the same size $n \geq 2$; and in the latter case they mix between disclosing one element and none with a probability that uniquely solves the equation for the payoff equivalence of the two options. When the two players' initial possibility sets may have multiple elements in common with small enough a probability, the expected payoff from each feasible choice at every Markov state is a continuous function of such probability (presuming the other player behaves according to σ^*), preserving optimality of σ^* except at Markov states at which σ^* prescribes mixing between disclosing one element and none.[11] With such mixing probabilities fine-tuned to retain their payoff equivalence, therefore, σ^* continues to be an equilibrium. However, efficiency is impaired because the common element disclosed by the other player may not be the true state of the world, albeit with a

[11]Additional Markov states arise when a player discloses multiple elements which are also in the remaining possibility set of the other player. In such states, it is optimal for the latter player to do nothing in all future periods. Arising with such a small probability, these additional states do not affect optimality of σ^*.

small probability.

Discreteness of the possibility sets is not necessary for our analysis, either. For instance, suppose that the two players' initial possibility sets are two continua with intersection X that has mass $x > 0$, and the game ends when either player takes an action in X. Then, by taking a random action in a set $S \supset X$ a player takes an action in X with a probability $x/m(S)$ where $m(S)$ is the mass of S; and the probability that a player discloses an element of X by disclosing a fraction f of $S \supset X$ is a decreasing function that converges to $x/m(S)$ as f tends to 0. In this setting, it is straightforward to verify that an equilibrium exists in which players disclose appropriate fractions so that the probability of disclosing an element of X in each round is the same as that in σ^*, supported by the off-equilibrium belief that disclosing any other fraction would induce no further disclosure.[12]

One may wonder what would happen if players may not disclose elements in their possibility set S_i, but only those in the complement of S_i. In this case, no information disclosure may take place that leads to identification of the true state ω^* with a positive probability. To see this, suppose to the contrary that an equilibrium existed where ω^* is taken with a positive probability. Given a finite state space Ω, there is a finite sequence of disclosures after which ω^* is taken with a positive probability. Suppose player 1 is to disclose in the last stage of this disclosure sequence: by doing so, player 1 risks losing the game in the case that all remaining elements in $S_2 \setminus \{\omega^*\}$ are disclosed, without anything to gain even if player 2 didn't take ω^* because that fact wouldn't eliminate any element of S_1 from being the true state. Thus, player 1 should find it suboptimal to disclose anything in the last stage of disclosure, upsetting the presumed equilibrium. That is, the screening motive is absent and the last disclosure is unsustainable when players may only disclosure elements outside of their possibility sets, precluding any beneficial information exchange.

We have considered disclosures backed by "hard" evidence. One might also wonder what would happen if the players communicated by cheap talk. In that case, the no common labeling assumption needs to be abandoned in order for there to be a language for communication. Then, when a player has two or three elements in his possibility

[12] If $x = 0$, then the screening motive is absent and agents alternate disclosures indefinitely in equilibrium. However, $x = 0$ seems unrealistic if we think of the continuum as a representation of a large number of finite states in a discrete world.

set, the screening motive continues to incentivize the agent to truthfully disclose all but one element because the expected payoff from doing so exceeds that from falsely disclosing an element outside his possibility set. In particular, σ^* is an equilibrium when both players have two elements. But, incentives do not work the same way for disclosures that take place before the last disclosure. When both players have three elements, for instance, σ^* ceases to be an equilibrium because, conditional on the opponent behaving according to σ^*, a player would benefit by falsely disclosing first an element outside of his possibility set, as it would not expose him to an immediate risk of losing yet induces the opponent to disclose two elements truthfully, increasing his expected payoff. By the same token, any longer sequence of truthful disclosures by cheap talk is unsustainable. If and when cheap talk may be enough to induce information exchanges between rivals, is a research agenda we leave for the future.

Appendix

Proof of Proposition 1: For both classes of Markov equilibria considered in the statement of the proposition, we have described the equilibrium strategy along the equilibrium path and verified its optimality in the main text prior to the statement of the proposition. We now summarize the equilibrium fully for all Markov states. We use $\mathbf{1}_i^t$ as a shorthand for $\mathbf{1}(h_i^t)$, and i and j as the two players of the game.

- $\#(S_i^t) = \#(S_j^t) = 2$: the two players' planned choices are as described in the proposition depending on the equilibrium.
- $\#(S_i^t) = 1,\ \#(S_j^t) = 2$: i takes action ω^* if $\mathbf{1}_i^t = 1$ and does nothing if $\mathbf{1}_i^t = 0$. j takes action ω^* if $\mathbf{1}_j^t = 1$, and mixes between disclosing one element, disclosing two elements, and doing nothing if $\mathbf{1}_j^t = 0$ (Cf. Remark 1).
- $\#(S_i^t) = \#(S_j^t) = 1,\ \mathbf{1}_i^t = 0$: i mixes between disclosing one element and doing nothing in a certain manner.
- $\#(S_i^t) = \#(S_j^t) = 1,\ \mathbf{1}_i^t = 1$: i takes the action that he has identified as ω^*.
- $\#(S_i^t) = 0$ and $\#(S_j^t) = 1$: i does nothing unless $\mathbf{1}_i^t = 1$ in which case he takes the action that he has identified as ω^*, and j takes the action that he has identified as ω^*.
- $\#(S_i^t) = \#(S_j^t) = 0$: both players take action ω^* because $\mathbf{1}_i^t = \mathbf{1}_j^t = 1$.

The above exhaust all possible Markov states (including such off-equilibrium contingencies as a player having taken an action even if $\mathbf{1}_i^t = 0$). Lastly, stipulate a belief profile as follows: for each history h_i^t, player i's posterior is uniform among elements in $S_i^t \cup D_i^{t-1}$. We have verified optimality of the above strategy along the equilibrium path in the main text leading to Proposition 1; optimality off the equilibrium path is straightforward. $\square$

Proof of Lemma 6: For each part (a)-(d), the equilibrium strategy is described for all possible Markov states in the statement of the Lemma, except for $\#(S_1^t) = 3$ and $\#(S_2^t) = 1$, in which case both players take action ω^* if identified (i.e., if $\mathbf{1}(h_i^t) = 1$) and do nothing otherwise. Recall from the discussion in the main text that we have already verified that (a) and (b) are the only possible symmetric Markov equilibria in which at least one player adopts a pure planned choice in period 1, and the corresponding optimality along the equilibrium path. Optimality on off-equilibrium path is straightforward to verify and is left to the reader.

Consider the remaining possibility that both players adopt mixed planned choices in the initial period in a symmetric Markov equilibrium. Since taking an action is suboptimal (Lemma 2), player 2 must mix between disclosing one element and none with probabilities $p \in (0, 1)$ and $1 - p$, respectively. Then, player 1's payoff from disclosing none, one, and two elements are, respectively,

$$\begin{aligned} U_0 &= p\delta\frac{\alpha + \delta\beta}{2} + (1-p)\delta V_{3,2}, \\ U_1 &= p\delta\frac{(\alpha+\beta)/2 + 2\alpha + \beta + 2\delta\beta}{6} + (1-p)\delta\frac{\beta + 2V_{2,2}}{3}, \\ U_2 &= p\delta\frac{2(\alpha+\beta)/2 + \alpha + 2\beta + \delta(\alpha+\beta)/2}{6} + (1-p)\delta\frac{2\beta + \delta\alpha}{3} \end{aligned}$$

where $V_{3,2}$ is the equilibrium payoff of player 1. Comparing U_1 and U_2, in particular, from

$$(\alpha+\beta)/2+2\alpha+\beta+2\delta\beta-\alpha-\beta-\alpha-2\beta-\delta(\alpha+\beta)/2 = (1-\delta)(\alpha+\beta)/2-2\beta(1-\delta) > 0 \tag{24}$$

$$\text{and} \quad \frac{\beta + 2V_{2,2} - 2\beta - \delta\alpha}{1-\delta} \begin{cases} \to \frac{-(\alpha+\beta)(\alpha+5\beta)}{6\alpha-2\beta} & \text{as } \delta \to 1, \\ = \frac{-(1-\delta)\left(\beta+9\delta\beta+\sqrt{\beta^2(1+34\delta+\delta^2)}\right)}{8\delta} & \text{if } \alpha = -5\beta \end{cases} \tag{25}$$

we deduce that, for large enough $\delta < 1$,

$$\begin{cases} \text{if } \alpha + 5\beta \leq 0, \text{ then } U_1 > U_2 \text{ for all } p, \text{ and} \\ \text{if } \alpha + 5\beta > 0, \text{ then there is a unique } \tilde{p}(\delta) \in (0,1) \text{ such that} \\ \qquad U_1 > U_2 \ \text{ if } \ p > \tilde{p}(\delta), \ \text{ and } \ U_1 < U_2 \ \text{ if } \ p < \tilde{p}(\delta). \end{cases} \tag{26}$$

In the equilibrium currently under consideration, it is not possible that player 1 mixes only between disclosing one and two elements because then it is easily verified that player 2 would strictly prefer disclosing none to disclosing one element. So, there are three possibilities for player 1, (i)–(iii) below.

(i) Player 1 mixes between disclosing one and none with probabilities q and $1-q$, respectively. In this case, we should have $V_{3,2} = U_0 = U_1$, i.e.,

$$V_{3,2} = p\delta\frac{\alpha+\delta\beta}{2} + (1-p)\delta V_{3,2} = p\delta\frac{(\alpha+\beta)/2 + 2\alpha+\beta+2\delta\beta}{6} + (1-p)\delta\frac{\beta+2V_{2,2}}{3}. \tag{27}$$

As a solution to this simultaneous equation system, we derive potential equilibrium values of p and $V_{3,2}$. If $\alpha + 5\beta \leq 0$ or if the obtained value of p is in $[\tilde{p}(\delta), 1)$ and $\alpha + 5\beta > 0$, then by (26) it is indeed optimal for player 1 to mix between disclosing one and none. Note that $p \to 0$ as $\delta \to 1$, because otherwise the second equality of (27) would be violated given that the first equality would imply $\lim_{\delta\to1} V_{3,2} = \frac{\alpha+\delta\beta}{2} > \lim_{\delta\to1}\frac{\beta+2V_{2,2}}{3}$.

