Too Big to Jail? Key-Player Policies When the Network is Endogenous

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Discussion Paper 17 / 693

20 December 2017

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Abstract

This paper endogenizes the network for the seminal model presented in Ballester et al. (2006) by way of a simple simultaneous move game. Agents choose with whom to associate and how much effort to exert. Effort levels display local strategic complementarities and global strategic substitutes. I show that all pairwise Nash equilibrium networks are nested split graphs. As in Ballester et al. (2006), agents' equilibrium effort levels are proportional to Bonacich centrality. However, their ranking now coincides with a simpler measure, which is also easier to identify: degree centrality. I then study key player policies, which aim at minimizing aggregate effort levels via the elimination of an agent. In the spirit of network formation, after an agent was eliminated from a pairwise Nash equilibrium network, the remaining agents may revise their effort decisions and adapt their linking behavior. It is shown that, if the parameter governing global strategic substitutes is sufficiently small, then eliminating a most central agent also decreases aggregate effort levels most. This mirrors results obtained by Ballester et al. (2006). However, when global strategic substitutes are large, then, different from Ballester et al. (2006), eliminating a most central agent may not be optimal. Eliminating a most central agent, who in equilibrium also exerts highest criminal effort, decreases competition/congestion effects and increases incentives of the remaining agents to create new links. The latter effect on the aggregate level of crime may outweigh the former. These results are relevant for a wide range of applications, such as juvenile delinquency and crime, R&D expenditure of firms, bank bailouts and trade.

Key Words: Strategic network formation, peer effects, local strategic complements, global strategic substitutes, positive externalities, negative externalities. **JEL Codes:** D62, D85.

*I thank Fernando Vega-Redondo, Coralio Ballester, Gonçalo Pina, participants of the BiNoMa workshop 2017, the internal Bristol theory working group and seminar participants at Oxford University for helpful comments. All remaining errors are mine. Contact: timo.hiller@bristol.ac.uk, Address: Department of Economics, University of Bristol, 8 Woodland Road, BS8 1TN, UK.

1 Introduction

Delinquency and crime are regularly considered to be among the most pressing social problems.[1] A better understanding of its determinants and the design of effective policy has the potential to yield significant social and economic benefits. In the U.S. current yearly expenditure on crime control amounts to approximately 270 billion USD, of which more than 80 billion USD are spent on incarceration alone, with a staggering 2.2 million imprisoned individuals nationwide.[2] While crime rates are declining, the number of prisoners continues to rise. Brute force policies, such as "zero tolerance," are reaching their limits and simply locking away more individuals for longer ceases to be a viable strategy to control crime. One may then ask, what are suitable alternatives? Current efforts include the use of randomized severity of punishment, a concentration of enforcement resources, and granting convicts probation and parole (see, for example, Kleinman, 2009).

Recently, the economics literature has produced interesting insights, which can be applied to the design of more effective and efficient policy to fight crime. This approach rests on two key ideas. First, building on Becker (1968), criminals are assumed to take expected costs and benefits into account when committing a crime or delinquency. Second, peer effects and the structure of bilateral relationships play a crucial role for determining individual criminal and delinquent behavior.[3] So-called key player policies have been derived, which identify the agent to be optimally targeted in a network of criminals, so as to decrease aggregate criminal activity maximally. Empirical studies show that these policies, which take the whole network into account in a non-trivial way, outperform traditional approaches.[4] However, theoretical results obtained to date postulate that crime networks are fixed. That is, after a criminal is apprehended, the remaining agents are assumed to not revise their linking decisions. In contrast, in this paper a model is presented, for which the network is assumed to be endogenous and, after an agent is eliminated, the remaining agents may revise not only their decisions regarding criminal activity, but also with whom to associate. It is shown that introducing endogenous network formation matters. In particular, the key player policy that is optimal when the network is fixed, may not be optimal when the network is endogenous and agents may now also adjust their linking decisions. The intuition is that, if the key player policy for a fixed network is applied to a network that is, in fact, endogenous, then the eliminated agent exerts the highest level of criminal activity in the network. However, removing such an agent also decreases global congestion/competition the most and the remaining agents may subsequently find it profitable to create new links. This effect may outweigh the effect of removing an agent exerting the highest criminal effort.

At this point it is worth to briefly discuss the implementability of these type of policies. One requirement is knowledge of the network and it may appear that this is difficult to obtain

[1] See, for example, https://www.nytimes.com/interactive/2017/02/27/us/politics/most-important-problem-gallup-polling-question.html.

[2] See https://obamawhitehouse.archives.gov/sites/default/files/page/files/20160423_cea_incarceration_criminal_justice.pdf

[3] See, for example, Sutherland (1947), Sarnecki (2001), Warr (2002), Haynie (2001) and Patacchini and Zenou (2012).

[4] For an interesting empirical study see Lindquist and Zenou (2014). The authors study changes in criminal activity over time, as criminals leave different crime networks, either due to imprisonment or death. The key player policy outperforms a random policy by 6.4%, removing the most active player by 2.4%, removing the player with the highest betweenness centrality by 6.4% and the policy of removing the player with the highest eigenvector centrality by 16%.

in the context of crime. However, such data exists or can often be obtained. For example, Sarnecki (2001) constructs a criminal network in Sweden by using police records, which register each time two (or more) individuals are suspected of a crime. Similar data is available in many countries.[5] Once a criminal network is obtained, there are different approaches how one may implement a key player policy in practice. Obviously, the state can and should not imprison criminals unless they are proven guilty of a crime. The state may, however, offer incentives for targeted criminals to leave the network. This can be achieved through heightened monitoring, providing job opportunities, employment and educational training, or even organizing geographic relocation. Policies of this sort have been implemented in the U.S. and Canada (Tremblay et al., 1996).

The seminal contribution in economics, on which not only the literature on crime networks, but also much of the recent literature on R&D networks builds, with further applications in inter-bank lending and trade, is Ballester et. al (2006). The authors study games with linear-quadratic utilities, where agents choose effort levels simultaneously. These games can be interpreted as games on a fixed network, with local strategic complements in effort levels for direct neighbors in the network, a globally uniform payoff substitutability component and an own-concavity effect. Before presenting the main results of Ballester et. al (2006) and relating them to the present paper, I briefly describe two underlying assumptions, which yield a model with local strategic complementarities and global substitutes for the application considered here. A formal derivation of the payoff function is provided in the model description. First, a criminal or delinquent's probability of being caught is lower, the higher the criminal activities of her direct neighbors in the crime network and, second, there is global competition for crime opportunities. Note that the most common argument for the former is that delinquents learn how to become more efficient criminals due to a direct know-how transfer.

Ballester et al. (2006) offers two main insights. The first, striking result is that, in the unique Nash equilibrium, effort levels are proportional to Bonacich centrality. An agent's Bonacich centrality is determined by the sum of all weighted paths of different lengths emanating from the agent in the network, where longer paths are weighted by less.[6] Note, however, that the ranking of agents in terms of their Bonacich centrality is typically dependent on the decay parameter chosen. Furthermore, for a given network it is often not easy to spot which agent is the one with the highest Bonacich centrality. The second main result is concerned with key player policies when the network is fixed. That is, after an elimination, the remaining agents may adjust their effort levels, but cannot revise their linking decisions. The authors show that a planner, whose aim is to maximally decrease aggregate effort levels, should remove an agent with the highest inter-centrality. An agent's inter-centrality is closely related to Bonacich centrality; it is her Bonacich centrality and the agent's contribution to the Bonacich centrality of all other agents. Note that agents do not anticipate being targeted. That is, the targeting strategy of the planner does not enter agents' payoff functions. However, if a planner were to, say, switch from a zero tolerance to a key player policy, then arguably agents would learn about it over time, which in turn may change incentives.

[5] Tayebi et al. (2011) use a data set provided by the Royal Canadian Mounted Police (RCMP), which comprises five years of arrest-data and is available for research purposes. Coplink (Hauck et al., 2002) is a large scale research project in crime data mining in the United States. It uses information from various sources, such as habits of criminals and close associations in crime to capture network connections. Mastrobuoni and Patacchini (2012) use a data set from the Federal Bureau of Narcotics on U.S. mafia members, which allows the authors to construct a criminal network.

[6] A formal definition of Bonacich centrality is provided in Appendix B.

Although interesting, I follow the approach of Ballester et al. (2006) and view the latter considerations as outside of the scope of the present paper.

Next a brief description of the model considered here is provided, together with the main results. I first propose a simple simultaneous move game, in which agents choose a non-negative, continuous effort level and announce to whom they want to be linked. A bilateral link is created when the announcement is mutual. Gross payoffs are based on the payoff function in Ballester et al. (2006), while links are assumed to be unweighted, undirected and to incur a linear cost.[7,8] The equilibrium concept used is pairwise Nash equilibrium. Pairwise Nash equilibrium refines Nash equilibrium and allows for deviations in which agents simultaneously create a link (and best respond to each other's effort level). This rules out configurations in which pairs of agents are not connected, but both agents find it profitable to create a link among themselves. I show that all pairwise Nash equilibrium networks are nested split graphs and that a pairwise Nash equilibrium always exists. Nested split graphs are a particular case of core-periphery networks and have recently drawn increased attention in the economics literature on networks.[9] The defining feature of nested split graphs is nestedness. Neighborhoods are nested in the following sense: agents with a higher number of links are connected to all agents to which an agent with fewer links is connected. Note that the structure of empirically observed crime networks appears to depend on the type of criminal activity. Canter (2004), for example, finds that networks of hooligans are less structured than property crime and drug networks. However, the presence of a core group is described as the most recognized structural feature.[10] Nested split graphs are also interesting from a theoretical point of view in our context, as for these networks the ranking of Bonacich centralities (and therefore Nash equilibrium effort levels) coincides with inter-centrality and agents' number of links. That is, relative to Ballester et al. (2006), the ranking of effort levels and the set of key players is summarized by a much simpler network measure than Bonacich centrality or inter-centrality: the agents' number of links.

I then turn to key player policies when the network is endogenous. Starting from a pairwise Nash equilibrium network, a planner may eliminate one agent, after which a pairwise best response dynamics with the following properties ensues. As in pairwise Nash equilibrium (and pairwise stability), link formation is separated from link deletion. In the link formation stage, any link is added that is profitable for a pair of agents in isolation, given the current network and corresponding vector of Nash equilibrium effort levels. In the link deletion stage, agents best respond to the current network by deleting any subset of links. In between link formation and link deletion stages, agents adjust their effort levels to the Nash equilibrium effort level of the current network, i.e. after links were added/links deleted. A key player policy is said to exist when the best response dynamic converges. A key player policy then prescribes eliminating an agent such that the sum of effort levels is minimal for the network to which the pairwise best response dynamics converge. I show that, when the parameter governing global substitution effects is sufficiently low, then there does not exist a pair of agents that finds it profitable to create a link and the process always converges. Furthermore, the key player policy coincides with the one proposed by Ballester et al. (2006) and, in the

[7] I relate the payoff function to the one used in Ballester et al. (2006) formally in Appendix A.

[8] If links are unweighted, then they are assumed to be all of same strength or intensity. If links are undirected, then they are all bidirectional.

[9] Goyal and Joshi (2003) is a very early paper that features nested split graphs (the authors call them interlinked stars). For a good discussion of nested split graphs, see König et al., 2014.

[10] See also Dorn and South (1990), Dorn, Murji and South (1992), Ruggiero and South (1997) and Johnston (2000).

pairwise Nash equilibrium networks obtained, this implies that an agent with the highest effort level (and the highest number of links) is eliminated. However, this is not always the case when the parameter governing global substitution effects is large. More specifically, an example is provided, where eliminating an agent with the highest inter-centrality, as proposed by Ballester et al. (2006), is not optimal. That is, when designing key player policies, taking into account how the network will react to an elimination can be important.

Below I briefly relate my paper to the networks literature in economics. Ballester et al. (2006) was the starting point for a rich body of theoretical and empirical research (see, for example, Calvó-Armengol et al., 2009, Ballester et al., 2010 and Helsley and Zenou, 2014). However, endogenizing the network proved difficult. Recent efforts have focused on models of network formation and action choices in a dynamic setting with myopic agents (see, for example, König et al., 2014, König et al., 2014 and Cohen-Cole et al., 2015). In these papers agents cannot revise their whole linking strategy (and only create at most one link at zero cost), while the deletion of links is not strategic and occurs due to decay over time. It is then shown that the stochastically stable networks of the dynamic process are nested split graphs. One of the advantages of these models is that they can be brought to the data. In contrast, my paper is the first to show that these network configurations can be sustained as Nash equilibria for the type of payoff functions considered in Ballester et al. (2006).[11] Another attractive feature is that I consider two-sided network formation, where both agents involved in a link incur a cost, while the aforementioned papers all assume unilateral creation of links. Hiller (2017) studies a similar setting as the one considered in the present paper, with a class of payoff functions for which the linear-quadratic specification is a special case, but without the global substitution term.[12] Introducing these congestion or competition effects not only complicates the analysis, but is also important for many of the applications, as disregarding them often requires adopting overly strong assumptions. For example, in the case of R&D agreements it implies that firms operate in entirely separate markets, while for interbank lending one needs to assume that agents are not at all able to substitute across loans. To the best of my knowledge, this is also the first paper to study key player policies when the network is endogenous.

The paper is organized as follows. Section 2 provides the model description, while Section 3 shows that all pairwise Nash equilibria are nested split graphs and that a pairwise Nash equilibrium always exists. Section 4 introduces pairwise best response dynamics, provides a general result for the case when the parameter governing global substitution effects is small and, finally, an example for when global substitution effects can be considered as large. Section 6 concludes. All proofs are relegated to Appendix B. The relationship between Ballester et al. (2006) and the payoff function considered here is explained in detail in Appendix A, while Bonacich centrality is defined in Appendix B.

[11] Joshi and Mahmud (2016) present a two-stage game, in which agents create links in the first stage and play Nash equilibrium effort levels in the second stage, and show that for limiting cases of parameter values pairwise stable networks are nested split graphs. However, no existence result is provided and pairwise stable networks are not necessarily Nash equilibria.

[12] For further work on network formation games with simultaneous action and linking choices, see Bätz (2014), Galeotti and Goyal (2010) and Kinateder and Merlino (2017).

2 Model Description

Let $N = \{1, 2, ..., n\}$ be the set of players with $n \geq 3$. Each agent i chooses an effort level $x_i \in X$ and announces a set of agents to whom the agent wishes to be linked, which is represented by a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n-1})$, with $g_{i,j} \in \{0, 1\}$ for each $j \in N\backslash\{i\}$. An entry $g_{i,j} = 1$ in $\mathbf{g}_i$ is interpreted as agent i announcing a link to agent j, while an entry $g_{i,j} = 0$ in $\mathbf{g}_i$ is taken to mean that agent i does not announce a link to agent j. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of agent i's strategies is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ then specifies the individual effort level for each player, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and the set of intended links, $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. A link between agents i and j, denoted by $\bar{g}_{i,j} = 1$, is created if and only if *both* agents i and j announce the link. That is, $\bar{g}_{i,j} = 1$ if and only if $g_{i,j} = g_{j,i} = 1$ (and $\bar{g}_{i,j} = 0$ otherwise) and therefore $\bar{g}_{i,j} = \bar{g}_{j,i}$. I define the undirected graph $\bar{\mathbf{g}}$ as $\bar{\mathbf{g}} = \{\{i, j\} \in N : \bar{g}_{i,j} = 1\}$. That is, $\bar{\mathbf{g}}$ is a collection of links, which are listed as subsets of N of size 2. We write $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ to indicate that $\{\{i, j\} \in N : \{i, j\} \in \bar{\mathbf{g}}\} \subset \{\{i, j\} \in N : \{i, j\} \in \hat{\bar{\mathbf{g}}}\}$ and write $\bar{\mathbf{g}} = \hat{\bar{\mathbf{g}}}$ for $\{\{i, j\} \in N : \{i, j\} \in \bar{\mathbf{g}}\} = \{\{i, j\} \in N : \{i, j\} \in \hat{\bar{\mathbf{g}}}\}$. We write $\bar{\mathbf{g}} \subseteq \hat{\bar{\mathbf{g}}}$ for $\{\{i, j\} \in N : \{i, j\} \in \bar{\mathbf{g}}\} \subseteq \{\{i, j\} \in N : \{i, j\} \in \hat{\bar{\mathbf{g}}}\}$. The presence of a link $\bar{g}_{i,j} = 1$ allows players to directly benefit from the effort level exerted by the respective other agent involved in the link. Denote the set of i's neighbors in $\bar{\mathbf{g}}$ with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ and the corresponding cardinality with $\eta_i(\bar{\mathbf{g}}) = |N_i(\bar{\mathbf{g}})|$.[13] The aggregate effort level of agent i's neighbors in $\bar{\mathbf{g}}$, i.e. the effort level accessed, is written as $y_i(\bar{\mathbf{g}}) = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. The aggregate effort level of all agents other than i is written as $z_i(\bar{\mathbf{g}}) = \sum_{j \in N\backslash\{i\}} x_j$. We sometimes write y_i for $y_i(\bar{\mathbf{g}})$ and z_i for $z_i(\bar{\mathbf{g}})$ when it is clear from the context. Given a network $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} + \bar{g}_{i,j}$ and $\bar{\mathbf{g}} - \bar{g}_{i,j}$ have the following interpretation. When $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j}$ adds the link $\bar{g}_{i,j} = 1$, while if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j} = \bar{\mathbf{g}}$. Similarly, if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j}$ deletes the link $\bar{g}_{i,j}$, while if $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j} = \bar{\mathbf{g}}$. The network is called empty and denoted by $\bar{\mathbf{g}}^e$ if $\bar{g}_{i,j} = 0\ \forall i, j \in N$ and complete and denoted by $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1\ \forall i, j \in N$. For our analysis it will be useful to define paths, components and sub-components. A path between agents i and j in network $\bar{\mathbf{g}}$ is a sequence of links $\bar{g}_{1,2}, \bar{g}_{2,3}, ..., \bar{g}_{K-1,K}$ such that $\bar{g}_{k,k+1} = 1$ for each $k \in \{1, 2, ..., K-1\}$, with $i = 1$ and $j = K$ and such that each agent in the sequence $1, 2, ..., K$ is distinct. Components are maximal subsets of agents $N^s \subset N$, such that for every $i, j \in N^s$, there exists a path between i and j. A component is called complete, if all links between all agents in a component are present. Note that the binary relationship of "being connected by a network path" is an equivalence relationship and therefore components partition the set of agents. With some abuse of notation we write $k \in \bar{\mathbf{g}}^s$ to denote that agent k lies in component N^s in network $\bar{\mathbf{g}}$. We sometimes use subscripts to distinguish different components. A network is said to be connected if there is only one component.[14]

Payoffs of player i under strategy profile $\mathbf{s} = (\mathbf{x},\mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi_i(\mathbf{x},\bar{\mathbf{g}}) - \eta_i(\bar{\mathbf{g}})\kappa,$$

where κ denotes linking cost with $\kappa > 0$. Gross payoffs, i.e. payoffs excluding linking cost, $\pi_i(\mathbf{x},\bar{\mathbf{g}})$, are given by the frequently employed linear-quadratic payoff function with local complementarities and global substitutes (Ballester et al., 2006). That is,

[13] Agents are not linked to themselves and not included in their own neighborhood.

[14] Jackson (2010) and Vega-Redondo (2007)

$$\pi_i(\mathbf{x},\bar{\mathbf{g}}) = \alpha x_i - \tfrac{1}{2}\beta x_i^2 + \lambda x_i \sum_{j\in N_i(\bar{\mathbf{g}})} x_j - \gamma x_i \sum_{j\in N\setminus\{i\}} x_j \ \forall i \in N.$$

Note that Ballester et al. (2006) allow for weighted graphs, while the focus here is on unweighted graphs. I relate the specification in Ballester et al. (2006) in more detail to the one presented in this paper in Appendix A. Gross payoffs $\pi_i(\mathbf{x},\bar{\mathbf{g}})$ can be written as a function of own effort, x_i, the sum of effort levels of direct neighbors, $y_i(\bar{\mathbf{g}}) = \sum_{j\in N_i(\bar{\mathbf{g}})} x_j$ and the sum of effort levels of all agents different from i, $z_i(\bar{\mathbf{g}}) = \sum_{j\in N\setminus\{i\}} x_j$. For ease of notation we sometimes write $\pi_i(x_i, y_i, z_i)$ and drop the subscripts when they are clear from the context.

Note that effort induces a local positive externality, since $\partial\pi(x, y, z)/\partial y > 0$, and, if $\lambda > \gamma$, then effort levels are strict local strategic complements, i.e. $\partial^2\pi(x, y, z)/\partial x\partial y > 0$ $\forall x, y, z$. We assume $\lambda > \gamma \geq 0$ throughout, unless specified otherwise. Furthermore, for $\gamma > 0$, effort levels of agents that are not direct neighbors in $\bar{\mathbf{g}}$ induce negative externalities, due to $\partial\pi(x, y, z)/\partial z < 0$. Moreover, the payoff function displays global strategic substitutes, since $\partial^2\pi(x, y, z)/\partial x\partial z < 0$ holds $\forall x, y, z$. To guarantee the existence and uniqueness of an (interior) Nash equilibrium on any fixed network $\bar{\mathbf{g}}$, I can build on a result by Ballester et al. (2006) and assume that $\frac{\lambda}{\beta} < \frac{1}{n-1}$ holds. Next the best response function and the value function are derived, which will be useful for our equilibrium characterization.

Best response function. The unique best response of player i to the vector of effort levels $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\bar{x}_i(\mathbf{x}_{-i}, \bar{\mathbf{g}}) = \bar{x}_i(y_i, z_i) = \tfrac{1}{\beta}\left(\alpha + \lambda \sum_{j\in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j\in N\setminus\{i\}} x_j\right).$$

Value function. The maximized gross payoff under activity $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\pi_i(\bar{x}_i, \mathbf{x}_{-i}, \bar{\mathbf{g}}) = v(y_i, z_i) = \tfrac{1}{2\beta}\left(\alpha + \lambda \sum_{j\in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j\in N\setminus\{i\}} x_j\right)^2.$$

Before defining pairwise Nash equilibrium, I briefly derive the above payoff function in the context of crime, based on Jackson and Zenou (2014). Assume that expected gains of crime to agent i are given by

$$\pi_i(\mathbf{x}, \mathbf{g}) = b_i(\mathbf{x}) - p_i(\mathbf{x}, \mathbf{g})f,$$

with

$$\begin{cases} b_i(\mathbf{x}) = \alpha' x_i - \tfrac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \sum_{j\in N} x_j \\ p_i(\mathbf{x}, \mathbf{g}) = p_0 x_i(A - \lambda' \sum_{j\in N_i(\bar{\mathbf{g}})} x_j). \end{cases}$$

Expected cost of criminal activity, $p_i(\mathbf{x}, \mathbf{g})f$, increases in own criminal activity, x_i, since being involved in more criminal activities increases the chance of being caught. Local strategic complementarities stem from a decrease in the apprehension probability in direct neighbors' involvement in crime, due to direct know-how transfer. Note that I assume A to be sufficiently large, so that the apprehension probability is always positive for all criminals.[15] Finally, global strategic substitutes are due to congestion effects for crime opportunities, captured by $\gamma x_i \sum_{j\in N} x_j$ in the expression for $b_i(\mathbf{x})$.[16]

[15] See König, Liu and Zenou (2014) for how to calculate an appropriate lower bound on A.

[16] One way to argue for as to why congestion effects should affect agents with higher criminal activity

Direct substitution yields

$$\pi_i(\mathbf{x},\mathbf{g}) = (\alpha' - p_0 f A)x_i - \tfrac{1}{2}\beta x_i^2 + p_0 f \lambda' x_i \sum_{j\in N_i(\bar{\mathbf{g}})} x_j - \gamma x_i \sum_{j\in N\setminus\{i\}} x_j.$$

For $\alpha = \alpha' - p_0 f A > 0$ and $\lambda = p_0 f \lambda'$ these payoffs are equivalent to the specification used in Ballester et al. (2006).

Next *pairwise Nash equilibrium* (PNE) is defined in the presence of simultaneous moves and effort choice. I assume that when agents i and j deviate to create a link, then deviation effort levels are mutual best responses (while the remaining agent's effort levels are assumed to remain unchanged). The corresponding deviation effort levels are denoted by $x_i' = \bar{x}(y_i(\bar{\mathbf{g}}) + x_j', z_i(\bar{\mathbf{g}}) + x_j' - x_j)$. To simplify notation I sometimes write $x_i'(\bar{\mathbf{g}} + \bar{g}_{i,j}^+)$ to denote agent i's effort level when i and j create a link in network $\bar{\mathbf{g}}$.

A strategy profile $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{g}^*)$ is a pairwise Nash equilibrium *iff*

- for any $i \in N$ and every $\mathbf{s}_i \in S_i$, $\Pi_i(\mathbf{s}^*) \geq \Pi_i(\mathbf{s}_i, \mathbf{s}_{-i}^*)$;
- for all $\bar{g}_{i,j} = 0$, if $\Pi_i(x_i', x_j', \mathbf{x}^*{}_{-i,-j}, \bar{\mathbf{g}}^* + \bar{g}_{i,j}) > \Pi_i(\mathbf{s}^*)$,

  then $\Pi_j(x_i', x_j', \mathbf{x}^*{}_{-i,-j}, \bar{\mathbf{g}}^* + \bar{g}_{i,j}) < \Pi_j(\mathbf{s}^*)$.

A pairwise Nash equilibrium is both a Nash equilibrium and pairwise stable and therefore refines Nash equilibrium. Pairwise Nash equilibrium allows for deviations where a pair of agents creates a link (and deviating agents best respond to each other's effort level). Furthermore, pairwise Nash equilibrium allows for deviations in which an agent deletes any subset of existing links (and adjusts her effort level). However, deviations where a pair of agents creates a link and/or adjusts effort levels and *simultaneously* deletes any subset of existing links are not considered. An agent i's deviation strategy regarding the announcement of links is denoted with $\mathbf{g}_i'$ and the network after proposed deviation with $\bar{\mathbf{g}}'$.

3 Analysis - Network Formation

Note first that for $(\mathbf{x}, \bar{\mathbf{g}})$ to be a pairwise Nash equilibrium, we need that agents play Nash equilibrium effort levels on the network $\bar{\mathbf{g}}$. To show existence of a unique NE effort levels, I can resort to Theorem 1 in Ballester et al. (2006). I then show that Nash equilibrium effort levels must be equal for all players in a complete component. Furthermore, singleton agents display same effort levels.

Proposition 1: *For any fixed network,* $\bar{\mathbf{g}}$*, there exists a unique* NE *in effort levels. Furthermore, (i)* NE *effort levels are equal for all agents in a complete component, (ii)* NE *effort levels are equal for all singleton agents.*

Before presenting the existence result, two cost thresholds are defined, $\underline{\kappa}$ and $\overline{\kappa}$. The lower threshold, $\underline{\kappa}$, is given by the gross marginal payoff when a pair of agents creates a link in the empty network, $\bar{\mathbf{g}}^e$. The higher threshold, $\overline{\kappa}$, is defined as the average gross marginal payoff of linking to $n-1$ agents in the complete network, $\bar{\mathbf{g}}^c$. Denote the unique Nash equilibrium

more, as reflected in the term $\gamma x_i \sum_{j\in N} x_j$, is that when aggregate crime levels are higher, the public may become more vigilant, which in turn has a higher impact on agents with high individual levels of criminal activity.

effort level in the complete network, $\bar{\mathbf{g}}^c$, by $x(\bar{\mathbf{g}}^c)$ and the unique Nash equilibrium effort level in the empty network, $\bar{\mathbf{g}}^e$, by $x(\bar{\mathbf{g}}^e)$. Furthermore, denote the corresponding vectors of Nash equilibrium effort levels with $\mathbf{x}(\bar{\mathbf{g}}^c)$ and $\mathbf{x}(\bar{\mathbf{g}}^e)$.