Optimality further requires that player 2 be indifferent between disclosing one and none:

$$V_{2,3} = q\delta\frac{\alpha+V_{2,2}}{3} + (1-q)\delta V_{2,3} = q\delta\frac{(\alpha+\beta)/2+2\beta+\alpha+2\delta\alpha}{6} + (1-q)\delta\frac{\beta+\delta\alpha}{2}$$

where $V_{2,3}$ is the equilibrium payoff of player 2. Solving this simultaneous equation system, we derive

$$V_{2,3} = \frac{\delta q}{(1-\delta+\delta q)}\Big(\frac{\alpha+2V_{2,2}}{3}\Big) = \delta q\Big(\frac{3\alpha-2\delta\alpha-\beta}{12}\Big) + \delta\frac{\beta+\delta\alpha}{2} \tag{28}$$

where

$$q = \frac{1}{2\delta(3\alpha-2\delta\alpha-\beta)}\Big(\alpha+\beta+5\alpha\delta-7\beta\delta-8\alpha\delta^2+8V_{2,2} - \sqrt{24(1-\delta)\delta(\beta+\alpha\delta)(\beta-\alpha(3-2\delta)) + (\alpha+\beta+5\alpha\delta-7\beta\delta-8\alpha\delta^2+8V_{2,2})^2}\Big) \tag{29}$$

If this value of q is in $(0, 1)$, it is indeed optimal for player 1 to mix as presumed, establishing a symmetric Markov equilibrium. Note that $q \to 0$ as $\delta \to 1$.

Derivation of the set of parameter values for which such an equilibrium exists is straightforward, albeit lengthy, which we omit here because it is not important for the purpose of this paper. Instead, we deduce from above that $V_{3,2} \to (\alpha+2\beta)/3$ and $V_{2,3} \to (\alpha+\beta)/2$ as $\delta \to 1$ in such equilibria because $p, q \to 0$. This proves part (c) of the Lemma.

(ii) Player 1 mixes between disclosing two and none with probabilities q and $1-q$, respectively. In this case, we should have $\alpha+5\beta>0$ by (26) and $V_{3,2}=U_0=U_2$, i.e.,

$$V_{3,2}=p\delta\frac{\alpha+\delta\beta}{2}+(1-p)\delta V_{3,2}=p\delta\frac{2(\alpha+\beta)/2+\alpha+2\beta+\delta(\alpha+\beta)/2}{6}+(1-p)\delta\frac{2\beta+\delta\alpha}{3}$$

As a solution to this simultaneous equation system, we derive potential equilibrium values of p and $V_{3,2}$. If the obtained value of p is in $(0,\tilde{p}(\delta)]$, it is indeed optimal for player 1 to mix between disclosing two and none. Note that $p\to 0$ as $\delta\to 1$ for the same reason as above.

Optimality requires that player 2 be indifferent between disclosing one and none:

$$V_{2,3}=q\delta\frac{2\alpha+\delta\beta}{3}+(1-q)\delta V_{2,3}=q\delta\frac{(\alpha+\beta)+2\alpha+\beta+\delta(\alpha+\beta)/2}{6}+(1-q)\delta\frac{\beta+\delta\alpha}{2}$$

where $V_{2,3}$ is the equilibrium payoff of player 2. Solving this simultaneous equation system, we derive

$$V_{2,3}=\frac{\delta q}{(1-\delta+\delta q)}\Big(\frac{2\alpha+\delta\beta}{3}\Big)=\delta q\Big(\frac{6\alpha-2\beta-\delta(5\alpha+\beta)}{12}\Big)+\delta\frac{\beta+\delta\alpha}{2} \tag{30}$$

where

$$q=\frac{1}{2\delta(6\alpha-2\beta-\delta(5\alpha+\beta))}\Big(2\alpha+2\beta+11\alpha\delta-3\beta\delta-11\alpha\delta^2-\beta\delta^2 \tag{31}$$
$$-\sqrt{24(1-\delta)\delta(\beta+\alpha\delta)(\beta(2+\delta)-\alpha(6-5\delta))+(\beta(2-3\delta-\delta^2)+\alpha(2+11\delta-11\delta^2))^2}\Big).$$

As a solution to this simultaneous equation system, we derive potential equilibrium values of q and $V_{2,3}$. If the obtained value of q is in $(0,1)$, it is indeed optimal for player 1 to mix as presumed, establishing a symmetric Markov equilibrium. Again, note that $q\to 0$ as $\delta\to 1$.

As before, we do not derive the set of parameter values for which such an equilibrium exists, but deduce that $V_{3,2}\to(\alpha+2\beta)/3$ and $V_{2,3}\to(\alpha+\beta)/2$ as $\delta\to 1$ in such equilibria. This proves part (d) of the Lemma.

(iii) Player 1 mixes among disclosing one, two, and none. For this, we need $V_{3,2}=U_0=U_1=U_2$ and also $\alpha+5\beta>0$ by (26). Note that if $\lim_{\delta\to 1}p>0$, then $V_{3,2}=U_0$ would imply $V_{3,2}\to\frac{\alpha+\beta}{2}$ as $\delta\to 1$, but this would contradict $V_{3,2}=U_1=U_2$ because U_1 and U_2 are bounded away from $\frac{\alpha+\beta}{2}$. Hence, $\lim_{\delta\to 1}p=0$. But, together with (24) and (25), this would imply $U_2>U_1$ as $\delta\to 1$. Therefore, no such equilibrium exists for large enough δ. This completes the proof. □

Proof of Proposition 2: Part (a) This part of the proposition has been proved in the main text except for the assertion that the symmetric Markov equilibrium described in part (a) is not viable if the equilibria of Lemma 6 (c) or (d) govern the continuation that ensues after one player disclosed one element. Here we prove this assertion, starting with Lemma 6 (c).

Suppose the equilibrium of Lemma 6 (c) governs the continuation. Then, $V_{2,3}$ is as derived in (28), which we denote by $V_{2,3}^c$. Similarly, denote $V_{3,2}$ for Lemma 6 (c) as $V_{3,2}^c$. Note that $\lim_{\delta\to 1} V_{2,3}^c = (\alpha+\beta)/2$ but $\lim_{\delta\to 1} V_{3,2}^c = (\alpha+2\beta)/3 = (\alpha+\beta)/2 - \frac{\alpha-\beta}{6}$ as shown earlier.

Recall that we obtained $\hat{p}$ as the crossing point of the graphs of (9) and (10), and p^* as that of the graphs of (9) and (11). The value of (11) exceeds that of (9) by an amount that converges to (15) as $\delta \to 1$ when $V_{3,2} = \delta(\alpha + \delta\beta)/2$, i.e, when the equilibrium of Lemma 6 (b) governs the continuation. When the equilibrium of Lemma 6 (c) governs the continuation instead, (10) does not change, but (11) drops due to a drop of $V_{3,2}$ to $V_{3,2}^c$, and (9) also changes due to a change from $V_{2,3}$ to $V_{2,3}^c$. We will show that for large enough $\delta < 1$, these changes maintain the feature that the value of p at which the graphs of (9) and (10) cross is greater than that at which the graphs of (9) and (11) cross.

First, note that

$$
\begin{aligned}
&\lim_{\delta\to 1} \frac{p^*}{1-\delta+p^*\delta} = \lim_{\delta\to 1} \frac{\beta+2V_{2,3}}{a+2V_{3,2}} = \frac{\alpha+2\beta}{2\alpha+\beta};\\
&\lim_{\delta\to 1} \frac{p^*}{1-\delta+p^*\delta} = \lim_{\delta\to 1} \frac{1}{(1-\delta)/p^*+\delta} \quad \Rightarrow \quad \lim_{\delta\to 1} \frac{p^*}{1-\delta} = \frac{\alpha+2\beta}{\alpha-\beta};\\
&\lim_{\delta\to 1} \frac{\hat{p}}{1-\delta+\hat{p}\delta} = \frac{6(\alpha+\beta)}{7\alpha+5\beta} \quad \text{and} \quad \lim_{\delta\to 1} \frac{\hat{p}}{1-\delta} = \frac{6(\alpha+\beta)}{\alpha-\beta};\\
&\lim_{\delta\to 1} \frac{\hat{p}}{p^*} = \frac{6(\alpha+\beta)}{\alpha+2\beta} \; > \; 6.
\end{aligned}
$$

The value of $\frac{(10)-(9)}{1-\delta}$ evaluated at $p = 0$, obtained by subtracting the increment of (9) as p increases from 0 to $\hat{p}$, from that of (10), converges as $\delta \to 1$ to

$$
\begin{aligned}
&\lim_{\delta\to 1} \frac{\delta\hat{p}}{1-\delta}\Big(\frac{5\alpha-\beta+8V_{2,2}-12V_{2,3}}{18} - \frac{2\alpha-\delta\alpha-\beta}{9}\Big)\\
&= \lim_{\delta\to 1} \delta\hat{p}\frac{\alpha-5\beta+4\delta\beta-\delta(1-\delta)(4\alpha-(3\alpha-\beta)p_2^*)}{18(1-\delta)}\\
&= \lim_{\delta\to 1} \frac{\delta\hat{p}}{1-\delta}\Big(\frac{\alpha-5\beta+4\delta\beta}{18}\Big) - \lim_{\delta\to 1} \delta\hat{p}\frac{\delta(4\alpha-(3\alpha-\beta)p_2^*)}{18}\\
&= \frac{6(\alpha+\beta)}{\alpha-\beta}\Big(\frac{\alpha-\beta}{18}\Big) = \frac{\alpha+\beta}{3}.
\end{aligned}
$$

The amount of drop in (11) at $\hat{p}$ converges to $\frac{6(\alpha+\beta)}{7\alpha+5\beta}\frac{2}{3}\frac{\alpha-\beta}{6} = \frac{2(\alpha+\beta)(\alpha-\beta)}{3(7\alpha+5\beta)}$ as $\delta \to 1$, which is a fraction

$$\frac{6(\alpha+\beta)}{\alpha-\beta} \tag{32}$$

of (15).