Definition 1: $\underline{\kappa} = v\left(x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}^+), x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}^+) + (n-2)x(\bar{\mathbf{g}}^e)\right) - v\left(0, (n-1)x(\bar{\mathbf{g}}^e)\right)$

$$= \frac{\alpha^2\beta(2(\beta+\gamma)-\lambda)\lambda}{2(\beta+(n-1)\gamma)^2(\beta+\gamma-\lambda)^2} \text{ and}$$

$$\overline{\kappa} = \frac{1}{n-1}\left(v\left((n-1)x(\bar{\mathbf{g}}^c), (n-1)x(\bar{\mathbf{g}}^c)\right) - v(0, (n-1)x(\bar{\mathbf{g}}^c))\right)$$

$$= \frac{\alpha^2\lambda(2\beta-(n-1)\lambda)}{2\beta(\beta+(n-1)(\gamma-\lambda))^2}.$$

Note first that that one can show that for the parameter ranges considered $\underline{\kappa} < \overline{\kappa}$ holds. The lower of the two bounds, $\underline{\kappa}$, is defined as the marginal payoffs of a pair of agents creating a link in the empty network. Therefore, for linking cost $\kappa \geq \underline{\kappa}$ the empty network is a pairwise Nash equilibrium. The higher of the bounds, $\overline{\kappa}$, is given by the average gross marginal payoff of linking to $n-1$ agents in the complete network. Since the value function is convex in effort level accessed (for any fixed value of z_i) and since effort levels are the same for all agents, an agent in the complete network either finds it profitable do delete all links, or none. Therefore, if $\kappa < \underline{\kappa}$, then no agent finds it profitable to delete any links and the complete network is a pairwise Nash equilibrium. Finally, $\underline{\kappa} < \overline{\kappa}$ guarantees that a pairwise Nash equilibrium exists, as summarized in the statement of Proposition 2.

Proposition 2: *$\underline{\kappa} < \overline{\kappa}$ holds. Furthermore, (i) if $\kappa < \underline{\kappa}$ then $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ is a PNE, (ii) if $\kappa > \overline{\kappa}$ then $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ is a PNE and (iii) if $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ then $(\mathbf{x}(\bar{\mathbf{g}}^c), \bar{\mathbf{g}}^c)$ and $(\mathbf{x}(\bar{\mathbf{g}}^e), \bar{\mathbf{g}}^e)$ are PNE.*

Before presenting the first main result, I formally define nested split graphs below, which are a strict subset of core-periphery networks.[17,18] Note that the star, the complete and the empty network are nested split graphs.

Definition 2: A network $\bar{\mathbf{g}}$ is a *nested split graph* if and only if

$$[\bar{g}_{i,l} = 1 \textit{ and } \eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})] \Rightarrow \bar{g}_{i,k} = 1.$$

In the following I provide intuition for Theorem 1. Assume first that $\gamma = 0$. Note first that, due to strategic complementarities, agents who access higher effort levels also exert higher effort levels. Due to the strict convexity of the value function, agents who access higher effort levels benefit more from linking to any particular agent. Conversely, agents prefer to link to agents with higher effort levels. Therefore, in any pairwise Nash equilibrium, agents with higher effort levels accessed (and therefore higher own effort levels) must be linked to all agents to which agents with lower effort levels accessed (and therefore lower own effort levels) are linked to. It is this reinforcing mechanism that generates nestedness. Agents with

[17] A network $\bar{\mathbf{g}}$ is a core-periphery network if the set of agents N can be partitioned into two sets, $C(\bar{\mathbf{g}})$ (the core) and $P(\bar{\mathbf{g}})$ (the periphery), such that $\bar{g}_{i,j} = 1\ \forall i,j \in C(\bar{\mathbf{g}})$ and $\bar{g}_{i,j} = 0\ \forall i,j \in P(\bar{\mathbf{g}})$.

[18] For a formal proof that all nested split graphs are core-periphery networks see Chvátal and Hammer (1977).

higher effort levels display a higher number of links for the same reason. As the network is nested in any pairwise Nash equilibrium, a higher number of links also implies a higher effort level accessed and, since the value function is increasing, higher gross payoffs.

The case when $\gamma > 0$ is more involved, but builds on a similar intuition. Note first that an agent's value function is strictly convex in the sum of direct neighbors' effort level, y_i, for any *fixed* aggregate effort level of the remaining agents, z_i. However, since $\lambda > \gamma$ and since agents then always increase effort levels when creating a new link, one can show that the payoffs from accessing effort level by creating new links is strictly convex. That is, while there is now a difference between accessing effort level via new links vs. direct neighbors increasing effort levels, much of the underlying intuition also applies to this case. Again reinforcing incentives to create and sustain links yield nested split graphs as the only pairwise Nash equilibrium networks.

Theorem 1: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*, the network* $\bar{\mathbf{g}}$ *is a nested split graph such that* $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}}) \Leftrightarrow \pi_i < \pi_k$ *holds.*

4 Key Player Policy

In the following I study key-player policies when the network is endogenous. Let us start by introducing pairwise best response dynamics. First note that that the configuration $\bar{\mathbf{g}}$ and $\mathbf{x}(\bar{\mathbf{g}})$ at the outset is assumed to be a pairwise Nash equilibrium. A planner can then eliminate one agent i from the network. Denote the network after agent i was eliminated from $\bar{\mathbf{g}}$ with $\bar{\mathbf{g}}_0^{-i}$. Agents adjust their efforts to the Nash equilibrium levels in $\bar{\mathbf{g}}_0^{-i}$, denoted with $\mathbf{x}(\bar{\mathbf{g}}_0^{-i})$. In the first time period, all links are added to the network that are profitable in isolation in configuration $\bar{\mathbf{g}}_0^{-i}$, $\mathbf{x}(\bar{\mathbf{g}}_0^{-i})$. That is, any link not already present, such that a pair of agent can profitably deviate by creating a link, given $\mathbf{x}(\bar{\mathbf{g}}_0^{-i})$ and $\bar{\mathbf{g}}_0^{-i}$, is added to the network $\bar{\mathbf{g}}_0^{-i}$. This yields the network $\bar{\mathbf{g}}_1^{-i}$. Again agents update their effort levels to Nash equilibrium effort levels $\mathbf{x}(\bar{\mathbf{g}}_1^{-i})$. Given $\mathbf{x}(\bar{\mathbf{g}}_1^{-i})$ and $\bar{\mathbf{g}}_1^{-i}$, I assume agents play a minimal optimal deletion strategy. That is, agents play a deviation strategy such that payoffs are maximal (including the strategy where no links are deleted) and, if there are multiple such deviation strategies, the strategy chosen is such that, loosely speaking, the number of links in an agent's deviation strategy is minimal. (A formal definition of minimal optimal deletion strategies is presented in Appendix B). The procedure is repeated until the process converges, i.e. until some time period t' such that $\bar{\mathbf{g}}_{t'}^{-i} = \bar{\mathbf{g}}_{t'+1}^{-i} = \bar{\mathbf{g}}_{t'+2}^{-i}$. Below I summarize the above procedure.

Pairwise best response dynamics:

- At $t = 0$ start with $\bar{\mathbf{g}}_0^{-i}$ and the corresponding vector of Nash equilibrium effort levels $\mathbf{x}(\bar{\mathbf{g}}_0^{-i})$. Define $\bar{\mathbf{g}}_t^{-i}$ iteratively as follows.
- **Step 1:** Add all links links that are profitable for any pair of agents j and k , given $\bar{\mathbf{g}}_t^{-i}$ and $\mathbf{x}(\bar{\mathbf{g}}_t^{-i})$. Denote the resulting network by $\bar{\mathbf{g}}_{t+1}^{-i}$ and corresponding effort levels by $\mathbf{x}(\bar{\mathbf{g}}_{t+1}^{-i})$.
- **Step 2:** Agents play minimal optimal link deletion strategies given $\bar{\mathbf{g}}_{t+1}^{-i}$ and $\mathbf{x}(\bar{\mathbf{g}}_{t+1}^{-i})$. Denote the resulting network by $\bar{\mathbf{g}}_{t+2}^{-i}$ and corresponding effort levels by $\mathbf{x}(\bar{\mathbf{g}}_{t+2}^{-i})$.
- **Repeat** Steps 1 and 2 until convergence is reached.

Next a key player policy is defined in the context of an endogenous network. More precisely, the key player policy aims at identifying the set of agents for which, once eliminated, aggregate effort levels of the remaining agents is lowest after above procedure has converged. Note that alternatively one could assume that the planner aims at minimizing the discounted sum of criminal activity over an infinite time horizon. The result in Theorem 2 then also goes through, as long as the discount value is sufficiently close to 1. Finally, a key player policy is said to exist if the pairwise best response dynamics converge, and to not exist otherwise.

Key player policy: Pick an agent i such that $min\{\sum_{j\in N\setminus\{i\}} x_j(\overline{\mathbf{g}}_{t'}^{-i}) \mid i = 1, ..., n\}$.

Theorem 2 shows that, if the parameter governing global substitution effects is sufficiently small, then not only a key player policy exists, but it also prescribes eliminating an agent with the highest number of links. As the initial network is a nested split graph, this is also the agent with the highest Bonacich centrality and the highest inter-centrality. Therefore, the key player policy presented here coincides with the key player policy in Ballester et al. (2006). Moreover, the key player is particularly easy to identify; it is an agent with the highest number of links. The intuition for Theorem 2 is simple. One can show that if γ is sufficiently small, then each agent's effort level is smaller in $\overline{\mathbf{g}}_0^{-i}$ than in $\overline{\mathbf{g}}$ and incentives to create new links in $\overline{\mathbf{g}}_0^{-i}$ are strictly lower than in $\overline{\mathbf{g}}$. Since $\overline{\mathbf{g}}$ is a pairwise Nash equilibrium and no pair of agent that is not connected finds it profitable to create a new link, no pair of agent that is not connected in $\overline{\mathbf{g}}_0^{-i}$ finds it profitable to create a new link. As effort levels are lower for each agent and (some) agents sustain fewer links, incentives to delete links are higher in $\overline{\mathbf{g}}_0^{-i}$ than in $\overline{\mathbf{g}}$. Using this argument iteratively, one can then show that agents never find it profitable to create new links, but may find it profitable to delete links. Therefore the process converges, as it is bounded below by the empty network. Since an agent with the highest number of links also has the highest effort level, incentives to delete links are highest in $\overline{\mathbf{g}}_0^{-i}$, and at every iteration the resulting network can be considered as minimal.[19] From Theorem 2 in Ballester et al. (2006) we know that if one network covers another network (i.e. if all links are present in one network that are also present in the other network, and some additional links), then the aggregate effort level is higher in the former than in the latter. That is, eliminating an agent with the highest number of links is optimal. It is worth noting that this result appears robust to, for example, drawing at random either pairs of agents that are presented with the opportunity to create a new link, or individual agents that may update their strategies, as we can apply the same reasoning as above.

Theorem 2: *If γ is sufficiently small, then a key player policy always exists and it prescribes eliminating an agent with the highest number of links. Furthermore, $\sum_{j\in N\setminus\{i\}} x_j(\overline{\mathbf{g}}_{t'}^{-i}) < \sum_{j\in N} x_j(\overline{\mathbf{g}})$ holds.*

Finally, I provide an example, depicted in Figure 1 below, for which the key player policy prescribed in Theorem 2, and therefore the key player policy in Ballester et al. (2006), does not select the agent that minimizes aggregate effort levels. Assume that $n = 7$, $\alpha = 1$, $\beta = 5$, $\lambda = 0.2$ and $\gamma = 0.8$. One can show that the network on the left is a pairwise Nash equilibrium and I now compare eliminating a more central agent c with a more peripheral player p. Note that when a peripheral player p is eliminated, then the remaining agents do not find it profitable to create additional links and network $\overline{\mathbf{g}}_0^{-p}$ is a pairwise Nash equilibrium network. That is, proposed pairwise best response dynamics converge to $\overline{\mathbf{g}}_0^{-p}$. If instead a

[19] Note that agents are relabeled, so that networks in which different agents were removed can be compared.

more central agent c is eliminated, yielding $\overline{\mathbf{g}}_0^{-c}$, then the remaining peripheral player find it profitable to create links among themselves. This yields the complete network, which in turn is a pairwise Nash equilibrium. Since $\overline{\mathbf{g}}_0^{-p} \subset \overline{\mathbf{g}}_1^{-c}$ holds, we know from Theorem 2 in Ballester et al. (2006) that the sum of criminal activity is higher in $\overline{\mathbf{g}}_1^{-c}$ than in $\overline{\mathbf{g}}_0^{-p}$. The intuition is that the aggregate criminal activity is lower in $\overline{\mathbf{g}}_0^{-c}$ than in $\overline{\mathbf{g}}_0^{-p}$ (since $\overline{\mathbf{g}}_0^{-c} \subset \overline{\mathbf{g}}_0^{-p}$), thereby increasing incentives to link. Furthermore, since γ is low, it is not necessarily the case that all agents have a lower effort level in $\overline{\mathbf{g}}_0^{-c}$ than in $\overline{\mathbf{g}}_0^{-p}$. Combining the two, creating new links in a network such as $\overline{\mathbf{g}}_0^{-c}$ may then be more profitable than in a network such as $\overline{\mathbf{g}}_0^{-p}$.

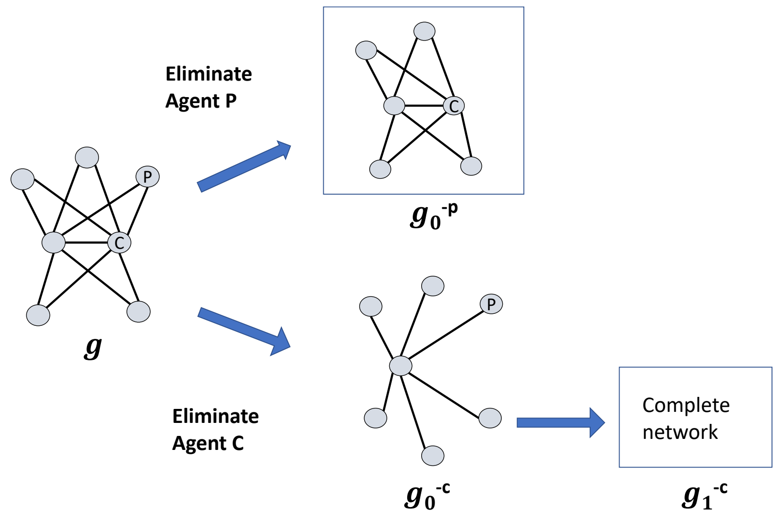

Figure 1

5 Conclusion

This paper makes two main contributions. First, it endogenizes the network for the seminal work by Ballester et al. (2006). It is shown that all pairwise Nash equilibria are nested split graphs. These networks have the interesting property that Bonacich, inter-centrality and degree centrality all coincide. I then turn to key player policies when the network is endogenous and show that, if the parameter governing strategic substitutes is sufficiently low, then, as in Ballester et al. (2006), it is optimal to eliminate an agent that is most central and therefore criminal activity is also highest. In the equilibrium networks obtained, this translates into a particularly simple policy: remove the agent with the highest number of links/the highest effort level. However, if the parameter governing strategic substitutes is large, then, different from Ballester et al. (2006), it may be optimal to eliminate a less central agent, who does not display the highest criminal activity. That is, these results indicate that when designing key player policies in practice, one should take into account possible changes to the network after an agent was eliminated, in particular when the competition or congestion effects are considered to be large.

6 Appendix A

Relationship to BCZ (2006) Payoffs given by

$$u_i(x_1, ..., x_n) = \alpha x_i + \tfrac{1}{2}\sigma x_i^2 + \textstyle\sum_{j \neq i} \sigma_{ij} x_i x_j,$$

with $\alpha > 0$ and $\sigma < 0$. BCZ (2006) define $\underline{\sigma} = min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = max\{\sigma_{ij} \mid i \neq j\}$ and rewrite payoffs as

$$u_i(\mathbf{x}) = \alpha x_i - \tfrac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \textstyle\sum_{j \in N} x_j + \lambda \textstyle\sum_{j \in N} g_{ij} x_i x_j,$$

where $\gamma = -min\{\underline{\sigma}, 0\} \geq 0$, $\lambda = \overline{\sigma} + \gamma \geq 0$ and $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$.

Note that when $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$ then adjacency matrix $\mathbf{G}$ is a symmetric $(0, 1)$ matrix and $\mathbf{g}$ is undirected and unweighted.

Note further that when $\overline{\sigma} > 0$, then $\lambda > \gamma$.

7 Appendix B

Proof of ***Proposition 1.*** The payoff function considered is a special case of Ballester et al. (2006) and we can therefore rely on the following result (Theorem 1 in Ballester et al., 2006): A NE exists and is unique if and only if $\beta < \lambda \frac{1}{\mu_1(\bar{\mathbf{g}})}$, where $\mu_1(\bar{\mathbf{g}})$ is the largest eigenvalue of the adjacency matrix of $\bar{\mathbf{g}}$. Note that the largest eigenvalue for a graph lies between the following bounds $max\{d_{avg}(\bar{\mathbf{g}}), \sqrt{d_{max}(\bar{\mathbf{g}})}\} \leq \mu_1(\bar{\mathbf{g}}) \leq d_{max}(\bar{\mathbf{g}})$, where $d_{max}(\bar{\mathbf{g}})$ is the maximum degree and $d_{avg}(\bar{\mathbf{g}})$ the average degree in network $\bar{\mathbf{g}}$.[20] The largest eigenvalue for a graph is then maximal and equal to $n-1$ in the complete network, $\bar{\mathbf{g}}^c$. The existence of a unique NE is therefore guaranteed by the assumption that $\frac{\lambda}{\beta} < \frac{1}{n-1}$.

Part (i): Assume to the contrary that there exists a NE, $\mathbf{x}(\bar{\mathbf{g}})$, such that a pair of players k and l are in a complete component with $x_k \neq x_l$ and assume without loss of generality that $x_k > x_l$. Note that in a complete component $N_k(\bar{\mathbf{g}}) \setminus \{l\} = N_l(\bar{\mathbf{g}}) \setminus \{k\}$ holds and therefore $\sum_{j \in N_l(\bar{\mathbf{g}})} x_j = \sum_{j \in N_k(\bar{\mathbf{g}})} x_j + (x_k - x_l) > \sum_{j \in N_k(\bar{\mathbf{g}})} x_j$. Note further that $\sum_{j \in N \setminus \{l\}} x_j = \sum_{j \in N \setminus \{k\}} x_j + (x_k - x_l)$. Plugging the above into the best response functions for agent k and l, respectively, we obtain

$$\begin{aligned}
\bar{x}_l(\mathbf{x}_{-l}, \bar{\mathbf{g}}) &= \tfrac{1}{\beta} \cdot \left(\alpha + \lambda \textstyle\sum_{j \in N_l(\bar{\mathbf{g}})} x_j - \gamma \textstyle\sum_{j \in N \setminus \{l\}} x_j\right) \\
&= \tfrac{1}{\beta} \cdot \left(\alpha + \lambda \left(\textstyle\sum_{j \in N_k(\bar{\mathbf{g}})} x_j + (x_k - x_l)\right) - \gamma \left(\textstyle\sum_{j \in N \setminus \{k\}} x_j + (x_k - x_l)\right)\right) \\
&> \tfrac{1}{\beta} \cdot \left(\alpha + \lambda \textstyle\sum_{j \in N_k(\bar{\mathbf{g}})} x_j - \gamma \textstyle\sum_{j \in N \setminus \{k\}} x_j\right) = \bar{x}_k(\mathbf{x}_{-k}, \bar{\mathbf{g}}),
\end{aligned}$$

where the inequality follows from $x_k - x_l > 0$ and $\lambda > \gamma \geq 0$. We have therefore reached a contradiction.

Part (ii): The result follows from an analogous argument to the one provided in Part (i). *Q.E.D.*

[20] See, for example, L. Lovasz, Geometric Representations of Graphs (2009).

Proof of ***Proposition 2.*** We first derive the two bounds on linking cost, $\underline{\kappa}$ and $\overline{\kappa}$. $\overline{\kappa}$ is given by the average marginal payoff per link of an agent in the complete network, $\bar{\mathbf{g}}^c$. Note that the effort level of an agent in the complete network, $\bar{\mathbf{g}}^c$, is easily calculated and given by $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c) = \alpha/(\beta + (n-1)(\gamma - \lambda))$ $\forall i \in N$. We then obtain $\overline{\kappa}$, i.e. the average gross marginal payoff of linking to $n-1$ agents in the complete network, $\bar{\mathbf{g}}^c$, by substituting $x(\bar{\mathbf{g}}^c)$ into the expression for $\overline{\kappa}$, provided in Definition 1. In turn, $\underline{\kappa}$ is given by the marginal payoff of two agents creating a new link in the empty network, $\bar{\mathbf{g}}^e$. The effort level in the empty network, $x(\bar{\mathbf{g}}^e)$, is given by $x(\bar{\mathbf{g}}^e) = x_i(\bar{\mathbf{g}}^e) = \alpha/(\beta + (n-1)\gamma)$ $\forall i \in N$ and the effort level of a pair of agents i and j deviating by creating a link is given by $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = x_j'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = \alpha(\beta+\gamma)/\left((\beta + (n-1)\gamma)(\beta - (\lambda - \gamma))\right)$. To obtain $\underline{\kappa}$ we substitute $x(\bar{\mathbf{g}}^e)$ and $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ in the expression for $\overline{\kappa}$, provided in Definition 1. Note next that we can write $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = \frac{(\beta+\gamma)}{(\beta+\gamma-\lambda)} \cdot x(\bar{\mathbf{g}}^e)$. Note also that from Ballester et al. (2006) we know that, given our assumptions on parameters, Nash equilibrium effort levels are interior for any $\bar{\mathbf{g}}$. From $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c) = \alpha/(\beta - (n-1)(\lambda - \gamma)) > 0$ we then know that $\beta - (n-1)(\lambda - \gamma) > 0$ holds and therefore $\beta - (\lambda - \gamma) > 0$ also holds. To see this, recall that $\lambda > \gamma$. Therefore, $\frac{(\beta+\gamma)}{(\beta+\gamma-\lambda)} > 0$ and $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) > x(\bar{\mathbf{g}}^e)$ holds. Finally, note that from the expression of Nash equilibrium effort levels in the empty network, $\bar{x}_i(0, (n-1)x(\bar{\mathbf{g}}^e)) = \frac{1}{\beta} \cdot (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))$, we also know that $\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e) > 0$ holds. We can then write $\underline{\kappa}$ as follows

$$\underline{\kappa} = \frac{1}{2\beta} \cdot \left(\left(\alpha + (\lambda - \gamma)\frac{(\beta+\gamma)}{(\beta+\gamma-\lambda)} x(\bar{\mathbf{g}}^e) - \gamma(n-2)x(\bar{\mathbf{g}}^e) \right)^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))^2 \right).$$

From $\lambda - \gamma > 0$ it follows that $\underline{\kappa} > 0$. Note next that the expression for $x(\bar{\mathbf{g}}^c) = x_i(\bar{\mathbf{g}}^c) = \alpha/(\beta - (n-1)(\lambda - \gamma))$ is increasing in n in the range of $[0, n]$. Recall also that, since Nash equilibrium effort levels are interior for any $\bar{\mathbf{g}}$, so that $\beta - (n-1)(\lambda - \gamma) > 0$ holds. Denote with $x(\bar{\mathbf{g}}^c_{n=2})$ when setting $n = 2$ in the expression for $x(\bar{\mathbf{g}}^c)$, i.e. $x(\bar{\mathbf{g}}^c_{n=2}) = \frac{\alpha}{(\beta - (\lambda - \gamma))}$ and note that $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^c_{n=2})$. Let us now compare $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$, where in accordance with our model we assume $n \geq 3$, with the expression $x(\bar{\mathbf{g}}^c_{n=2})$. Recall that $x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) = \alpha(\beta + \gamma)/\left((\beta + (n-1)\gamma)(\beta - (\lambda - \gamma))\right)$. Note that $(\beta + \gamma)/(\beta + (n-1)\gamma) < 1$ (since $n \geq 3$) and therefore $x(\bar{\mathbf{g}}^c_{n=2}) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$ holds. Since $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^c_{n=2})$ holds, $x(\bar{\mathbf{g}}^c) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) > x(\bar{\mathbf{g}}^e)$ also holds. Finally, note that

$$\begin{aligned}
\overline{\kappa} &= \tfrac{1}{2\beta} \cdot \left((\alpha + (\lambda - \gamma)(n-1)x(\bar{\mathbf{g}}^c))^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^c))^2 \right) / (n-1) \\
&> \tfrac{1}{2\beta} \cdot \left((\alpha + (\lambda - \gamma)(n-1)x(\bar{\mathbf{g}}^c))^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))^2 \right) / (n-1) \\
&> \tfrac{1}{2\beta} \cdot \left((\alpha + (\lambda - \gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}))^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))^2 \right) / (n-1) \\
&> \tfrac{1}{2\beta} \cdot \left((\alpha + (\lambda - \gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e))^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))^2 \right) / (n-1) \\
&> \tfrac{1}{2\beta} \cdot \left((\alpha + (\lambda - \gamma)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e))^2 - (\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e))^2 \right) = \underline{\kappa}.
\end{aligned}$$

The first inequality follows from $x(\bar{\mathbf{g}}^c) > x(\bar{\mathbf{g}}^e)$, while the second inequality follows from $x(\bar{\mathbf{g}}^c) > x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j})$. To see that the third inequality holds, note first that $\alpha - \gamma(n-1)x(\bar{\mathbf{g}}^e) > 0$ holds (since Nash equilibrium effort levels are interior in $\bar{\mathbf{g}}^e$), so that $\alpha + (\lambda - \gamma)(n-1)x_i'(\bar{\mathbf{g}}^e + \bar{g}_{i,j}) - \gamma(n-2)x(\bar{\mathbf{g}}^e) > 0$ also holds. The inequality then follows from $\gamma(n-2)x(\bar{\mathbf{g}}^e) > 0$. Finally, the last inequality follows immediately from the quadratic functional form. Therefore, $\overline{\kappa} > \underline{\kappa} > 0$. Note next that, if $\kappa \leq \overline{\kappa}$, then an agent in the complete network does not find it profitable to delete all his links. As $v(y_i, z_i)$, is convex in y_i, deleting any subset of links is also not profitable. Therefore, for $\kappa \leq \overline{\kappa}$, the complete network is a PNE. If $\kappa \geq \overline{\kappa}$, then no pair of agents finds it profitable to create a link in the empty network, and therefore the empty network is a PNE. *Q.E.D.*

Proof of ***Theorem 1.*** We first provide four auxiliary lemmas, which we then use to show that every PNE is a nested split graph.

Auxiliary Lemma: *For any network* $\bar{\mathbf{g}}$ *and corresponding vector of NE effort levels,* $\mathbf{x}(\bar{\mathbf{g}})$*, if* $\bar{g}_{i,j} = 0$ *and* $\bar{g}'_{i,j} = 1$*, then* $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_i(\bar{\mathbf{g}})$ *and* $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_j(\bar{\mathbf{g}})$ *holds.*

Proof of ***Auxiliary Lemma.*** Recall that the best response functions for $x_i(\bar{\mathbf{g}})$ and $x_j(\bar{\mathbf{g}})$ are given by $x_i(\bar{\mathbf{g}}) = \frac{1}{\beta} \cdot (\alpha + \lambda y_i(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}}))$ and $x_j(\bar{\mathbf{g}}) = \frac{1}{\beta} \cdot (\alpha + \lambda y_j(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}}))$, while the best response functions for $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j})$ and $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j})$ can be written as $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) = \frac{1}{\beta} \cdot \left(\alpha + \lambda \left(y_i(\bar{\mathbf{g}}) + x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j})\right) - \gamma \left(z_i(\bar{\mathbf{g}}) + (x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) - x_j(\bar{\mathbf{g}}))\right)\right)$ and $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) = \frac{1}{\beta} \cdot \left(\alpha + \lambda \left(y_j(\bar{\mathbf{g}}) + x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j})\right) - \gamma \left(z_j(\bar{\mathbf{g}}) + (x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) - x_i(\bar{\mathbf{g}}))\right)\right)$. We can now rewrite the latter expressions as $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) = x_i(\bar{\mathbf{g}}) + \frac{1}{\beta}\left((\lambda - \gamma)x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) + \gamma x_j(\bar{\mathbf{g}})\right)$ and $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) = x_j(\bar{\mathbf{g}}) + \frac{1}{\beta}\left((\lambda - \gamma)x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) + \gamma x_i(\bar{\mathbf{g}})\right)$, respectively. Assume first that $0 \leq x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) \leq x_i(\bar{\mathbf{g}})$ and $0 \leq x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) \leq x_j(\bar{\mathbf{g}})$ hold. But then $(\lambda - \gamma)x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) + \gamma x_j(\bar{\mathbf{g}}) > 0$ and therefore $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_i(\bar{\mathbf{g}})$ must hold. We have reached a contradiction. Assume next, and without loss of generality, that $0 \leq x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) \leq x_i(\bar{\mathbf{g}})$ and $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_j(\bar{\mathbf{g}})$ holds. But then $(\lambda - \gamma)x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) + \gamma x_j(\bar{\mathbf{g}}) > 0$ holds and therefore $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_i(\bar{\mathbf{g}})$ must hold. We have again reached a contradiction. Therefore, $x'_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_i(\bar{\mathbf{g}})$ and $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x_j(\bar{\mathbf{g}})$ hold. *Q.E.D.*

Lemma 1: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*, if* $\bar{g}_{i,l} = 1$*, then* $\bar{g}_{i,k} = 1$ *for all agents* k *with* $x_k \geq x_l$*.*