Consider the graphs in the 2-dimensional space where the horizontal axis measures $\frac{p}{1-\delta}$ and the vertical axis measures the values of (9)$\sim$(11) divided by $(1-\delta)$. Then, at the limit $\delta \to 1$, when Lemma 6 (b) prevails, (10) is linear with a slope $\frac{\alpha-\beta}{9}$; (9) is lower at $p=0$ by an amount that converges to $\frac{\alpha+\beta}{3}$ and is also linear with a steeper slope $\frac{\alpha-\beta}{6}$ and crosses (10) at $\frac{\hat{p}}{1-\delta}$; and (11) starts from 0 at $p=0$ and is strictly concave and crosses (9) at $\frac{p^*}{1-\delta} < \frac{\hat{p}}{6(1-\delta)}$, and at $\frac{\hat{p}}{1-\delta}$ its value exceeds (9) and (10) by an amount of (15) divided by $(1-\delta)$, i.e., $\frac{(\alpha-\beta)(5\alpha+4\beta)}{3(7\alpha+5\beta)(1-\delta)}$, which explodes as $\delta \to 1$. Hence, (11) crosses (10) at a point arbitrarily close to $\lim_{\delta\to 1}\frac{p^*}{1-\delta} = \frac{\alpha+2\beta}{\alpha-\beta}$ as $\delta \to 1$.

When the equilibrium of Lemma 6 (c) governs the continuation instead, (10) remains the same, and (11) drops less than a fraction (32) of $\frac{(\alpha-\beta)(5\alpha+4\beta)}{3(7\alpha+5\beta)(1-\delta)}$ at $\frac{p}{1-\delta} < \frac{\hat{p}}{1-\delta}$, and continues to be strictly concave. As $\delta \to 1$, therefore, (11) crosses (10) at a point arbitrarily close to a fraction (32) toward $\lim_{\delta\to 1}\frac{\hat{p}}{1-\delta} = \frac{6(\alpha+\beta)}{\alpha-\beta}$ from $\lim_{\delta\to 1}\frac{p^*}{1-\delta} = \frac{\alpha+2\beta}{\alpha-\beta}$, i.e., $\frac{\alpha+2\beta}{\alpha-\beta} + \frac{6(\alpha+\beta)}{\alpha-\beta}(\frac{6(\alpha+\beta)}{\alpha-\beta} - \frac{\alpha+2\beta}{\alpha-\beta}) = \frac{3\alpha+4\beta}{\alpha-\beta} < \frac{1}{2}\frac{6(\alpha+\beta)}{\alpha-\beta}$. Therefore, if (9) under Lemma 6 (c) is larger than under Lemma 6 (b) by less than one half of $\frac{\alpha+\beta}{3}$ at $p=0$ at the limit as $\delta \to 1$, then (9) crosses (11) before it crosses (10) under Lemma 6 (c) as desired.

Finally, we show that (9) under Lemma 6 (c) is larger than under Lemma 6 (b) by less than one half of $\frac{\alpha+\beta}{3}$ at $p=0$ at the limit as $\delta \to 1$. By differentiating (28) with respect to δ and rearranging, we get

$$\begin{aligned}
&\delta q'\Big(\frac{3\alpha-2\delta\alpha-\beta}{12} - \frac{(1-\delta)(\alpha+2V_{2,2})}{3(1-\delta+\delta q)^2}\Big) \\
= \;& \frac{q}{3(1-\delta+\delta q)}\Big(\frac{\alpha+2V_{2,2}}{1-\delta+\delta q} + \delta 2V'_{2,2}\Big) - q\frac{3\alpha-4\delta\alpha-\beta}{12} - \delta\alpha - \frac{\beta}{2}
\end{aligned}$$

where q' and $V'_{2,2}$ denote the derivatives with respect to δ. Taking the limit as $\delta \to 1$ and rearranging,

$$(\lim_{\delta\to 1} q')\frac{\alpha-\beta}{12} - \Big(\lim_{\delta\to 1}\frac{q+\delta(1-\delta)q'}{(1-\delta+\delta q)^2}\Big)\frac{\alpha+2V_{2,2}}{3} = \lim_{\delta\to 1}\frac{2\delta q V'_{2,2}}{3(1-\delta+\delta q)} - \frac{2\alpha+\beta}{2}. \tag{33}$$

In addition, from taking the limit of (28), as $\lim_{\delta\to 1} q = 0$, we have

$$\lim_{\delta\to 1}\frac{\delta q}{1-\delta+\delta q} = \frac{3(\alpha+\beta)}{2(2\alpha+\beta)};\quad \lim_{\delta\to 1}\frac{1-\delta}{1-\delta+\delta q} = \frac{\alpha-\beta}{2(2\alpha+\beta)};\quad \lim_{\delta\to 1}\frac{q}{1-\delta} = \frac{3(\alpha+\beta)}{\alpha-\beta}. \tag{34}$$

Hence, (33) becomes

$$(\lim_{\delta\to1} q')\frac{\alpha-\beta}{12}-\Big(\lim_{\delta\to1}\frac{(1-\delta)(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q')}{(1-\delta+\delta q)^2}\Big)\frac{\alpha+2V_{2,2}}{3}=\frac{(\alpha+\beta)(\lim_{\delta\to1}V'_{2,2})}{2\alpha+\beta}-\frac{2\alpha+\beta}{2}.$$

Since the RHS is finite, so must be the LHS. If $\lim_{\delta\to1}(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q')\neq0$, then $\lim_{\delta\to1}q'$ must be $-\infty$ for the equality to hold, but $\lim_{\delta\to1}q'=-\infty$ is impossible from (29). Hence, we must have $\lim_{\delta\to1}(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q')=0$ so that $\lim_{\delta\to1}q'=-\frac{3(\alpha+\beta)}{\alpha-\beta}$. Consequently, from (28),

$$\frac{dV^c_{2,3}}{d\delta}=(q+\delta q')\frac{3\alpha-2\delta\alpha-\beta}{12}-\delta q\frac{\alpha}{6}+\frac{\beta}{2}+\delta\alpha\ \to\ \frac{3\alpha+\beta}{4}\ \text{ as }\ \delta\to1.$$

Hence, $V^c_{2,3}(\delta)$ is first-order approximated as $\frac{3\alpha+\beta}{4}\delta-\frac{\alpha-\beta}{4}$ near $\delta=1$ because $V^c_{2,3}(1)=\frac{\alpha+\beta}{2}$. Since $V_{2,3}=\delta(\beta+\delta\alpha)/2$ under Lemma 6 (b),

$$\lim_{\delta\to1}\frac{V^c_{2,3}-V_{2,3}}{1-\delta}=\lim_{\delta\to1}\frac{(3\alpha+\beta)\delta-(\alpha-\beta)-2\delta(\beta+\delta\alpha)}{4(1-\delta)}=\lim_{\delta\to1}\frac{2\alpha\delta-\alpha+\beta}{4}=\frac{\alpha+\beta}{4}.$$

Therefore, since the equilibrium of Lemma 6 (c) governs the continuation rather than (b), (9) increases less than two-thirds of this at $p=0$ at the limit as $\delta\to1$, i.e., less than $\frac{\alpha+\beta}{6}$ which is less than one half of $\frac{\alpha+\beta}{3}$, as desired. This completes the proof that the equilibrium described in part (a) of the Proposition 2 is not viable if equilibrium of Lemma 6 (c) governs the continuation.

Next, suppose that the equilibrium of Lemma 6 (d) governs the continuation. Then, $V_{2,3}$ is as derived in (30), which we denote by $V^d_{2,3}$. Similarly, we denote $V_{3,2}$ for Lemma 6 (d) as $V^d_{3,2}$. Note that $\lim_{\delta\to1}V^d_{2,3}=(\alpha+\beta)/2$ but $\lim_{\delta\to1}V^d_{3,2}=(\alpha+2\beta)/3=(\alpha+\beta)/2-\frac{\alpha-\beta}{6}$ as shown earlier. Note that these limit values are the same as when the equilibrium of Lemma 6 (c) governs the continuation. Hence, the same arguments as above apply here identically except for some details in the last stage of showing that (9) under Lemma 6 (d) is larger than under Lemma 6 (b) by less than one half of $\frac{\alpha+\beta}{3}$ at $p=0$ at the limit as $\delta\to1$. To do this differentiate (30) with respect to δ and rearrange to get

$$\begin{aligned}&\delta q'\Big(\frac{6\alpha-2\beta-\delta(5\alpha+\beta)}{12}-\frac{(1-\delta)(2\alpha+\delta\beta)}{3(1-\delta+\delta q)^2}\Big)\\&=\ \frac{q}{3(1-\delta+\delta q)}\Big(\frac{2\alpha+\delta\beta}{1-\delta+\delta q}+\delta\beta\Big)-q\frac{6\alpha-2\beta-2\delta(5\alpha+\beta)}{12}-\delta\alpha-\frac{\beta}{2}.\end{aligned}$$

Taking the limit as $\delta\to1$ and rearranging,

$$(\lim_{\delta\to1}q')\frac{\alpha-3\beta}{12}-\Big(\lim_{\delta\to1}\frac{q+\delta(1-\delta)q'}{(1-\delta+\delta q)^2}\Big)\frac{2\alpha+\delta\beta}{3}=\lim_{\delta\to1}\frac{\delta q\beta}{3(1-\delta+\delta q)}-\frac{2\alpha+\beta}{2}.\quad(35)$$

In addition, from taking the limit of (30), as $\lim_{\delta\to 1} q = 0$, we have

$$\lim_{\delta\to 1}\frac{\delta q}{1-\delta+\delta q} = \frac{3(\alpha+\beta)}{2(2\alpha+\beta)};\ \lim_{\delta\to 1}\frac{1-\delta}{1-\delta+\delta q} = \frac{\alpha-\beta}{2(2\alpha+\beta)};\ \lim_{\delta\to 1}\frac{q}{1-\delta} = \frac{3(\alpha+\beta)}{\alpha-\beta}.$$