Proof of ***Lemma 1.*** Assume that $(\mathbf{x}, \bar{\mathbf{g}})$ is a PNE and, contrary to the above, that $\bar{g}_{i,l} = 1$ and $\bar{g}_{i,k} = 0$ for some agent k with $x_k \geq x_l$. Note first that for $\bar{g}_{i,l} = 1$ to be part of a PNE, it must be that $v(y_i, z_i) - v(y_i - x_l, z_i) \geq \kappa$ holds, as otherwise agent i could profitably deviate by deleting the link with agent l (and adjust her effort level). Next we show that, if the latter condition holds, then agent i also finds it profitable to create the link $\bar{g}'_{i,k} = 1$. From Lemma 0 we know that $x'_k > x_k$ holds. Since the value function is strictly convex in the first argument for all z_i, $v(y_i + x_l, z_i) - v(y_i, z_i) > v(y_i, z_i) - v(y_i - x_l, z_i)$ holds. Furthermore, since $x_k \geq x_l$ holds, the following also holds $v(y_i + x_k, z_i) - v(y_i, z_i) \geq v(y_i + x_l, z_i) - v(y_i, z_i)$. For ease of notation in the following we sometimes use y'_i to denote $y'_i = y_i + x'_k$ and z'_i to denote $z'_i = z_i + (x'_k - x_k)$. Note next that we can write $v(y_i + x_k, z_i)$ as $v(y_i + x_k, z_i) = \pi(\overline{x}(y_i + x_k, z_i), y_i + x_k, z_i)$. From $\lambda > \gamma$ we know that $\pi(\overline{x}(y_i + x_k, z_i), y'_i, z'_i) > \pi(\overline{x}(y_i + x_k, z_i), y_i + x_k, z_i)$ holds. Finally, note that $\pi(\overline{x}(y'_i, z'_i), y'_i, z'_i) \geq \pi(\overline{x}(y_i + x_k, z_i), y'_i, z'_i)$ holds (since $\overline{x}(y_i + x_k, z_i)$ is not optimal given y'_i and z'_i). Since by definition $v(y'_i, z'_i) = \pi(\overline{x}(y'_i, z'_i), y'_i, z'_i)$, we therefore know that $v(y_i + x'_k, z_i + (x'_k - x_k)) - v(y_i, z_i) > v(y_i, z_i) - v(y_i - x_l, z_i) \geq \kappa$ holds. That is, if agent i does not find it profitable to delete the link with agent l, then agent i finds it profitable to create the link with agent k. For $\bar{g}_{i,k} = 0$ to hold in a PNE, we would therefore need that agent k does not find it profitable to link to agent i. In the following we show that this cannot be the case. Note that for $\bar{g}_{i,l} = 1$ to hold, $v(y_l, z_l) - v(y_l - x_i, z_l) \geq \kappa$ must hold, as otherwise agent l could profitably deviate by deleting the link with agent i (and adjust her effort level). From the convexity of the value function in the first argument, we know that $v(y_l + x_i, z_l) - v(y_l, z_l) > v(y_l, z_l) - v(y_l - x_i, z_l)$ holds. Note that we can write z_k and z_l as $z_k = \sum_{j \in N} x_j - x_k$ and $z_l = \sum_{j \in N} x_j - x_l$, respectively. Note that, if $x_k \geq x_l$ holds, then $z_l \geq z_k$ also holds. We can now write the best response functions as $\bar{x}_k(\bar{\mathbf{g}}) = \frac{1}{\beta} \cdot \Big(\alpha + \lambda y_k(\bar{\mathbf{g}}) - \gamma(\sum_{j \in N} x_j(\bar{\mathbf{g}}) - \bar{x}_k(\bar{\mathbf{g}}))\Big)$ and $\bar{x}_l(\bar{\mathbf{g}}) = \frac{1}{\beta} \cdot \Big(\alpha + \lambda y_l(\bar{\mathbf{g}}) - \gamma(\sum_{j \in N} x_j(\bar{\mathbf{g}}) - \bar{x}_l(\bar{\mathbf{g}}))\Big)$. Taking the difference we obtain $\bar{x}_k(\bar{\mathbf{g}}) - \bar{x}_l(\bar{\mathbf{g}}) = \frac{\lambda}{\beta - \gamma} \cdot (y_k(\bar{\mathbf{g}}) - y_l(\bar{\mathbf{g}})) \geq 0$. Rewrite $\frac{\lambda}{\beta} < \frac{1}{n-1}$ as $\beta > (n-1)\lambda$. From $\lambda > \gamma \geq 0$ it then follows that $\frac{\lambda}{\beta - \gamma} > 0$ and therefore $y_k \geq y_l$ holds. Finally, from $\partial^2 v(y,z)/\partial y \partial z = -\lambda\gamma/\beta \leq 0$, $z_l \geq z_k$, the convexity of the value function in the first argument and $y_k \geq y_l$, it then follows that $v(y_k + x_i, z_k) - v(y_k, z_k) \geq v(y_l + x_i, z_l) - v(y_l, z_l)$ holds. By an argument analogous to the one presented above, it then follows that $v(y_k + x_i', z_k + (x_i' - x_i)) - v(y_k, z_k) > v(y_k + x_i, z_k) - v(y_k, z_k) > \kappa$. That is, agent k finds it profitable to link to agent i and proposed deviation is profitable. Therefore, in any PNE, if $\bar{g}_{i,l} = 1$, then $\bar{g}_{i,k} = 1$ for all agents k with $x_k \geq x_l$. *Q.E.D.*

Before presenting Lemma 2, we define Bonacich centrality, following the notation in Ballester et al. (2006). Denote with $\bar{g}_{i,j}^{[k]} \geq 0$ the number of paths of length $k \geq 1$ in $\bar{\mathbf{g}}$ from i to j.[21] Define $m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{k=0}^{+\infty} (\frac{\lambda}{\beta})^k \bar{g}_{i,j}^{[k]}$, so that $m_{i,j}(\bar{\mathbf{g}}, a)$ counts the number of paths in $\bar{\mathbf{g}}$ that start at i and end at j, and paths of length k are weighted by a^k. The Bonacich centrality of agent i is $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{j=1}^{n} m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$, so that $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ counts the total number of paths that start at i (and are weighted by $(\frac{\lambda}{\beta})^k$). Furthermore, define $b(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{i \in N} b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$.

Lemma 2: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*,* $x_i = x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$*.*

Proof of ***Lemma 2.*** First we show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i = x_k$. Note that from Theorem 1 in Ballester et al. (2006) we can write effort levels in a given network $\bar{\mathbf{g}}$, and for given parameter values, as $x_i(\bar{\mathbf{g}}) = \alpha b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) / \Big(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\Big)$.[22] Note that, if $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$, then $b_i(\bar{\mathbf{g}}) = b_k(\bar{\mathbf{g}})$ also holds and therefore $x_i = x_k$. Next we show that $x_i = x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i = x_k$ and $N_i(\bar{\mathbf{g}}) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}) \setminus \{i\}$. There must then exist an agent l such that either $k \in N_l(\bar{\mathbf{g}})$ and $i \notin N_l(\bar{\mathbf{g}})$, or $k \notin N_l(\bar{\mathbf{g}})$ and $i \in N_l(\bar{\mathbf{g}})$. Since $x_i = x_k$, this contradicts Lemma 1 and therefore $N_i(\bar{\mathbf{g}}) \setminus \{k\} = N_k(\bar{\mathbf{g}}) \setminus \{i\}$. *Q.E.D.*

Lemma 3: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*,* $x_i < x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$*.*

Proof of ***Lemma 3.*** First we show that $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\} \Rightarrow x_i < x_k$. Note that if $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$, then $b_i(\bar{\mathbf{g}}) < b_k(\bar{\mathbf{g}})$ and therefore by the above argument $x_i(\bar{\mathbf{g}}) < x_k(\bar{\mathbf{g}})$ holds. Next we show that $x_i < x_k \Rightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$. Assume to the contrary that $x_i < x_k$, but $N_i(\bar{\mathbf{g}}) \setminus \{k\} \subset N_k(\bar{\mathbf{g}}) \setminus \{i\}$ does not hold. We distinguish two

[21] A path of length k from i to j is a sequence $(i_0, ..., i_k)$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$ and $\bar{g}_{i_p, i_{p+1}} = 1$ for all $0 \leq k \leq k-1$, i.e. agents i_p and i_{p+1} are directly connected in $\bar{\mathbf{g}}$ (Ballester et al., 2006).

[22] The vector of Bonacich centralities of parameter a in $\bar{\mathbf{g}}$, $\mathbf{b}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$, can be obtained by the following expression $\mathbf{b}(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = [\mathbf{I} - \frac{\lambda}{\beta} \bar{\mathbf{G}}]^{-1} \cdot \mathbf{1}$, where $\mathbf{I}$ is the n by n identity matrix, with slight abuse of notation we take $\bar{\mathbf{G}}$ to be the adjacency matrix of $\bar{\mathbf{g}}$ and $\mathbf{1}$ is the n dimensional vector of ones. From Theorem 1 in Ballester et al. (2006) we know that the vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}})$, for network $\bar{\mathbf{g}}$ is obtained by $\mathbf{x}(\bar{\mathbf{g}}) = \Big(\alpha / \Big(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\Big)\Big) \cdot \mathbf{b}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$. Furthermore, an agent i's Nash equilibrium effort level can be written as $x_i(\bar{\mathbf{g}}) = \Big(b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) / b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\Big) \cdot x(\bar{\mathbf{g}})$, where $x(\bar{\mathbf{g}}) = \sum_{j \in N} x_j(\bar{\mathbf{g}})$. We can now write $\mathbf{x}(\bar{\mathbf{g}}) \cdot \mathbf{1} = \Big(\alpha / \Big(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\Big)\Big) \cdot b(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = x(\bar{\mathbf{g}})$ and substitute this into the expression for $x_i(\bar{\mathbf{g}})$ to obtain $x_i(\bar{\mathbf{g}}) = \alpha b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) / \Big(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\Big)$.

subcases. Assume first that $x_i < x_k$ and $N_k(\bar{\mathbf{g}})\setminus\{i\} = N_i(\bar{\mathbf{g}})\setminus\{k\}$. This contradicts Lemma 2. Next, assume $x_i < x_k$ and $N_k(\bar{\mathbf{g}})\setminus\{i\} \neq N_i(\bar{\mathbf{g}})\setminus\{k\}$ holds, while $N_i(\bar{\mathbf{g}})\setminus\{k\} \subset N_k(\bar{\mathbf{g}})\setminus\{i\}$ does not hold. There then exists an agent m such that $m \in N_i(\bar{\mathbf{g}})\setminus\{k\}$ and $m \notin N_k(\bar{\mathbf{g}})\setminus\{i\}$. But then, since $x_i < x_k$, $\bar{g}_{i,m} = 1$ by Lemma 1 and we have reached a contradiction. *Q.E.D.*

Lemma 4: *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$, $x_i \leq x_k \Leftrightarrow N_i(\bar{\mathbf{g}}) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}) \setminus \{i\}$. *Furthermore,* $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}})$, $x_i \leq x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) \leq \eta_k(\bar{\mathbf{g}})$ *and* $x_i < x_k \Leftrightarrow \pi_i < \pi_k$.

Proof of ***Lemma 4.*** This first three equivalence relationships follow directly from the lemmas above. Concerning the fourth equivalence relationship, assume without loss of generality that $x_i < x_k$ and $\pi_i \geq \pi_k$. From $x_i < x_k$ and $z_i = \sum_{j\in N} x_j - x_i$ and $z_k = \sum_{j\in N} x_j - x_k$ it follows that $z_k < z_i$. For $x_i < x_k$ to hold, we know from the best response functions that $y_i < y_k$ must hold. To see this, we can write the best response functions as $x_i = \frac{1}{\beta}\cdot\Big(\alpha + \lambda y_i - \gamma(\sum_{j\in N} x_j - x_i)\Big)$ and $x_k = \frac{1}{\beta}\cdot\Big(\alpha + \lambda y_k - \gamma(\sum_{j\in N} x_j - x_k)\Big)$. We can rewrite the expressions as $x_i =$ Note that in a PNE $\pi_i = v(y_i, z_i)$ and $v(y_k, z_k) = \pi_k$ hold. The value function is increasing in the first argument and decreasing in the second and therefore $\pi_i < \pi_k$ holds. We have reached a contradiction. Assume next that $\pi_i < \pi_k$ and $x_i \geq x_k$ holds. From $x_i \geq x_k$ and $z_i = \sum_{j\in N} x_j - x_i$ and $z_k = \sum_{j\in N} x_j - x_k$ it follows that $z_i \leq z_k$. Note next that, for $x_i \geq x_k$ to hold, we know from the best response functions that $y_i \geq y_k$ must also hold. To see this, we can again take the difference between the best responses x_i and x_k (as in Lemma 1) to obtain $x_i - x_k = \frac{\lambda}{\beta-\gamma}\cdot(y_i - y_k) \geq 0$. As shown above, $\frac{\lambda}{\beta-\gamma} > 0$ holds, and therefore $y_i \geq y_k$ also holds. Since in a PNE we have that $\pi_i = v(y_i, z_i)$ and $\pi_k = v(y_k, z_k)$, it follows directly from $y_i \geq y_k$, $z_i \leq z_k$ and the properties of v that $\pi_i \geq \pi_k$. *Q.E.D.*

In any PNE the network is a nested split graph.

In any PNE if $\bar{g}_{i,l} = 1$ and $\eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})$, then $x_k \geq x_l$ by Lemma 4 and $\bar{g}_{i,k} = 1$ by Lemma 1. That is, $\bar{\mathbf{g}}$ is a nested split graph. *Q.E.D.*

Lemma 5: *If* $\bar{\mathbf{g}}$ *is a nested split graph, then* $\bar{\mathbf{g}}^{-k}$ *is a nested split graph for all* $k \in N$.

Proof of ***Lemma 5.*** Note that $\bar{\mathbf{g}}$ is a nested split graph if and only if for any pair of agents i and j either $N_j(\bar{\mathbf{g}})\setminus\{i\} \subset N_i(\bar{\mathbf{g}})\setminus\{j\} \Leftrightarrow \eta_j(\bar{\mathbf{g}}) < \eta_i(\bar{\mathbf{g}})$ or $N_j(\bar{\mathbf{g}})\setminus\{i\} = N_i(\bar{\mathbf{g}})\setminus\{j\} \Leftrightarrow \eta_j(\bar{\mathbf{g}}) = \eta_i(\bar{\mathbf{g}})$ holds. Assume next that an agent k is eliminated from $\bar{\mathbf{g}}$, yielding the network $\bar{\mathbf{g}}^{-k}$. For a pair of agents i and j, there are there cases to distinguish. Without loss of generality, assume $\eta_i(\bar{\mathbf{g}}) \geq \eta_j(\bar{\mathbf{g}})$. Consider first the case when $\bar{g}_{i,k} = \bar{g}_{j,k} = 0$. Then after eliminating agent k, $N_j(\bar{\mathbf{g}})\setminus\{i\} = N_j(\bar{\mathbf{g}}^{-k})\setminus\{i\}$, $N_i(\bar{\mathbf{g}})\setminus\{j\} = N_i(\bar{\mathbf{g}}^{-k})\setminus\{j\}$, $\eta_j(\bar{\mathbf{g}}) = \eta_j(\bar{\mathbf{g}}^{-k})$ and $\eta_i(\bar{\mathbf{g}}) = \eta_i(\bar{\mathbf{g}}^{-k})$ holds. Therefore, if $\eta_i(\bar{\mathbf{g}}) > \eta_j(\bar{\mathbf{g}})$, then $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} \subset N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) < \eta_i(\bar{\mathbf{g}}^{-k})$ holds, while if $\eta_i(\bar{\mathbf{g}}) = \eta_j(\bar{\mathbf{g}})$, then $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} = N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) = \eta_i(\bar{\mathbf{g}}^{-k})$ holds. Next, assume that $\bar{g}_{i,k} = 1$ and $g_{j,k} = 1$. By an analogous argument as the one used in the previous case, we have again that, if $\eta_i(\bar{\mathbf{g}}) > \eta_j(\bar{\mathbf{g}})$, then $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} \subset N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) < \eta_i(\bar{\mathbf{g}}^{-k})$ holds, while if $\eta_i(\bar{\mathbf{g}}^{-k}) = \eta_j(\bar{\mathbf{g}}^{-k})$, then $N_j(\bar{\mathbf{g}}^{-k})\setminus\{i\} = N_i(\bar{\mathbf{g}}^{-k})\setminus\{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) = \eta_i(\bar{\mathbf{g}}^{-k})$ holds. Finally, assume that one agent is connected to k, while the other is not. Without loss of generality assume $\bar{g}_{i,k} = 1$ and $\bar{g}_{j,k} = 0$. Note that, since $\bar{\mathbf{g}}$ is a nested split graph, $\eta_i(\bar{\mathbf{g}}) > \eta_j(\bar{\mathbf{g}})$ must also hold. We distinguish two subcases. Assume first that $\eta_i(\bar{\mathbf{g}}) > \eta_j(\bar{\mathbf{g}}) + 1$. Then $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} \subset N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) < \eta_i(\bar{\mathbf{g}}^{-k})$ holds. Assume next that $\eta_i(\bar{\mathbf{g}}) = \eta_j(\bar{\mathbf{g}}) + 1$. Then $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} =$

$N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) = \eta_i(\bar{\mathbf{g}}^{-k})$. That is, when deleting an agent k from $\bar{\mathbf{g}}$, either $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} \subset N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) < \eta_i(\bar{\mathbf{g}}^{-k})$ or $N_j(\bar{\mathbf{g}}^{-k}) \setminus \{i\} = N_i(\bar{\mathbf{g}}^{-k}) \setminus \{j\}$ and $\eta_j(\bar{\mathbf{g}}^{-k}) = \eta_i(\bar{\mathbf{g}}^{-k})$ holds. Therefore, $\bar{\mathbf{g}}^{-k}$ is therefore a nested split graph. *Q.E.D.*

In order to compare pairs of networks when different agents k and l are eliminated, we relabel the eliminated agents. More specifically, when comparing $\bar{\mathbf{g}}^{-k}$ and $\bar{\mathbf{g}}^{-l}$, we relabel agents follows: $r_{\{k,l\}}(l) = d$ in $\bar{\mathbf{g}}^{-k}$ and $r_{\{k,l\}}(k) = d$ in $\bar{\mathbf{g}}^{-l}$. That is, when agent k is eliminating from $\bar{\mathbf{g}}$, then agent l is relabeled as agent d in $\bar{\mathbf{g}}^{-k}$. Likewise, when agent l is eliminated from $\bar{\mathbf{g}}$, then agent k is relabeled as agent d in $\bar{\mathbf{g}}^{-l}$. Relabeling agents in this way ensures that the set of agents are the same in $\bar{\mathbf{g}}^{-k}$ and $\bar{\mathbf{g}}^{-l}$. We adopt the following notation $N_{\{k,l\}} = \{N \setminus \{k,l\}\} \cup \{d\}$.

Lemma 6: *If $\bar{\mathbf{g}}$ is a nested split graph and $\eta_k(\bar{\mathbf{g}}) = \eta_l(\bar{\mathbf{g}})$, then $\bar{\mathbf{g}}^{-k} = \bar{\mathbf{g}}^{-l}$, while if $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$ then $\bar{\mathbf{g}}^{-l} \subset \bar{\mathbf{g}}^{-k}$ holds.*

*Proof of **Lemma 6**.* We first show that, if $\eta_k(\bar{\mathbf{g}}) = \eta_l(\bar{\mathbf{g}})$ then $\bar{\mathbf{g}}^{-k} = \bar{\mathbf{g}}^{-l}$ holds. We distinguish two sub-cases. Assume first that $k = l$. It is then easy to see that $\bar{g}_{i,j} = \bar{g}^{-k}_{i,j} = \bar{g}^{-l}_{i,j}$ $\forall i,j \in N_{\{k,l\}}$ and therefore $\bar{\mathbf{g}}^{-k} = \bar{\mathbf{g}}^{-l}$. Assume next that $k \neq l$. Relabel agents such that $r_{\{k,l\}}(l) = d$ in $\bar{\mathbf{g}}^{-k}$ and $r_{\{k,l\}}(k) = d$ in $\bar{\mathbf{g}}^{-l}$. Note that, since $\bar{\mathbf{g}}$ is a nested split graph and since $\eta_k(\bar{\mathbf{g}}) = \eta_l(\bar{\mathbf{g}})$, we also know that $N_k(\bar{\mathbf{g}}) \setminus \{l\} = N_l(\bar{\mathbf{g}}) \setminus \{k\}$ holds. Therefore, $N_d(\bar{\mathbf{g}}^{-k}) = N_d(\bar{\mathbf{g}}^{-l})$ holds. Note further that $\bar{g}_{i,j} = \bar{g}^{-k}_{i,j} = \bar{g}^{-l}_{i,j}$ $\forall i,j \notin \{k,l\}$ and therefore $\bar{\mathbf{g}}^{-k} = \bar{\mathbf{g}}^{-l}$. Next we show that if $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$ then $\bar{\mathbf{g}}^{-k} \subset \bar{\mathbf{g}}^{-l}$ holds. Since $\bar{\mathbf{g}}$ is a nested split graph and $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$ hold, we know that $N_l(\bar{\mathbf{g}}) \subset N_k(\bar{\mathbf{g}})$ holds. From $N_l(\bar{\mathbf{g}}) \subset N_k(\bar{\mathbf{g}})$ it follows that if $\bar{g}^{-k}_{d,j} = 1$ then $\bar{g}^{-l}_{d,j} = 1$, while there exists an agent j such that $\bar{g}^{-k}_{d,j} = 0$ and $\bar{g}^{-l}_{d,j} = 1$. Furthermore, $\bar{g}_{i,j} = \bar{g}^{-k}_{i,j} = \bar{g}^{-l}_{i,j}$ $\forall i,j \notin \{k,j\}$ and therefore $\bar{\mathbf{g}}^{-k} \subset \bar{\mathbf{g}}^{-l}$ hold. *Q.E.D.*

In the following we write gross payoffs as a class of parametrized functions in γ, while assuming that the other parameter values remain fixed. That is, we write $\pi(x,y,z,\gamma)$, so that an agent i's payoff function is given by $\pi_i(x_i,y_i,z_i,\gamma)$. Define $\boldsymbol{\theta}_i = (y_i,z_i,\gamma)$ with $\boldsymbol{\theta}_i \in \Theta$ and $\Theta = \mathbb{R}^3_{\geq 0}$. This allows us to write $\pi_i(x_i,\boldsymbol{\theta}_i)$ and, similarly, $\bar{x}_i(\boldsymbol{\theta}_i)$ and $v_i(\boldsymbol{\theta}_i)$. Below we use the Theorem of the Maximum to show that the corresponding best response functions, $\bar{x}_i(\boldsymbol{\theta}_i)$ and $v_i(\boldsymbol{\theta}_i)$ are continuous in $\boldsymbol{\theta}_i$. We then write the Nash equilibrium effort levels for a given network $\bar{\mathbf{g}}$ as a function of γ, i.e. $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$. Using the Implicit Function Theorem and Sard's Lemma we then show that the vector of Nash equilibrium effort levels, $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$, generically changes continuously in γ. For convenience we provide formal definitions for both theorems below.

For $X \subseteq \mathbb{R}^l$, define $\mathrm{K}(X) = \{K \subseteq X : K \neq \emptyset \text{ is compact}\}$. For a set X, denote $X^n = X_1 \times ... \times X_n$.

Definition 4: (Theorem of the Maximum). For $\Theta \subseteq \mathbb{R}^k$, $X \subseteq \mathbb{R}$, $\Psi : \Theta \to \mathrm{K}(X)$ a correspondence and $\pi : X \times \Theta \to \mathbb{R}$, define the value function $\boldsymbol{\theta} \to v(\boldsymbol{\theta}) = max_{x \in \Psi(\boldsymbol{\theta})} \pi(x,\boldsymbol{\theta})$, and the argmax correspondence $\boldsymbol{\theta} \to \bar{x}(\boldsymbol{\theta}) = \{x \in \Psi(\boldsymbol{\theta}) : \forall x' \in \Psi(\boldsymbol{\theta}),\ \pi(x,\boldsymbol{\theta}) \geq \pi(x',\boldsymbol{\theta})\}$. If π is (jointly) continuous and $\Psi(\boldsymbol{\theta})$ is continuous then (i) $v(\cdot)$ is continuous and (ii) $\bar{x}(\cdot)$ is hemicontinuous. Furthermore, if $\bar{x}(\cdot)$ is always singleton valued, then $\boldsymbol{\theta} \to \bar{\mathbf{x}}(\boldsymbol{\theta})$ is continuous.[23]

[23]See Corbae et al. (2009), p. 151.

Definition 5: (Implicit Function Theorem). Let $f_{\bar{\mathbf{g}}} : \mathbb{R}^{n+nk} \supseteq X^n \times \Theta^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function on an open set $X^n \times \Theta^n$. Consider the system of equations $f_{\bar{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{0}$ and assume it has a solution at $\mathbf{x}^0 \in X^n$ for given parameter values $\boldsymbol{\theta}^0 \in \Theta^n$. If the determinant of the Jacobian of endogenous variables is not zero at $(\mathbf{x}^0, \boldsymbol{\theta}^0)$, that is, if $\mid J(\mathbf{x}^0, \boldsymbol{\theta}^0) \mid = \mid D_x f_{\bar{\mathbf{g}}}(\mathbf{x}^0, \boldsymbol{\theta}^0) \mid \neq 0$ then (i) there exist open sets U in $\mathbb{R}^{n+nk}$ and $U_{\boldsymbol{\theta}}$ in $\mathbb{R}^{nk}$ with $(\mathbf{x}^0, \boldsymbol{\theta}^0) \subseteq U$ and $\boldsymbol{\theta}^0 \subseteq U_{\boldsymbol{\theta}}$ such that for each $\boldsymbol{\theta}$ in $U_{\boldsymbol{\theta}}$ there exists a unique $\mathbf{x}_{\boldsymbol{\theta}}$ such that $(\mathbf{x}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \in U$ and $f_{\bar{\mathbf{g}}}(\mathbf{x}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \mathbf{0}$. That is, the correspondence from $U_{\boldsymbol{\theta}}$ to X^n defined by $\mathbf{x}(\boldsymbol{\theta}) = \mathbf{x}_{\boldsymbol{\theta}}$ is a well-defined function when restricted to U. (ii) The solution function $\mathbf{x}(\cdot) : U_{\boldsymbol{\theta}} \rightarrow \mathbb{R}^n$ is continuously differentiable. (iii) If $f_{\bar{\mathbf{g}}}$ is C^k, so is $\mathbf{x}(\cdot)$.[24]

Lemma 7: *An agent's best response function $\bar{x}_i(\boldsymbol{\theta}_i)$ and value function $v_i(\boldsymbol{\theta}_i)$ are continuous in $\boldsymbol{\theta}_i$. Furthermore, the vector of Nash equilibrium effort levels for a given network $\bar{\mathbf{g}}$, $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$, generically changes continuously in γ (and therefore, generically, $lim_{\gamma \rightarrow 0} \mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma}) = \mathbf{x}_{\bar{\mathbf{g}}}(0)$).*

*Proof of **Lemma 7.*** Define $\tilde{X} = [0, M]$ and assume M sufficiently large so that for all $\bar{\mathbf{g}}$ and all $\gamma \in [0, m]$, all vectors of Nash equilibrium effort levels, $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$, are in the interior of $[0, M]^n$. Note that for any $\bar{\mathbf{g}}$ and m, we can find a corresponding M. Furthermore, assume $\Psi(\tilde{\boldsymbol{\theta}}_i) = [0, M]$ $\forall \tilde{\boldsymbol{\theta}}_i \in \tilde{\Theta}$. Note that $\Psi(\tilde{\boldsymbol{\theta}}_i) = [0, M]$ $\forall \tilde{\boldsymbol{\theta}}_i \in \tilde{\Theta}$ imposes a restriction on agents' strategies, since $x_i \in [0, M]$ must hold $\forall i \in N$. However, any PNE is also a PNE in the presence of the restriction, since PNE effort levels $\mathbf{x}$ are feasible when $\Psi(\boldsymbol{\theta}_i) = [0, M]$ $\forall \boldsymbol{\theta}_i \in \Theta$, while the set of available deviations is restricted. Note further that $\Psi(\boldsymbol{\theta}_i)$ is continuous. Moreover, π is (jointly) continuous and $\bar{x}_i(\boldsymbol{\theta}_i)$ is singleton valued by assumption. From the Theorem of the Maximum (Definition 4) it then follows directly that $v_i(\boldsymbol{\theta}_i)$ and $\bar{x}_i(\boldsymbol{\theta}_i)$ are continuous in $\tilde{\boldsymbol{\theta}}_i$. Next, assume $(\mathbf{x}, \bar{\mathbf{g}})$ is a pairwise Nash equilibrium for some $\bar{\mathbf{g}}$ and for some γ with $\gamma \in \Gamma$ and $\Gamma = \mathbb{R}_{\geq 0}$. Define $\tilde{X}' = (0, M)$ and the function $f_{\bar{\mathbf{g}}} :\ \tilde{X}'^n \times \Gamma \rightarrow \mathbb{R}^n$ as

$$f_{\bar{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\gamma}) = \begin{pmatrix} \partial \pi_1(x_1, y_1, z_1, \gamma)/\partial x_1 \\ \vdots \\ \partial \pi_n(x_n, y_n, z_n, \gamma)/\partial x_n \end{pmatrix}.$$