Hence, (35) becomes

$$(\lim_{\delta\to 1} q')\frac{\alpha-3\beta}{12} - \Big(\lim_{\delta\to 1}\frac{(1-\delta)(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q')}{(1-\delta+\delta q)^2}\Big)\frac{2\alpha+\delta\beta}{3} = \frac{(\alpha+\beta)\beta}{2(2\alpha+\beta)} - \frac{2\alpha+\beta}{2}.$$

Since the RHS is finite, so must be the LHS. If $\lim_{\delta\to 1}(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q') \neq 0$, then $\lim_{\delta\to 1} q'$ must be $-\infty$ for the equality to hold, but $\lim_{\delta\to 1} q' = -\infty$ is impossible from (31). Hence, we must have $\lim_{\delta\to 1}(\frac{3(\alpha+\beta)}{\alpha-\beta}+\delta q') = 0$ so that $\lim_{\delta\to 1} q' = -\frac{3(\alpha+\beta)}{\alpha-\beta}$. Consequently, from (30),

$$\frac{dV^d_{2,3}}{d\delta} = (q+\delta q')\frac{6\alpha-2\beta-\delta(5\alpha+\beta)}{12} - \delta q\frac{5\alpha+\beta}{12} + \frac{\beta}{2} + \delta\alpha \ \to\ \frac{3\alpha+\beta}{4} \ \text{ as } \ \delta\to 1.$$

Hence, $V^d_{2,3}(\delta)$ is first-order approximated as $\frac{3\alpha+\beta}{4}\delta - \frac{\alpha-\beta}{4}$ near $\delta = 1$ because $V^d_{2,3}(1) = \frac{\alpha+\beta}{2}$. Since $V_{2,3} = \delta(\beta+\delta\alpha)/2$ under Lemma 6 (b),

$$\lim_{\delta\to 1}\frac{V^d_{2,3}-V_{2,3}}{1-\delta} = \lim_{\delta\to 1}\frac{(3\alpha+\beta)\delta-(\alpha-\beta)-2\delta(\beta+\delta\alpha)}{4(1-\delta)} = \lim_{\delta\to 1}\frac{2\alpha\delta-\alpha+\beta}{4} = \frac{\alpha+\beta}{4}.$$

Therefore, since the equilibrium of Lemma 6 (d) governs the continuation rather than (b), (9) increases less than two-thirds of this at $p = 0$ at the limit as $\delta\to 1$, i.e., less than $\frac{\alpha+\beta}{6}$ which is less than one half of $\frac{\alpha+\beta}{3}$ as desired, completing the proof.

Part (b) This part of the proposition has been proved in the main text except for the assertion that the inequality (18) is satisfied at p^* for large enough δ if any equilibrium of Lemma 6 (c)–(d) governs the continuation that ensues after one player disclosed one element. Here we prove this assertion, starting with Lemma 6 (c).

Note from (28) that $V_{2,3} \to (\alpha+\beta)/2$ as $\delta\to 1$ with the derivative

$$\frac{\partial V_{2,3}}{\partial\delta}\bigg|_{\delta=1} \ = \ \frac{(\alpha^2-\beta^2)}{4(\alpha+3\beta-4V_{2,2})} + \frac{\beta}{2} + \alpha \ = \ \frac{3\alpha-\beta}{4}.$$

Hence, $V_{2,3}$ is approximated by $\tilde{V}_{2,3} = (\alpha+\beta)/2 - (1-\delta)(3\alpha-\beta)/4$ for δ near 1. For $\tilde{V}_{2,3}$, the LHS of (19) exceeds the RHS when evaluated at $\hat{q}$ (the difference converges to $\frac{(\alpha-\beta)(8\alpha+\beta)}{6(5\alpha+\beta)}$ as $\delta\to 1$), which implies that $q^* < \hat{q}$. This means that the inequality (18) is satisfied at q^* when $V_{2,3} = \tilde{V}_{2,3}$, thus when $V_{2,3}$ is as in (28), for large enough δ.

For Lemma 6 (d), note from (30) that $V_{2,3} \to (\alpha+\beta)/2$ as $\delta \to 1$ with

$$\left.\frac{\partial V_{2,3}}{\partial \delta}\right|_{\delta=1} = -\frac{(\alpha-3\beta)(\alpha+\beta)}{4(\alpha-\beta)} + \frac{\beta+2\alpha}{2} = \frac{3a^2+b^2}{4(\alpha-\beta)}.$$

Hence, $V_{2,3}$ is approximated by $\widehat{V}_{2,3} = (\alpha+\beta)/2 - (1-\delta)(3a^2+b^2)/(4(\alpha-\beta))$ for δ near 1. For $\widehat{V}_{2,3}$, the LHS of (19) exceeds the RHS when evaluated at $\hat{q}$ (the difference converges to $\frac{8\alpha^3+\beta^3-3\alpha\beta(\alpha+\beta)}{30\alpha^2+12\alpha\beta-6\beta^2} > 0$ as $\delta \to 1$), which implies that $q^* < \hat{q}$. This means that the inequality (18) is satisfied at q^* when $V_{2,3} = \widehat{V}_{2,3}$, thus when $V_{2,3}$ is as in (30), for large enough δ. □

In the proof of Proposition 3 we will make use of the following lemma. We use $V^*_{n,k}$ to denote the expected payoff of agent i when $\#(S_i) = n$, $\#(S_{-i}) = k$, and the two agents play according to σ^*.

Lemma 7 *Suppose $\#(S_1) = n \geq \#(S_2) = k > 2$. For $d = 1, \cdots, k-1$, let $\Psi(d)$ be agent 1's payoff when he discloses $n-k+d$ elements first, after which both agents behave according to σ^*. Then, there exist $\delta(n,k) < 1$ such that $\Psi(1) > \Psi(d)$ for $d = 2, \cdots, k-1$ if $\delta(n,k) < \delta < 1$.*

Proof: The lemma clearly holds for $n = 3$: in that case $k = 3$ and thus, for large δ player 1's expected payoff from disclosing two elements is approximately $\frac{2}{3}\beta + \frac{1}{3}\alpha$ while that from disclosing one element is approximately $\frac{1}{3}\beta + \frac{2}{3}\left(\frac{2}{3}\alpha + \frac{1}{3}\beta\right) > \frac{2}{3}\beta + \frac{1}{3}\alpha$.

Assume $n > 3$ below. Observe that

$$\Psi(1) = \frac{n-k+1}{n}\delta\beta + \frac{k-1}{n}\delta^2\phi(k) \quad \text{and} \quad \Psi(d) = \frac{n-k+d}{n}\delta\beta + \frac{k-d}{n}\delta^2 V^*_{k-d,k}. \tag{36}$$

If $k - d = 1$, then $\Psi(d) = \delta((n-1)\beta + \delta\alpha)/n < \Psi(1)$ is clear for large enough δ.

Hence, consider a $d \geq 2$ such that $k - d \geq 2$. First, suppose that agent 2 would disclose none if the game does not end after player 1 disclosed $k-d$ elements. If agent 1 would disclose none in that case as well, then $V^*_{k-d,k} = 0$; if agent 1 would disclose all but one in that case, then $V^*_{k-d,k} = \delta\left(\frac{k-d-1}{k-d}\beta + \frac{1}{k-d}\delta\alpha\right) < \phi(k)$ for sufficiently large δ, where the inequality follows because $k - d \geq 2$ while $\psi(k) < \phi(k)$ and $\psi(k) + \phi(k) \to \alpha + \beta$ as $\delta \to 1$ and thus, $\lim_{\delta\to 1}\phi(k) \geq (\alpha+\beta)/2$. In either case, $\Psi(1) > \Psi(d)$ follows from (36).

Next, consider $d \geq 2$ such that $k-d \geq 2$ and agent 2 would disclose $d+1$ if the game does not end after player 1 disclosed $k-d$ elements. This would imply that if player 1 disclosed one less element, then agent 2 would disclose d elements subsequently, because $(d+1)\beta + \delta(k-d-1)\phi(k-d) > 0$ implies $d\beta + \delta(k-d)\phi(k-d+1) > 0$ if $d < k-2$, or $(k-1)\beta + \alpha > 0$ implies $(k-2)\beta + \delta 2\phi(3) > 0$ if $d = k-2$, for sufficiently large δ, as is easily verified by routine calculations. Due to this property, it suffices to show that $\Psi(d') > \Psi(d)$ where $d' = d-1$, which is done below.

If $(k-d')$ is an even number, the ex ante probabilities that agent 1 will eventually win the game when he initially discloses $n-k+d'$ and $n-k+d$ elements are, respectively,

$$\begin{aligned}\rho(d') &= \frac{k-d'}{n}\cdot\frac{d'+1}{k} + \frac{k-d'-2}{n}\cdot\frac{2}{k} + \cdots + \frac{4}{n}\cdot\frac{2}{k} + \frac{2}{n}\cdot\frac{2}{k} \\ &= \frac{2\big(2+4+\cdots+(k-d')\big) + (k-d')(d'-1)}{nk},\end{aligned}$$

and

$$\begin{aligned}\rho(d) &= \frac{k-d'-1}{n}\cdot\frac{d'+2}{k} + \frac{k-d'-3}{n}\cdot\frac{2}{k} + \cdots + \frac{3}{n}\cdot\frac{2}{k} + \frac{1}{n}\cdot\frac{2}{k} \\ &= \frac{2\big(1+3+\cdots+(k-d'-1)\big) + (k-d'-1)d'}{nk} \\ &= \frac{2\big(2+4+\cdots+(k-d')\big) + (k-d')(d'-1) - d'}{nk} \ < \ \rho(d').\end{aligned}$$

Since one of the agents eventually will win the game,

$$\Psi(d') \to \rho(d')\alpha + (1-\rho(d'))\beta \quad \text{and} \quad \Psi(d) \to \rho(d)\alpha + (1-\rho(d))\beta \quad \text{as} \quad \delta \to 1.$$

As desired, therefore, $\Psi(d') > \Psi(d)$ for sufficiently large δ because $\rho(d') > \rho(d)$.

Analogously, if $(k-d')$ is an odd number, the ex ante winning probabilities for agent 1 when he initially discloses $n-k+d'$ and $n-k+d$ elements are, respectively,

$$\begin{aligned}\rho(d') &= \frac{k-d'}{n}\cdot\frac{d'+1}{k} + \frac{k-d'-2}{n}\cdot\frac{2}{k} + \cdots + \frac{3}{n}\cdot\frac{2}{k} + \frac{1}{n}\cdot\frac{2}{k} \\ &= \frac{2\big(1+3+\cdots+(k-d')\big) + (k-d')(d'-1)}{nk} \\ &= \frac{2\big(2+4+\cdots+(k-d'-1)\big) + (k-d')d' + 1}{nk}\end{aligned}$$

and

$$\begin{aligned}\rho(d) &= \frac{k-d'-1}{n}\cdot\frac{d'+2}{k} + \frac{k-d'-3}{n}\cdot\frac{2}{k} + \cdots + \frac{4}{n}\cdot\frac{2}{k} + \frac{2}{n}\cdot\frac{2}{k} \\ &= \frac{2\big(2+4+\cdots+(k-d'-1)\big) + (k-d'-1)d'}{nk} \ < \ \rho(d').\end{aligned}$$

Since one of the agents eventually will win,

$$\Psi(d') \to \rho(d')\alpha + (1-\rho(d'))\beta \quad \text{and} \quad \Psi(d) \to \rho(d)\alpha + (1-\rho(d))\beta \quad \text{as} \quad \delta \to 1$$

and therefore, $\Psi(d') > \Psi(d)$ for sufficiently large δ as desired. $\square$

Proof of Proposition 3: By Proposition 2, when $N = 3$ the proposition holds and the equilibrium payoffs are

$$\lim_{\delta\to 1} V^*_{n,n} = \frac{\alpha+\beta}{2} \text{ for even } n \le N; \quad \lim_{\delta\to 1} V^*_{n,n} = \frac{(n^2-1)\alpha + (n^2+1)\beta}{2n^2} \text{ for odd } n \le N. \tag{37}$$

For induction purposes, suppose that the proposition and (37) hold for some $N \ge 3$. Below we prove that then the proposition and (37) hold when N is replaced by $N+1$ as well.

The induction hypothesis means that the equation system (23) has a unique solution, denoted by p^*_N and $V^*_{N,N}$, when $n = N$ and $V_{N-1,N-1} = V^*_{N-1,N-1}$. First, we verify that $p^*_N \to 0$ as $\delta \to 1$ from this equation system: If $\lim_{\delta\to 1} p^*_N > 0$, then $\lim_{\delta\to 1} V^*_{N,N} = \frac{\alpha+(N-1)\psi(N)}{N}$. Since $\frac{\alpha+(N-1)\psi(N)}{N} \ge \frac{\beta+(N-1)\phi(N)}{N}$ (otherwise, disclosing one element would be uniquely optimal when both players have N elements), $\frac{\alpha+(N-1)\psi(N)}{N} + \frac{\beta+(N-1)\phi(N)}{N} \to \alpha+\beta$ as $\delta \to 1$, and $\lim_{\delta\to 1} V^*_{N,N} \le \frac{\alpha+\beta}{2}$, it would follow that $\lim_{\delta\to 1} V^*_{N,N} = \frac{\alpha+(N-1)\psi(N)}{N} = \frac{\beta+(N-1)\phi(N)}{N} = \frac{\alpha+\beta}{2}$. Then, the second equality of (23) would be violated in light of (37). This proves that $p^*_N \to 0$ as $\delta \to 1$.

Now, we verify that σ^* constitutes an equilibrium for a $(N+1)\times(N+1)$ game under the induction hypothesis.

The first task is to verify that σ^* constitutes an equilibrium in the continuation, denoted by $G_{N+1,N}$, that starts in period t with $\#(S^t_1) = N+1$ and $\#(S^t_2) = N$ (conditional on the game does not end in period t). This verification consists of four steps as below.

<u>Step 1: player 1 prefers disclosing two elements rather than one in period t.</u>

Player 2 does nothing in period t according to σ^*. We consider the strategy of player 1 disclosing one element, denoted by ω_t, in period t, and compare it with what would have happened if he disclosed two instead. If $\omega_t = \omega^*$, then it does not matter if he disclosed one more in period t because he would lose the game in period $t+1$ either way.

Consider the contingency that $\omega_t \ne \omega^*$. Then, they start a $N \times N$ game in period $t+1$, and as they mix between disclosing one and none in the first period of this continuation, player 1 would get the equilibrium payoff $V^*_{N,N}$ by disclosing one in period $t+1$. If player 2 were to disclose none in period $t+1$ (which happens with a probability $1-p^*_N$), then player 1 would have done better by disclosing one more in period t because then, given that $\omega_t \ne \omega^*$, he would get $X(\delta) := \frac{\beta+(N-1)\phi(N)}{N}$

as of period $t+1$ rather than as of period $t+2$; If player 2 were to disclose one element in period $t+1$ (which happens with a probability p_N^*), player 1 would get $Y(\delta) := \delta \frac{(2N-1)(\alpha+\beta)/2+(N-1)^2 V_{N-1,N-1}^*}{N^2}$ as of period $t+1$, but if he disclosed one more in period t, then he would have gotten $X(\delta)$ as of period $t+1$ as explained above. Hence, the extra payoff for player 1 from disclosing two rather than one in period t is

$$\Delta u(\delta) := \frac{N}{N+1}\Big[\big(1-p_N^*(\delta)\big)(1-\delta)X(\delta) + p_N^*(\delta)\big(X(\delta)-Y(\delta)\big)\Big] \tag{38}$$

where $p_N^*(\delta)$ explicitly indicates dependence of p_N^* on δ. Differentiating with respect to δ,

$$\begin{aligned} \Delta u'(\delta) \quad = \quad & \frac{N}{N+1}\Big[-{p_N^*}'(\delta)(1-\delta)X(\delta) - (1-p_N^*(\delta))\big(X(\delta)-(1-\delta)X'(\delta)\big) \\ & \quad + {p_N^*}'(\delta)X(\delta) + p_N^*(\delta)X'(\delta) - {p_N^*}'(\delta)Y(\delta) - p_N^*(\delta)Y'(\delta)\Big]. \end{aligned}$$

Note that $|X'(\delta)| < \infty$ because $X(\delta)$ is a finite-order polynomial, and $p_N^*(\delta) \to 0$ as $\delta \to 1$ as asserted earlier. Therefore,

$$\begin{aligned} & \Delta u'(\delta) \;\to\; \frac{N}{N+1}\Big[-X(1) + \lim_{\delta\to 1}{p_N^*}'(\delta)\big(X(1)-Y(1)\big)\Big] \quad \text{as} \quad \delta \to 1 \\ & \text{if} \quad |Y'(1)| = |\lim_{\delta\to 1} Y'(\delta)| < \infty \quad \text{and} \quad -\infty < {p_N^*}'(1) = \lim_{\delta \to 1}{p_N^*}'(\delta) < 0. \end{aligned} \tag{39}$$

As $\Delta u(\delta) \to 0$ as $\delta \to 1$, if we show (39) and

$$-X(1) + \lim_{\delta\to 1}{p_N^*}'(\delta)\big(X(1)-Y(1)\big) < 0, \tag{40}$$

then $\Delta u(\delta) > 0$ for large enough $\delta < 1$ and thus, player 1 would prefer disclosing two rather than one element in period t.

From $\lim_{\delta\to 1} V_{N,N}^* = (\beta + (N-1)\phi^*(N))/N$ and $\phi^*(N) + \psi^*(N) = \alpha + \beta$ where $\phi^*(N) = \phi(N)|_{\delta=1}$ and $\psi^*(N) = \psi(N)|_{\delta=1}$, together with (37), we deduce that

$$\phi^*(N) = \frac{N\alpha + (N-2)\beta}{2(N-1)} \;\text{ and }\; \psi^*(N) = \frac{(N-2)\alpha + N\beta}{2(N-1)} \;\text{ if } N \text{ is even,} \tag{41}$$

and

$$\phi^*(N) = \frac{(N^2-1)\alpha + (N^2-2N+1)\beta}{2N(N-1)} \;\text{ and }\; \psi^*(N) = \frac{(N^2-2N+1)\alpha + (N^2-1)\beta}{2N(N-1)} \;\text{ if } N \text{ is odd.} \tag{42}$$

First, consider even $N \geq 4$. Then,

$$X(1) = \frac{\beta + (N-1)\frac{N\alpha+(N-2)\beta}{2(N-1)}}{N} = \frac{\alpha+\beta}{2} > 0$$

and

$$X(1) - Y(1) = \frac{\alpha+\beta}{2} - \frac{(N^2-1)\alpha + (N^2+1)\beta}{2N^2} = \frac{\alpha-\beta}{2N^2} > 0$$

Hence, $\Delta u(\delta) > 0$ for all large $\delta < 1$ according to (38) (so, no need to show (39)-(40)).