Given our assumptions on π, $f_{\bar{\mathbf{g}}}(\mathbf{x}, \gamma)$ is continuously differentiable and, since $\mathbf{x}$ is assumed to be the vector of NE effort levels, $f_{\bar{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\gamma}) = \mathbf{0}$ holds for the particular $\boldsymbol{\gamma}$ considered. We can now apply the Implicit Function Theorem. That is, if the Jacobian of $f_{\bar{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\gamma})$ is invertible, then the vector of NE equilibrium effort levels on a fixed network $\bar{\mathbf{g}}$, which we denote with $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$, is continuous in $\boldsymbol{\gamma}$. Note that assuming $\tilde{X}' = (0, M)$ is analytically innocuous, since the vector of NE effort levels, $\mathbf{x}$, is a solution in the interior of $[0, M]^n$ and therefore in the interior of $(0, M)^n$. From Sard's Lemma we know that the set of critical points of a sufficiently smooth function has Lebesque measure zero.[25] Our payoff function π is smooth and therefore the property that $(\mathbf{x}, \boldsymbol{\gamma})$ is a regular point of $f_{\bar{\mathbf{g}}}$ (and that the Jacobian of $f_{\bar{\mathbf{g}}}$ is invertible) is generic. That is, $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$ generically changes continuously in γ (and therefore, generically, $lim_{\gamma \rightarrow 0} \mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma}) = \mathbf{x}_{\bar{\mathbf{g}}}(0)$). *Q.E.D.*

[24] See de la Fuente (2009), p. 210.

[25] Sard's Lemma reads as follows (see de la Fuente (2009), p. 214.). Let $g : \mathbb{R}^n \supseteq X \rightarrow \mathbb{R}^m$ (X open) be a C^r function with $r > max\{0, n - m\}$ and let C_f be the set of critical points of g. Then $g(C_f)$ has Lebesque measure zero.

Lemma 8: *Assume $\gamma = 0$. Then $x_k(\bar{\mathbf{g}}^{-i}) < x_k(\bar{\mathbf{g}})$ for all agents k such that $i, k \in \bar{\mathbf{g}}^s$, while $x_k(\bar{\mathbf{g}}^{-i}) = x_k(\bar{\mathbf{g}})$ for all agents k such that $i \in \bar{\mathbf{g}}^s$ and $k \notin \bar{\mathbf{g}}^s$.*

Proof of ***Lemma 8.*** Note first that, since $\gamma = 0$, $\bar{x}_i(0, z_i) = \bar{x}_i(0, 0)$ $\forall z_i$ and effort levels are bounded below by $\bar{x}_i(0,0) = \bar{x}(0,0)$. Moreover, $\partial\bar{x}(y,z)/\partial y > 0$, while $\partial\bar{x}(y,z)/\partial z = 0$ $\forall y, z$. Note first that, since $\gamma = 0$, we can treat different components in $\bar{\mathbf{g}}^{-i}$ in isolation. We distinguish two cases. Assume first that $i, k \in \bar{\mathbf{g}}^s$. Then for any agent l with $\bar{g}_{i,l} = 1$, we have that $\sum_{j \in N_l(\bar{\mathbf{g}}^{-i})} x_j(\bar{\mathbf{g}}) < \sum_{j \in N_l(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}})$. Iterating on best responses, any agent l with $\bar{g}_{i,l} = 1$ strictly decreases her effort level and in turn any agent sustaining links with l strictly decrease their effort levels, and so forth. The effort level of each agent is a (weakly) decreasing sequence of real numbers (where each agent strictly decrease her effort level in some iteration), which is bounded below by $\bar{x}(0,0)$. Effort levels therefore converge to new NE effort levels with $x_k(\bar{\mathbf{g}}^{-i}) < x_k(\bar{\mathbf{g}})$ for all k such that $i, k \in \bar{\mathbf{g}}^s$. Note next that for any agent k such that $i \in \bar{\mathbf{g}}^s$ *and* $k \notin \bar{\mathbf{g}}^s$, $\sum_{j \in N_l(\bar{\mathbf{g}}^{-i})} x_j(\bar{\mathbf{g}}) = \sum_{j \in N_l(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}})$ holds for all agents l such that $k, l \in \bar{\mathbf{g}}^{-i\,s}$. That is, at every iteration, no agent $k \in \bar{\mathbf{g}}^{-i\,s}$ adjusts her effort level $x_k(\bar{\mathbf{g}})$ and therefore effort levels remain constant $Q.E.D.$

Lemma 9: *Assume $\gamma = 0$ and $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$. Then $x_k(\bar{\mathbf{g}}) < x_k(\hat{\bar{\mathbf{g}}})$ for all agents k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = 0$ for some $i, j \in \hat{\bar{\mathbf{g}}}^s$, while $x_k(\bar{\mathbf{g}}) = x_k(\hat{\bar{\mathbf{g}}})$ for all agents k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = \bar{g}_{i,j}$ for all $i, j \in \hat{\bar{\mathbf{g}}}^s$.*

Proof of ***Lemma 9.*** Note first that, since $\gamma = 0$, $\bar{x}_i(0, z_i) = \bar{x}_i(0, 0)$ $\forall z_i$ and effort levels are bounded below by $\bar{x}_i(0,0) = \bar{x}(0,0)$. Moreover, $\partial\bar{x}(y,z)/\partial y > 0$, while $\partial\bar{x}(y,z)/\partial z = 0$ $\forall y, z$. Note further that, since $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$, $\bar{\mathbf{g}}$ can be obtained by deleting all links such $\bar{g}_{i,j} = 0$ and $\hat{\bar{g}}_{i,j} = 1$ from $\hat{\bar{\mathbf{g}}}$. Consider players' best responses to the NE effort levels $\mathbf{x}(\hat{\bar{\mathbf{g}}})$ when when the network is $\bar{\mathbf{g}}$. Since $\gamma = 0$ and $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$, we can analyze components in $\hat{\bar{\mathbf{g}}}$ in isolation. Note first that for any agent k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = \bar{g}_{i,j}$ for all $i, j \in \hat{\bar{\mathbf{g}}}^s$, then $\sum_{j \in N_k(\bar{\mathbf{g}})} x_j(\hat{\bar{\mathbf{g}}}) = \sum_{j \in N_k(\hat{\bar{\mathbf{g}}})} x_j(\hat{\bar{\mathbf{g}}})$ holds. That is, at every iteration, no agent $k \in \hat{\bar{\mathbf{g}}}^s$ adjusts her effort level $x_k(\hat{\bar{\mathbf{g}}})$ and therefore effort levels remain constant. Next we consider agents k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = 0$ for some $i, j \in \hat{\bar{\mathbf{g}}}^s$. We distinguish two cases. Assume first that $k \in \hat{\bar{\mathbf{g}}}^s$, $k \in \bar{\mathbf{g}}^s$ and $\bar{\mathbf{g}}^s = \hat{\bar{\mathbf{g}}}^s$. Note that for any pair of agents i and j such that $\bar{g}_{i,j} = 0$ and $\hat{\bar{g}}_{i,j} = 1$, the initial best response is to strictly decrease effort levels, since $\sum_{j \in N_k(\bar{\mathbf{g}})} x_j(\hat{\bar{\mathbf{g}}}) < \sum_{j \in N_k(\hat{\bar{\mathbf{g}}})} x_j(\hat{\bar{\mathbf{g}}})$. Iterating on best responses, any agent l with $\hat{\bar{g}}_{i,j} = 1$ strictly decreases the effort level an in turn any agent sustaining links with l strictly decrease their effort levels, and so forth. Effort levels are a weakly decreasing sequence of real numbers for each agent, which is bounded below by $\bar{x}(0,0)$. Effort levels therefore converge to new NE effort levels in $\bar{\mathbf{g}}$ with $x_k(\bar{\mathbf{g}}) < x_k(\hat{\bar{\mathbf{g}}})$ for all k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $k \in \bar{\mathbf{g}}^s$ with $\bar{\mathbf{g}}^s = \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = 0$ for some $i, j \in \hat{\bar{\mathbf{g}}}^s$. Next consider the case that $k \in \hat{\bar{\mathbf{g}}}^s$, $k \in \bar{\mathbf{g}}^s$ and $\bar{\mathbf{g}}^s \subset \hat{\bar{\mathbf{g}}}^s$. Note that again any pair of agents i and j such that $\bar{g}_{i,j} = 0$ and $\hat{\bar{g}}_{i,j} = 1$ in $\bar{\mathbf{g}}^s$, the initial best response is to strictly decrease effort levels, since $\sum_{j \in N_k(\bar{\mathbf{g}})} x_j(\hat{\bar{\mathbf{g}}}) < \sum_{j \in N_k(\hat{\bar{\mathbf{g}}})} x_j(\hat{\bar{\mathbf{g}}})$. Note further that, since $\bar{\mathbf{g}}^s \subset \hat{\bar{\mathbf{g}}}^s$, there exists at least one agent $i \in \bar{\mathbf{g}}^s$ such that $\hat{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = 0$ (where we allow for $j \notin \bar{\mathbf{g}}^s$). By the same argument as above, $x_k(\bar{\mathbf{g}}) < x_k(\hat{\bar{\mathbf{g}}})$ holds for all k such that $k \in \hat{\bar{\mathbf{g}}}^s$ and $k \in \bar{\mathbf{g}}^s$ with $\bar{\mathbf{g}}^s \subset \hat{\bar{\mathbf{g}}}^s$ and $\hat{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = 0$ for some $i, j \in \hat{\bar{\mathbf{g}}}^s$. $Q.E.D.$

Lemma 10: *Assume γ is sufficiently small, $(\mathbf{x}, \bar{\mathbf{g}})$ is a pairwise Nash equilibrium and $\eta_i(\bar{\mathbf{g}}) \geq 1$. Then, generically, there does not exist a pair of agents j and k in $\bar{\mathbf{g}}^{-i}$, such that $\bar{g}_{j,k} = 0$ in $\bar{\mathbf{g}}$ and j and k find it profitable to create a link in $\bar{\mathbf{g}}^{-i}$.*

Proof of ***Lemma 10.*** Note that, since $\bar{\mathbf{g}}$ is a pairwise Nash equilibrium network, $\bar{\mathbf{g}}$ is a nested split graph. Therefore, there is at most one connected component (and possibly some singletons). Assume first that $\gamma = 0$ and that $\bar{g}_{j,k} = 0$. We show that the marginal payoffs of j and k creating the link $\bar{g}^{-i}_{j,k} = 1$ in $\bar{\mathbf{g}}^{-i}$ are weakly lower than when j and k create the link $\bar{g}_{j,k} = 1$ in $\bar{\mathbf{g}}$. From Lemma 9 we know that $x_j(\bar{\mathbf{g}}^{-i}) \leq x_j(\bar{\mathbf{g}})\ \forall j \in N\backslash\{i\}$ and therefore $y_j(\bar{\mathbf{g}}^{-i}) \leq y_j(\bar{\mathbf{g}})$ also holds $\forall j \in N \setminus \{i\}$. Note further that $x'_j(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) \leq x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k})$ and $x'_k(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) \leq x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k})$ holds. To see this, note that $x'_j(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) = \frac{1}{\beta} \cdot (\alpha + \lambda y_j(\bar{\mathbf{g}}^{-i}) + \lambda x'_k(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}))$ and $x'_k(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) = \frac{1}{\beta} \cdot \left(\alpha + \lambda y_k(\bar{\mathbf{g}}^{-i}) + \lambda x'_j(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k})\right)$, while $x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k}) = \frac{1}{\beta} \cdot (\alpha + \lambda y_j(\bar{\mathbf{g}}) + \lambda x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k}))$ and $x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k}) = \frac{1}{\beta} \cdot \left(\alpha + \lambda y_k(\bar{\mathbf{g}}) + \lambda x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k})\right)$. Solving for $x'_j(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k})$, $x'_k(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k})$, $x'_j(\bar{\mathbf{g}}+\bar{g}_{j,k})$ and $x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k})$ it is immediate that $x'_j(\bar{\mathbf{g}}^{-i}+\bar{g}_{j,k}) \leq x'_j(\bar{\mathbf{g}}+\bar{g}_{j,k})$ and $x'_k(\bar{\mathbf{g}}^{-i}+\bar{g}_{j,k}) \leq x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k})$ follows from $y_j(\bar{\mathbf{g}}^{-i}) \leq y_j(\bar{\mathbf{g}})\ \forall j \in N \setminus \{i\}$. Note next that when $\gamma = 0$, then we can write $v(y_i, z_i) = \frac{1}{2(\beta+\gamma)} \cdot \left(\alpha + \lambda \sum_{j\in N_i(\bar{\mathbf{g}})} x_j\right)^2$. Since $y_j(\bar{\mathbf{g}}^{-i}) \leq y_j(\bar{\mathbf{g}})\ \forall j \in N \setminus \{i\}$, $x'_j(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) \leq x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k})$ and $x'_k(\bar{\mathbf{g}}^{-i} + \bar{g}_{j,k}) \leq x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k})$ holds, marginal payoffs from creating the link a link are weakly lower in $\bar{\mathbf{g}}$ than in $\bar{\mathbf{g}}^{-i}$. Since we assumed that $\bar{\mathbf{g}}$ is a PNE network, creating the link $\bar{g}_{j,k} = 1$ is not profitable in $\bar{\mathbf{g}}$ and therefore creating the link $\bar{g}^{-i}_{j,k} = 1$ is also not profitable in $\bar{\mathbf{g}}^{-i}$. Assume next that γ is sufficiently small. Note that from Theorem 1 we know that $\bar{\mathbf{g}}$ is a nested split graph, while from Lemma 5 we know that $\bar{\mathbf{g}}^{-i}$ is a nested split graph. Note further that in a nested split graph, there is at most one component of more than one agent, while any agents not in the component are singleton agents. Recall that Lemma 7 we know that $\bar{x}_i(\boldsymbol{\theta}_i)$ and $v_i(\boldsymbol{\theta}_i)$ are continuous in $\boldsymbol{\theta}_i$, with $\boldsymbol{\theta}_i = (y_i, z_i, \gamma)$, while $\mathbf{x}_{\bar{\mathbf{g}}}(\boldsymbol{\gamma})$ changes continuously in γ. We distinguish three cases. Assume first that $j, k, i \in \bar{\mathbf{g}}^s$, i.e. j and k are in the same component as i in $\bar{\mathbf{g}}$. From Lemma 7 and Lemma 8 we know that if γ is sufficiently small, then $x_k(\bar{\mathbf{g}}^{-i}) < x_k(\bar{\mathbf{g}})$ for all agents k such that $i, k \in \bar{\mathbf{g}}^s$. By an analogous argument to the one used above, it then follows directly that for all agents j and k with $j, k, i \in \bar{\mathbf{g}}^s$, creating the link $\bar{g}^{-i}_{j,k} = 1$ is not profitable in $\bar{\mathbf{g}}^{-i}$. Assume next that j and k are such that $j, k \notin \bar{\mathbf{g}}^s$, while $i \in \bar{\mathbf{g}}^s$. Since $\eta_i(\bar{\mathbf{g}}) \geq 1$, $\bar{\mathbf{g}} \neq \bar{\mathbf{g}}^e$ holds and, since there is at most one connected component, if $j, k \notin \bar{\mathbf{g}}^s$, then j and k are singletons in $\bar{\mathbf{g}}$. For $\bar{\mathbf{g}}$ to be a PNE network, it must therefore be the case that the links $\bar{g}_{j,i} = 1$ and $\bar{g}_{k,i} = 1$ are not profitable in $\bar{\mathbf{g}}$. Marginal payoffs are the same for j and k and we therefore focus on j. There are two relevant cases to distinguish. First, assume agent i does not find it profitable to create the link with j in $\bar{\mathbf{g}}$. Note that $\eta_i(\bar{\mathbf{g}}) \geq 1$. Since $\eta_j(\bar{\mathbf{g}}) = \eta_k(\bar{\mathbf{g}}) = 0$, we know from Theorem 1 that $x_i(\bar{\mathbf{g}}) > x_j(\bar{\mathbf{g}})$. From the best response function we know that $\alpha+\lambda y_i(\bar{\mathbf{g}})-\gamma z_i(\bar{\mathbf{g}}) > \alpha+\lambda y_j(\bar{\mathbf{g}})-\gamma z_j(\bar{\mathbf{g}}) = \alpha+\lambda y_k(\bar{\mathbf{g}})-\gamma z_k(\bar{\mathbf{g}})$ must hold. Note further that from the above argument we know that $x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) > x'_k(\bar{\mathbf{g}} + \bar{g}^+_{j,k}) = x'_j(\bar{\mathbf{g}} + \bar{g}^+_{j,k})$, where the equality follows from symmetry. That is, the deviation effort level of agent i in a deviation where i and j create a link in $\bar{\mathbf{g}}$ is strictly larger than the deviation effort of agent k in a deviation where i and k create a link in $\bar{\mathbf{g}}$. The marginal payoffs of agent i, when linking to agent j in $\bar{\mathbf{g}}$, is given by $\Delta v_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j})$, where