Next, consider odd $N \geq 4$. From the simultaneous equation (23), the equation that determines $p_N^*(\delta)$ is

$$\begin{aligned}
&V_{N,N}^* N(1-\delta+\delta p_N^*(\delta)) \\
&= \delta p_N^*(\delta)(\alpha + (N-1)\psi(N)) = N(1-\delta+\delta p_N^*(\delta))\Big[p_N^*(\delta)Y(\delta) + (1-p_N^*(\delta))\delta X(\delta)\Big].
\end{aligned}$$

As $X(\delta) > 0$ and $(\alpha + (N-1)\psi(N))/N > Y(\delta)$ for large enough $\delta < 1$, there exists $p_N^*(\delta) \in (0,1)$ that solves this equation, because the LHS of the second equality falls short of the RHS when $p_N^*(\delta) = 0$ while the opposite holds when $p_N^*(\delta) = 1$. Differentiating both sides of this latter equality with respect to δ,

$$\begin{aligned}
&p_N^*(\delta)(\alpha + (N-1)\psi(N)) + \delta\Big[{p_N^*}'(\delta)(\alpha + (N-1)\psi(N)) + p_N^*(\delta)(N-1)\frac{\partial\psi(N)}{\partial\delta}\Big] \\
&= N(-1 + p_N^*(\delta) + \delta {p_N^*}'(\delta))\big(p_N^*(\delta)Y(\delta) + (1-p_N^*(\delta))\delta X(\delta)\big) \\
&\quad + N(1-\delta+\delta p_N^*(\delta))\Big[{p_N^*}'(\delta)Y(\delta) + p_N^*(\delta)Y'(\delta) - {p_N^*}'(\delta)\delta X(\delta) + (1-p_N^*(\delta))(X(\delta)+\delta X'(\delta))\Big]. \quad (43)
\end{aligned}$$

We now show that (39) holds by induction. Note that $\lim_{\delta\to 1}\frac{\partial V_{2,2}^*}{\partial\delta}$ in finite. For induction purposes, assume that $\lim_{\delta\to 1}\frac{\partial V_{n,n}^*}{\partial\delta}$ is finite for all $n < N$. Then, $Y'(1)$ is bounded. Note that the limits of $\frac{\partial\psi(N)}{\partial\delta}$, $\frac{\partial\phi(N)}{\partial\delta}$ and $X'(\delta)$ are bounded as $\delta \to 1$ because $\psi(N)$ and $\phi(N)$ are finite-order polynomials as functions of δ. Thus, given $p_N^*(\delta) \to 0$ as $\delta \to 1$ and (42), by rearranging (43), we have

$$\lim_{\delta\to 1} {p_N^*}'(\delta) = \lim_{\delta\to 1}\frac{-NX(\delta)}{\alpha + (N-1)\psi(N) - NX(\delta)} = \frac{-N^2X(1)}{\alpha-\beta} < 0,$$

which proves (39). Consequently, the LHS of (40) is calculated as $\frac{\alpha-\beta-(\alpha+\beta)N^2}{4N^2} < 0$, proving (40).

Up to now we have proved Step 1. In doing so, in addition to the induction hypothesis that the proposition and (37) hold for some $N \geq 3$, we verified and used the fact that $\lim_{\delta\to 1}\frac{\partial V_{n,n}^*}{\partial\delta}$ is bounded for all $n < N$. For the subsequent induction step, therefore, it is straightforward from (23) to verify that $\lim_{\delta\to 1}\frac{\partial V_{N,N}^*}{\partial\delta}$ is bounded as well.

Step 2: it is optimal for player 1 to disclose two elements in period t.

By Lemma 7, player 1 prefers disclosing two elements rather than more than two in period t. In addition, it is clearly suboptimal for for player 1 to do nothing in period

1 given that player 2 does nothing in the initial period of $G_{N+1,N}$. This confirms that it is optimal for player 1 to disclose two elements in period t for large enough $\delta < 1$.

Step 3: it is optimal for player 2 to do nothing in period t.

Given that player 1 discloses two elements in period t, player 2's payoff from disclosing one element is

$$\delta \frac{2(N-1)\alpha + 2(\alpha+\beta)/2 + (N-1)\beta + (N-1)^2 V^*_{N-1,N-1}}{N(N+1)}. \tag{44}$$

His payoff from doing nothing is

$$\begin{aligned} \delta \frac{2\alpha + (N-1)\psi(N)}{N+1} &\geq \delta \frac{2\alpha + (N-1)\delta \frac{\beta + (N-1)V^*_{N-1,N-1}}{N}}{(N+1)} \\ &\to \frac{2N\alpha + (N-1)\beta + (N-1)^2 V^*_{N-1,N-1}}{N(N+1)} \end{aligned}$$

which is larger than his payoff from disclosing one element obtained above. Here the inequality follows because in the continuation game with $\#(S_1^{t+1}) = N$ and $\#(S_2^{t+1}) = N-1$, it is optimal for player 1 to disclose two elements by induction hypothesis.

Note from (23) that, as $p^*_{N-1} \to 0$ as $\delta \to 1$, $V^*_{N-1,N-1}$ converges to $(\beta + (N-2)\phi(N-1))/(N-1)$ which is $\Psi(1)$ of Lemma 7 when $n = k = N-1$. This means that his payoff from disclosing one element in period t, (44), converges as $\delta \to 1$ to that from disclosing one element in period t and then he alone discloses another element in period $t+1$ if the game has not ended, followed by σ^*. By Lemma 7, this latter payoff is better than that from disclosing one element in period t and then he alone disclosing additional $d > 1$ elements in period $t+1$ if the game has not ended, followed by σ^*, which converges to the payoff from disclosing $d+1$ elements in period t as $\delta \to 1$. This proves Step 3.

Steps 1–3 confirm optimality of σ^* in the initial period of the continuation $G_{N+1,N}$. It remains to prove optimality of σ^* in all continuations of $G_{N+1,N}$.

Step 4: σ^* is optimal in all continuations of $G_{N+1,N}$

By induction hypothesis, σ^* is optimal in all continuations with $\#(S_1^t) \leq N$ and $\#(S_2^t) \leq N$. Hence, only need to consider continuations in which one player, say 1, starts with $N+1$ elements and the other with $K < N$ elements, denoted by $G_{N+1,K}$. In light of the induction hypothesis, we only need to verify optimality in the initial period of such continuations. This can be done inductively in K in the same manner as Steps 1-3 above, as outlined below.

Step 1 is to show where relevant that player 1 prefers disclosing $N-K+2$ elements rather than $N-K+1$, and is much simpler: His payoff from disclosing $N-K+2$

($N-K+1$, resp.) is approximated by that from disclosing one element in period t and then additional $N-K+1$ ($N-K$, resp.) elements in period $t+1$ if the game has not ended. But, induction hypothesis implies that should the game not end in period $t+1$, player 1 prefers disclosing additional $N-K+1$ elements rather than $N-K$. Hence, player 1 must prefer disclosing $N-K+2$ elements rather than $N-K+1$ in the first place as well for large enough δ. Steps 2 and 3 are analogous to above (for $G_{N+1,N}$) and are omitted here.

As the final step of the induction argument, we now show that the equilibrium value $p^*_{N+1} \in (0,1)$ uniquely exists that solves the simultaneous equation system (23) when $n = N+1$. Note that

$$V^*_{N+1,N+1} = \frac{p^*_{N+1}\delta Z(\delta)}{1-\delta(1-p)} \quad \text{where} \quad Z(\delta) := \frac{\alpha + N\psi(N+1)}{N+1}$$

so that p^*_{N+1} is a solution to

$$\frac{p\delta Z(\delta)}{1-\delta(1-p)} = p\delta Y_{N+1}(\delta) + (1-p)\delta X_{N+1}(\delta) \tag{45}$$

where

$$Y_{N+1}(\delta) = \frac{(2N+1)(\alpha+\beta)/2 + N^2 V^*_{N,N}}{(N+1)^2} \quad \text{and} \quad X_{N+1}(\delta) = \frac{\beta + N\phi(N+1)}{N+1}.$$

Note that $X_{N+1}(\delta)$ and $Z(\delta)$ are the payoffs of players 1 and 2, respectively, in a $(N+1)\times(N+1)$ game when player 1 discloses one element first after which the two players, starting with player 2, alternate in disclosing two elements at a time until ω^* is identified and taken as action; and $Y_{N+1}(\delta)$ is the payoff of both players when the two players disclose one element each in the first period, and then they play the symmetric Markov equilibrium σ^* for a $N \times N$ game from the next period if the game does not end then.

Note that if N is odd then $Y_{N+1}(\delta) < \frac{\alpha+\beta}{2} = X_{N+1}(\delta) = Z(\delta)$ as $\delta \to 1$. Thus, the LHS of (45) increases in p from 0 when $p=0$ to $\delta Z(\delta)$ when $p=1$, while the RHS decreases in p from $\delta X_{N+1}(\delta)$ when $p=0$ to $\delta Y_{N+1}(\delta)$ when $p=1$. Therefore, there is a unique solution $p \in (0,1)$ to (45).

If N is even, on the other hand, $Y_{N+1}(\delta) = \frac{\alpha+\beta}{2} > X_{N+1}(\delta)$ as $\delta \to 1$, so that the RHS of (45) increases linearly from $\delta X_{N+1}(\delta)$ when $p=0$ to $\delta Y_{N+1}(\delta)$ when $p=1$. The LHS also increases from 0 to $\delta Z(\delta)$, but it is concave and $Z(\delta) > Y_{N+1}(\delta)$ because

$Z(1) + X_{N+1}(1) = \frac{\alpha+\beta}{2}$ and consequently, there is a unique solution $p^*_{N+1} \in (0, 1)$ to (45). Hence, players are indifferent between disclosing one and none in the initial period of $(N + 1) \times (N + 1)$ game when the other player mixes between them with probabilities p^*_{N+1} and $1 - p^*_{N+1}$, respectively. As it follows from Lemma 7 that disclosing more is suboptimal in the initial period, the first part of the proposition is proved.

We now prove the second part: There exists an integer $\lambda \geq 2$ such that σ^* is efficient in the limit as $\delta \to 1$ provided $0 < |\#(S_1) - \#(S_2)| \leq \lambda$.

First, notice that if $|\#(S_1) - \#(S_2)| = 1$ then σ^* is efficient in the limit, because from the first period the two players alternate disclosing two elements each until ω^* is identified. Therefore, it suffices to show that σ^* is efficient in the limit if $|\#(S_1) - \#(S_2)| = 2$ and $\#(S_1), \#(S_2) \geq 2$. To show this, consider the cases that $\#(S_1) = \#(S_1) + 2$ and $\#(S_2) = k \geq 2$. If $k = 2$ or 3, then player 2's payoff from disclosing all but one element is positive at the limit as $\delta \to 1$ due to (1), conditional on player 1 doing nothing. Consequently, σ^* prescribes that either player 1 discloses 3 elements or player 2 discloses all but one element in the initial period, ensuring efficiency in the limit.

Next, consider the cases that $k > 3$. Then, player 1's payoff from disclosing 3 elements first converges as $\delta \to 1$ to that from disclosing 2 elements in the initial periods and then continuing with σ^* for $k \times k$ game. The payoff from the latter in the limit is $(2\beta+kV^*_{k,k})/(k+2)$. If k is an even number, this value is $(4\beta+(\alpha+\beta)k)/(4+2k)$ by (37), which is positive because $4\beta + (\alpha + \beta)k$ increases in k from a value of $4(\alpha + 2\beta) > 0$ when $k = 4$. If k is an odd number, this value is $\frac{\alpha(k^2-1)+\beta(k^2+4k+1)}{2k(k+2)}$ by (37), which is positive because the numerator is increases in k from a value of $24\alpha + 46\beta > 0$ when $k = 5$. Therefore, σ^* prescribes that player 1 discloses 3 elements in the initial period, followed by alternation of disclosing two elements each until ω^* is identified, ensuring efficiency in the limit. □

Proof of Theorem 2: Assume without loss of generality that $\#(S_1) = n \geq k = \#(S_2)$. Note that information disclosure takes place with certainty in every period until ω^* is identified according to σ^* if $n - k = 1$, or if $\frac{(k-1)\beta+\alpha}{k} > 0$, for large enough δ. If $n = k$, it is easy to verify that the following is an equilibrium: agent 1 discloses one element in period 1, then they follow σ^*. Thus, the theorem is established for these cases.

Below we consider remaining cases, i.e., $n-k \geq 2$ and $\frac{(k-1)\beta+\alpha}{k} \leq 0$. As the latter inequality implies $k \geq |\alpha/\beta| + 1 > 3$, we consider $k \geq 4$ below.

Let m be the largest integer such that $mk \leq n$ so that $0 \leq r = n - mk < k$. We consider six cases separately below, because minor adjustments need to be made in the Markov equilibrium to be constructed depending on the details of the parameter values.

Case 1: $m \geq 1$ and k is odd, but not $m = 1$ and $r > \hat{r} := \frac{k^2-3}{2k}$.

Consider the following Markov strategy along the equilibrium-path:

(*) Player 1 discloses $r + m$ elements and player 2 does nothing in period 1; then starting with player 2 the two players alternate disclosing 2 and $2m$ elements each, respectively, in alternating periods (player 2 in even periods and player 1 in odd periods) until ω^* is identified and taken as an action. Note that, since k is odd, the last period of possible disclosure is period $k-1$ in which player 2 would disclose all but one element.

(**) Off-equilibrium strategies are described below where n' and k' denote sizes of the remaining possibility sets of players 1 and 2, respectively, at the beginning of the relevant period. Let X_i, $i = 1, 2$, denote the set of all possible sizes of the remaining possibility set that player i starts with in some period along the equilibrium-path according to (*) above. For each $n' \geq 1$, let $\bar{n'}$ denote the smallest number in X_1 subject to being equal to or larger than n'; and $\underline{n'}$ denote the largest number in X_1 that is strictly lower than n' if exists, and $\underline{n'} = n'$ otherwise. For each $\bar{n'} \in X_1$, let $k(\bar{n'})$ denote the number of elements with which player 2 ends the period in which player 1 starts with $\bar{n'}$ and discloses no element according to (*). Analogously, for each $k' \geq 1$ let $\bar{k'}$ denote the smallest number in X_2 subject to being equal to or larger than k', and $\underline{k'}$ denote the largest number in X_2 that is strictly lower than k' if exists, and $\underline{k'} = k'$ otherwise. For each $\bar{k'} \in X_2$, let $n(\bar{k'})$ denote the number of elements with which player 1 ends the period in which player 2 starts with $\bar{k'}$ and discloses no element according to (*).

(i) If $k' \geq 2$ and $n' > n(\bar{k'})$, then player 1 discloses $n' - n(\bar{k'})$ elements if his expected payoff from doing so, followed by player 2 disclosing $k' - \underline{k'}$ elements if

$k' - \underline{k'} > 0$ and then (*), is positive, and player 2 does nothing; otherwise, player 1 does nothing and player 2 discloses all but one element if his payoff from doing so is positive, and does nothing otherwise.

(ii) If $k' \geq 2$ and $\min X_1 \leq n' \leq n(\bar{k}')$, then player 2 discloses $k' - k(\bar{n}')$ elements if his expected payoff from doing so, followed by player 1 disclosing $n' - \underline{n'}$ elements if $n' - \underline{n'} > 0$ and then (*), is positive, and player 1 does nothing; otherwise, player 2 does nothing and player 1 discloses all but one if his payoff from doing so is positive, and does nothing otherwise.

(iii) If $k' \geq 2$ and $1 < n' < \min X_1$, then player 2 discloses $k' - 1$ elements if his expected payoff from doing so, is positive, and player 1 does nothing; otherwise, player 2 does nothing and player 1 discloses all but one if his payoff from doing so is positive, and does nothing otherwise.

(iv) If $k' = 1$ or $n' = 1$, both players disclose no element and take action ω^* if identified.

Note that (**) is described generally enough so that it applies to subsequent cases as well. We now verify optimality of (*) and then that of (**). For this we analyse every possible period in the game presuming that neither player has identified ω^* at that point, which is taken for granted in the sequel. Also, we take it granted that $\delta < 1$ is large enough for the argument to be valid whenever pertinent.

Along the equilibrium-path, note that player 2 in even periods with k' elements faces a continuation equilibrium that is equivalent to σ^* when $\#(S_1) = k' - 1$ and $\#(S_2) = k'$, in terms of probabilities of winning and losing in subsequent periods, so his payoff is $\psi(k')$. Unless $k' = 1$, therefore, disclosing two elements is optimal for him because disclosing less will subject him to disclosing more in subsequent periods by (ii) above, and disclosing more would expose himself to a higher risk of losing before getting back to some future point of equilibrium-path by (i). If $k' = 1$, optimality of taking action ω^* as soon as identified is trivial. In odd periods along the equilibrium-path, doing nothing is clearly optimal for player 2 for the same reason that it is so in σ^* because the opponent is expected to disclose a positive number of elements (or take action ω^* if identified).

An analogous logic verifies optimality of player 1 along the equilibrium-path from period 2 onward. In period 1, as $\delta \to 1$, player 1's expected payoff converges to that when he disclosed $r+m$ elements in two consecutive installments, r elements first then m. This payoff conditional on the first r elements disclosed do not contain ω^*, converges to $V^*_{k,k}$ by (23) because $p^*_k \to 0$ as $\delta \to 1$. Therefore, player 1's payoff in period 1 according to (*) converges to

$$\frac{r\beta + mkV^*_{k,k}}{r+mk} \quad = \quad \frac{1}{r+mk}\Big(\frac{m(k^2-1)\alpha + (2kr+mk^2+m)\beta}{2k}\Big)$$

which is positive by (1) if $m \geq 2$, or $m=1$ and $r \leq \hat{r} = \frac{k^2-3}{2k}$, because given $k \geq 4$,

$$2m(k^2-1)-(2kr+mk^2+m) = m(k^2-3)-2kr \begin{cases} \geq \ 2(k(k-r)-3) > 0 & \text{if } \ m \geq 2 \\ = \ k(k-2r)-3 > 0 & \text{if } \ m=1, \ r \leq \hat{r}. \end{cases}$$

In period 1, disclosing less than $r+m$ elements only delays the process by (ii), and disclosing more is suboptimal by the same reason as above, verifying optimality of player 1 along the equilibrium-path.

The off-equilibrium strategy (i)–(iv) prescribes that the players get back to the equilibrium-path as quickly as possible if doing so gives a positive expected payoff, with the added incentive feature that the player who deviated by delaying the equilibrium exchange process bears the cost of getting back on the equilibrium-path. Verifying optimality of off-equilibrium strategy is straightforward from its description, which is omitted here.

Case 2: $m \geq 1$ and k is even, but not $m=1$ and $r > \hat{r}$.

Consider the following Markov strategy along the equilibrium-path, presuming that $r \neq 0$ (the analysis is the same when $r=0$ except that the first period below is redundant):

(*) Player 1 discloses r elements and player 2 does nothing in period 1; player 2 discloses one element and player 1 does nothing in period 2; then starting with player 1 the two players alternate disclosing $2m$ and 2 elements each period, respectively until ω^* is identified and taken as an action. Since k is even, player 2 discloses all but one in the last potential period of disclosure.

The same description of off-equilibrium strategy (**) applies here. The optimality is verified in the same manner as above, except for periods 1 and 2, which we explain below.

In period 2 players start a continuation equilibrium which is equivalent to player 2 disclosing one element while player 1 doing nothing in the initial period, followed by

σ^* in the $k \times (k-1)$ game characterised in Theorem 1. Hence, player 2's continuation payoff in period 2 is $\delta \frac{\beta + (k-1)\phi(k)}{k}$ which converges to $V^*_{k,k} > 0$ by (23) because $p^*_k \to 0$ as $\delta \to 1$. Therefore, disclosing one element is optimal for player 2 in period 2 because disclosing none only delays the process and disclosing more exposes himself to a higher risk before getting back on the equilibrium-path. The optimality of player 1 doing nothing in period 2 follows by an argument that must be straightforward by now. As k is even, $V^*_{k,k} = \delta(\alpha + \delta\beta)/2$ and thus, player 1's continuation payoff in period 2 converges to $\alpha + \beta - V^*_{k,k} = V^*_{k,k}$ as $\delta \to 1$.

Therefore, player 1's payoff in period 1 converges to

$$\frac{r\beta + mkV^*_{k,k}}{r + mk} \;=\; \frac{1}{r+mk}\Big(r\beta + mk\frac{\alpha+\beta}{2}\Big) \;=\; \frac{1}{r+mk}\Big(\frac{mk\alpha + (2r+mk)\beta}{2}\Big)$$

which is positive if $m \geq 2$, or $m = 1$ and $r \leq \hat{r}$, because

$$2mk - 2r - mk \;=\; mk - 2r \;\begin{cases} \geq \; 2(k-r) > 0 & \text{if } \; m \geq 2 \\ = \; k - 2r > 0 & \text{if } \; m = 1, \; r \leq \hat{r} < k/2. \end{cases}$$

In period 1, therefore, disclosing r elements is optimal for player 1 because disclosing less only delays the process by (ii) of (**), and disclosing more is suboptimal by the same reason as above.

Below we consider cases in which $m = 1$ and $r > \hat{r} = \frac{k^2-3}{2k}$. As $\hat{r} > k/3$, we use $k/3$ instead of $\hat{r}$ when it suffices and simplifies calculation.

<u>Case 3:</u> $m = 1$ so that $n = k + r < 2k$, $r > \hat{r}$, n is even and $n/2$ is odd.

Consider the following Markov strategy along the equilibrium-path:

(*) Player 2 discloses $k - n/2$ elements in period 1; player 1 discloses two elements in period 2; then starting with player 2 the two players alternate disclosing 2 and 4 elements each period, respectively until ω^* is identified and taken as an action. Since $n/2$ is odd, player 2 discloses all but one element in the last potential period of disclosure.

The same description of off-equilibrium strategy (**) applies here. The optimality is verified in the same manner as before, apart from some minor modification of details for periods 1 and 2, explained below.

In period 2 players start a continuation equilibrium which is equivalent to player 1 disclosing one element while player 2 doing nothing in the initial period, followed by σ^*

in the $(\frac{n}{2}-1)\times\frac{n}{2}$ game characterised in Theorem 1. Hence, player 1's continuation payoff is $\delta\frac{\beta+(n/2-1)\phi(n/2)}{n/2}$ which converges to $V^*_{n/2,n/2}$ by (23) because $p^*_{n/2}\to 0$ as $\delta\to 1$. Therefore, disclosing two elements is optimal for player 1 in period 2 because disclosing less only delays the process and disclosing more exposes himself to a higher risk before getting back on the equilibrium-path.

As player 2's continuation payoff in period 2 converges to $\alpha+\beta-V^*_{n/2,n/2}$, player 2's payoff in period 1 converges to

$$\frac{(k-n/2)\beta+n(\alpha+\beta-V^*_{n/2,n/2})/2}{k} \quad = \quad \frac{1}{2k}\Big(\frac{(n^2+4)\alpha+(4kn-n^2-4)\beta}{2n}\Big)$$

which is positive because

$$2n^2+8-(4kn-n^2-4)\geq -k^2+2kr+3(r^2+4)|_{r=k/3}=12>0.$$

In period 1, therefore, disclosing $k-n/2$ elements is optimal for player 2 because disclosing less only delays the process by (ii) of (**), and disclosing more is suboptimal by the same reason as above.

<u>Case 4:</u> $m=1$ so that $n=k+r<2k$, $r>\hat{r}$, and both n and $n/2$ are even.

Consider the following Markov strategy along the equilibrium-path:

(*) Player 2 discloses $k-n/2+1$ elements in period 1; then starting with player 1 the two players alternate disclosing 4 and 2 elements each period, respectively until ω^* is identified. Since $n/2-1$ is odd, player 2 discloses all but one in the last period of potential disclosure.

The same description of off-equilibrium strategy (**) applies here. The optimality is verified in the same manner as before, apart from a minor modification of details for period 1, explained below.

Player 2's payoff in period 1 is equivalent to that when he discloses $k-n/2$ elements and then another element before the alteration starts, as explained earlier. Therefore, player 2's payoff in period 1 converges to

$$\frac{(k-n/2)\beta+n(V_{n/2,n/2})/2}{k} \;=\; \frac{1}{2k}\Big((2k-n)\beta+n\frac{\alpha+\beta}{2}\Big) \;=\; \frac{1}{2k}\Big(\frac{n\alpha+(4k-n)\beta}{2}\Big)$$

which is positive because

$$2n-(4k-n)\geq -k+3r\Big|_{r=k/3}=0.$$

In period 1, therefore, disclosing $k - n/2 + 1$ elements is optimal for player 2 because disclosing less only delays the process by (ii) of (**), and disclosing more increases the risk of losing.

Case 5. $m = 1$ so that $n = k + r < 2k$, $r > \hat{r}$, n is odd and $\kappa = (n-1)/2$ is even. Note that $\kappa \geq 3$.

Consider the following Markov strategy along the equilibrium-path:

(*) Player 2 discloses $k - \kappa - 1$ elements in period 1; player 1 discloses one element in period 2; player 2 discloses two in period 3; then starting with player 1 the two players alternate disclosing 4 and 2 elements each period, respectively until ω^* is taken as an action. Since $\kappa + 1$ is odd, player 2 discloses all but one in the last period of potential disclosure.

The same description of off-equilibrium strategy (**) applies here. The optimality is verified in the same manner as before, apart from some minor modification of details for periods 1, 2 and 3, explained below.

Note that player 2's payoff in period 3 converges to

$$U = \frac{\beta + \kappa V^*_{\kappa,\kappa}}{\kappa + 1} = \frac{\kappa\alpha + (\kappa+2)\beta}{2(\kappa+1)}$$

which is positive because $\kappa \geq 3$. Hence, disclosing one is optimal for him in period 3 for the now usual reason.

Player 1's payoff in period 2 converges to

$$U' = \frac{\beta + (n-1)(\alpha + \beta - U)}{n} = \frac{1}{n}\Big(\frac{(n-1)(\kappa+2)\alpha + ((n+1)\kappa + 2)\beta}{2(\kappa+1)}\Big)$$

which is positive because

$$2(n-1)(\kappa+2) - ((n+1)\kappa+2) \geq (k^2 + 2k(r+2) + r^2 + 4r - 9)/2\big|_{r=k/3} = 8k^2/9 + 8k/3 - 9/2 > 0.$$

Hence, it is optimal for him to disclose one in period 2 again for the usual reason.

Player 2's payoff in period 1 converges to

$$\begin{aligned} & \frac{(k-\kappa-1)\beta + (\kappa+1)(\alpha+\beta-U')}{k} \\ = \ & \frac{1}{k}\Big((k-\kappa-1)\beta + (\kappa+1)\frac{((n+1)\kappa+2)\alpha + (n-1)(\kappa+2)\beta}{2n(\kappa+1)}\Big) \\ = \ & \frac{1}{k}\Big(\frac{(\kappa+1)((n+1)\kappa+2)\alpha + (\kappa+1)((n-1)(\kappa+2) + 2n(k-\kappa-1))\beta}{2n(\kappa+1)}\Big) \end{aligned}$$

which is positive because

$$\begin{aligned}
&2(\kappa+1)((n+1)\kappa+2)-(\kappa+1)((n-1)(\kappa+2)+2n(k-\kappa-1))\\
&= \big(-k^3+k^2(r-1)+k(5r^2+2r+9)+3(r^3+r^2+3r+3)\big)/4\\
&> \big(-k^3+k^2(r-1)+k(5r^2+2r+9)+3(r^3+r^2+3r+3)\big)/4\big|_{r=\hat{r}}\\
&= 3(3k^6+2k^5+3k^4+4k^3+21k^2+18k-27)/(32k^3) \ > \ 0
\end{aligned}$$

where the first inequality follows from the derivative of the LHS with respect to r being

$$(k^2+2k+10kr+9r^2+6r+9)/4 \ > \ 0.$$

In period 1, therefore, disclosing $k-\kappa-1$ elements is optimal for player 1 because disclosing less only delays the process by (ii) of (**), and disclosing more increases the risk of losing.

Case 6. Finally, suppose $m=1$ so that $n=k+r<2k,\ r>\hat{r}$, and both n and $\kappa=(n-1)/2$ are odd.

Consider the following Markov strategy along the equilibrium-path:

(*) Player 1 discloses one element in period 1; player 2 discloses $k-\kappa$ elements in period 2; player 1 discloses two elements in period 3; then starting with player 2 the two players alternate disclosing 2 and 4 elements each period, respectively until ω^* is taken as an action. Since κ is odd, player 2 discloses all but one in the last period of potential disclosure.

The same description of off-equilibrium strategy (**) applies here. The optimality is verified in the same manner as before, apart from some minor modification of details for periods 1 and 2, explained below.

In period 3, player 1's continuation payoff converges to $V^*_{\kappa,\kappa}>0$ as explained earlier, hence disclosing two elements is optimal for the now usual reason. Then, player 2's expected payoff in period 2 converges to

$$U''=\frac{(k-\kappa)\beta+\kappa(\alpha+\beta-V^*_{\kappa,\kappa})}{k} \quad = \quad \frac{1}{k}\Big(\frac{(\kappa^2+1)\alpha+(2k\kappa-\kappa^2-1)\beta}{2\kappa}\Big)$$

which is positive because

$$2(\kappa^2+1)-(2k\kappa-\kappa^2-1) \geq \frac{-k^2+2k(r-1)+3(r^2-2r+5)}{4}\Big|_{r=\hat{r}} = \frac{3k^4-20k^3+30k^2+36k+27}{16k^2} > 0.$$

Player 1's payoff in period 1 converges to, therefore,

$$\frac{\beta + (n-1)(\alpha + \beta - U'')}{n} \;=\; \frac{1}{n}\Big(\frac{(n-1)(2k\kappa - \kappa^2 - 1)\alpha + ((n-1)(\kappa^2+1) + 2k\kappa)\beta}{2k\kappa}\Big)$$

which is positive because

$$\begin{aligned}\Delta &= 2(n-1)(2k\kappa - \kappa^2 - 1) - ((n-1)(\kappa^2+1) + 2k\kappa)\\ &= \big(5k^3 + k^2(7r-11) - k(r^2+2r+9) - 3(r^3 - 3r^2 + 7r - 5)\big)/4\end{aligned}$$

is concave in r with a positive slope at $r = 1$ as

$$\frac{\partial\Delta}{\partial r}\Big|_{r=1} = \frac{7k^2 - 2k - 2kr - 3(3r^2 - 6r + 7)}{4}\Big|_{r=1} = \frac{7k^2}{4} - k - 3 > 0$$

and $\partial^2\Delta/\partial r^2 = (9 - k - 9r)/2 < 0$, and Δ is positive both at $r = 1$ and $r = k - 1$ for every $k \geq 4$:

$$\Delta|_{r=1} = k(5k^2 - 4k - 12)/4 > 0 \;\text{ and }\; \Delta|_{r=k-1} = 2(k^3 - 7k + 6) > 0.$$

In period 1, therefore, disclosing one element is optimal for player 1 because disclosing none only delays the process by (ii) of (**), and disclosing more increases the risk of losing.

In every equilibrium constructed above, in every period one player discloses at least one element for sure. Therefore, ω^* is identified and taken as an action in period $n + k + 1$ at the latest, and efficiency is achieved as $\delta \to 1$. This completes the proof. $\square$

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