$$\Delta v_i(\bar{\mathbf{g}} + \bar{g}^+_{i,j}) = v(y_i(\bar{\mathbf{g}}) + x'_j(\bar{\mathbf{g}} + \bar{g}^+_{i,j}), z_i(\bar{\mathbf{g}}) + x'_j(\bar{\mathbf{g}} + \bar{g}^+_{j,k}) - x_j(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})).$$

Substituting $x'_k(\bar{\mathbf{g}} + \bar{g}^+_{j,k}) = x'_j(\bar{\mathbf{g}} + \bar{g}^+_{j,k})$ we can write this as

$$\begin{aligned}\Delta v_i(\bar{\mathbf{g}} + \bar{g}^+_{i,k}) &= v(y_i(\bar{\mathbf{g}}) + x'_k(\bar{\mathbf{g}} + \bar{g}^+_{i,k}), z_i(\bar{\mathbf{g}}) + x'_k(\bar{\mathbf{g}} + \bar{g}^+_{i,k}) - x_k(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) \\ &= \tfrac{1}{2\beta}\cdot\left(\alpha + \lambda y_i(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}}) + (\lambda - \gamma)x'_k(\bar{\mathbf{g}} + \bar{g}^+_{j,k})) + \gamma x_k(\bar{\mathbf{g}})\right)^2 - \tfrac{1}{2\beta}\cdot(\alpha + \lambda y_i(\bar{\mathbf{g}}) - \gamma z_i(\bar{\mathbf{g}}))^2.\end{aligned}$$

The marginal payoff for agent j, when linking to agent k in $\bar{\mathbf{g}}$, is given by

$$\Delta v_j(\bar{\mathbf{g}}+\bar{g}^+_{j,k}) = v(y_j(\bar{\mathbf{g}})+x'_k(\bar{\mathbf{g}}+\bar{g}^+_{j,k}), z_i(\bar{\mathbf{g}})+x'_k(\bar{\mathbf{g}}+\bar{g}^+_{j,k})-x_k(\bar{\mathbf{g}}))-v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}}))$$
$$=\tfrac{1}{2\beta}\cdot\big(\alpha+\lambda y_j(\bar{\mathbf{g}})-\gamma z_j(\bar{\mathbf{g}})+(\lambda-\gamma)x'_k(\bar{\mathbf{g}}+\bar{g}^+_{j,k}))+\gamma x_k(\bar{\mathbf{g}})\big)^2-\tfrac{1}{2\beta}\cdot(\alpha+\lambda y_j(\bar{\mathbf{g}})-\gamma z_j(\bar{\mathbf{g}}))^2.$$

Since $\alpha+\lambda\sum_{l\in N_i(\bar{\mathbf{g}})}x_l-\gamma\sum_{l\in N\setminus\{i\}}x_l > \alpha+\lambda\sum_{l\in N_j(\bar{\mathbf{g}})}x_l-\gamma\sum_{l\in N\setminus\{j\}}x_l$, $x'_k(\bar{\mathbf{g}}+\bar{g}^+_{j,k}) = x'_j(\bar{\mathbf{g}}+\bar{g}^+_{j,k})$ and since $\bar{\mathbf{g}}$ is a PNE network, we know that $\kappa\geq\Delta v_i(\bar{\mathbf{g}}+\bar{g}^+_{i,k})$ and $\Delta v_i(\bar{\mathbf{g}}+\bar{g}^+_{i,k}) > \Delta v_j(\bar{\mathbf{g}}+\bar{g}^+_{j,k})$ holds. Note next that from Lemma 7 we know that generically, for γ sufficiently low, $\Delta v_j(\bar{\mathbf{g}}^{-i}+\bar{g}^+_{j,k})$ is arbitrarily close to $\Delta v_j(\bar{\mathbf{g}}+\bar{g}^+_{j,k})$ and therefore $\kappa > \Delta v_j(\bar{\mathbf{g}}^{-i}+\bar{g}^+_{j,k})$ holds for for γ sufficiently low. That is if agent i does not find it profitable to create a link to j in $\bar{\mathbf{g}}$, then j also does not find it profitable to create a link with k in $\bar{\mathbf{g}}^{-i}$ for γ sufficiently low. Assume next that agent j does not find it profitable to create the link with i in $\bar{\mathbf{g}}$. Since $x'_i(\bar{\mathbf{g}}+\bar{g}^+_{i,j}) > x'_k(\bar{\mathbf{g}}+\bar{g}^+_{j,k})$ holds, we can then use an analogous argument as the one above to show that, for γ sufficiently low, agent j then also does not find it profitable to create a link with k in $\bar{\mathbf{g}}^{-i}$. *Q.E.D.*

In the following we formally introduce minimal deletion best responses. Denote the set of agents to which a link is deleted with $D_i(\mathbf{g}'_i,\bar{\mathbf{g}}) = \{j\in N:\ j\in N_i(\bar{\mathbf{g}})\ and\ j\notin N_i(\bar{\mathbf{g}}')\}$ in a deviation $\mathbf{g}'_i$ in network $\bar{\mathbf{g}}$. Define with $\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}}) = \pi_i(\bar{x}_i(\bar{\mathbf{g}}'),\mathbf{x}_{-i}(\bar{\mathbf{g}}),\bar{\mathbf{g}}')-\eta_i(\bar{\mathbf{g}}')\kappa$ agent i's payoffs after deviation $\mathbf{g}'_i$, where agent i exerts the best response effort level $\bar{x}_i(\bar{\mathbf{g}}')$ to the resulting network $\bar{\mathbf{g}}'$. Write $\mathbf{g}'_i\subseteq\mathbf{g}_i$ if for all $j\in N\setminus\{i\}$ with $g'_{i,j}=1$ in $\mathbf{g}'_i$, $g_{i,j}=1$ also holds in $\mathbf{g}_i$. For a network $\bar{\mathbf{g}}$ and agent i, define a minimal deletion best response, $\mathbf{g}'^m_i$, as follows. If $\mathbf{g}_i\in argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, then $\mathbf{g}'^m_i=\mathbf{g}_i$. If $\mathbf{g}_i\notin argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, then $\mathbf{g}'^m_i$: $\mathbf{g}'^m_i\in argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ and $\mathbf{g}'^m_i\subseteq\mathbf{g}'_i$ $\forall\mathbf{g}'_i\in argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$. That is, if $\mathbf{g}_i$ is (part of) a best response, then $\mathbf{g}'^m_i=\mathbf{g}_i$, while if $\mathbf{g}_i$ is not (part of) a best response, then $\mathbf{g}'^m_i$ selects the deletion best response that can be considered minimal. Note that then $\bar{x}_i(\bar{\mathbf{g}}')$ is also minimal.

Lemma 11: *Assume agents play their Nash equilibrium effort levels,* $\mathbf{x}(\bar{\mathbf{g}})$*, in network* $\bar{\mathbf{g}}$*. Then a minimal optimal deletion strategy,* $\mathbf{g}'^m_i$*, always exists and is unique for every agent* $i\in N$*. Furthermore,* $\mathbf{g}'^m_i$ *is such that* $x_k(\bar{\mathbf{g}}') > x_l(\bar{\mathbf{g}}')$ $\forall k,l$ *such that* $k\in N_i(\bar{\mathbf{g}}')$*, while* $l\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$*.*

Proof of ***Lemma 11.*** Recall that $z_i(\bar{\mathbf{g}})=\sum_{j\in N\setminus\{i\}}x_j$, so that in any deviation, $z_i(\bar{\mathbf{g}}) = z_i(\bar{\mathbf{g}}')$ holds and for deletion deviation strategies we can therefore treat $v(y_i,z_i)$ as a strictly convex function in y_i. From $\mathbf{g}'^m_i\subseteq\mathbf{g}_i$ we know that $N_i(\bar{\mathbf{g}}')\subseteq N_i(\bar{\mathbf{g}})$ holds. We next show in two steps that any deletion best response $\mathbf{g}'_i\in argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ is such that for all j and k with $j\in N_i(\bar{\mathbf{g}}')$ and $k\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$ holds. Assume first that $\mathbf{g}'_i$ is a deletion best response, such that there exists a pair of agents j and k, such that $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$ with $j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ and $k\in N_i(\bar{\mathbf{g}}')$. But then deletion strategy $\mathbf{g}''_i=\mathbf{g}'_i+g_{i,j}-g_{i,k}$ yields strictly higher deviation payoffs, since $y_i(\bar{\mathbf{g}}'') > y_i(\bar{\mathbf{g}}')$, while $z_i(\bar{\mathbf{g}}'')=z_i(\bar{\mathbf{g}}')$ and v is strictly convex in its first argument. Next we show that if $j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, then $k\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ $\forall k\in N_i(\bar{\mathbf{g}})$: $x_k(\bar{\mathbf{g}})=x_j(\bar{\mathbf{g}})$. Assume that $\mathbf{g}'_i$ is an deletion best response and, to the contrary to the above statement, that $j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, but $k\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ and $x_k(\bar{\mathbf{g}})=x_j(\bar{\mathbf{g}})$. For $j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, it must be the case that $\kappa\geq v(y_i(\bar{\mathbf{g}}')+x_j(\bar{\mathbf{g}}),z_i(\bar{\mathbf{g}}))-v(y_i(\bar{\mathbf{g}}'),z_i(\bar{\mathbf{g}})$ holds. However, since v is strictly convex in the first argument, $v(y_i(\bar{\mathbf{g}}')+x_j(\bar{\mathbf{g}}),z_i(\bar{\mathbf{g}}))-v(y_i(\bar{\mathbf{g}}'),z_i(\bar{\mathbf{g}}) > v(y_i(\bar{\mathbf{g}}'),z_i(\bar{\mathbf{g}}))-v(y_i(\bar{\mathbf{g}}')-x_k(\bar{\mathbf{g}}),z_i(\bar{\mathbf{g}})$ also holds and therefore $\kappa > v(y_i(\bar{\mathbf{g}}'),z_i(\bar{\mathbf{g}}))-v(y_i(\bar{\mathbf{g}}')-x_k(\bar{\mathbf{g}}),z_i(\bar{\mathbf{g}})$ holds. That is, deviation payoffs are strictly larger in the deviation $\mathbf{g}''_i=\mathbf{g}'_i-g^-_{i,k}$ than in $\mathbf{g}'_i$ and $\mathbf{g}'_i$ is therefore not a deletion best response. That is, for any pair of agents j and k with $j,k\in N_i(\bar{\mathbf{g}})$ and $x_k(\bar{\mathbf{g}})=x_j(\bar{\mathbf{g}})$, then either $j,k\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ or $j,k\notin D_i(\mathbf{g}'_i,\bar{\mathbf{g}})$ in any deletion best response $\mathbf{g}'_i$. Therefore, in any deletion best response $\mathbf{g}'_i\in argmax_{\mathbf{g}'_i:\,\mathbf{g}'_i\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}'_i,\bar{\mathbf{g}})$, if

$k \in D_i(\mathbf{g}'_i, \bar{\mathbf{g}})$ and $j \in N_i(\bar{\mathbf{g}}')$, then $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$. The above allows us to characterize minimal deletion best responses as follows. Partition the set of agents in $N_i(\bar{\mathbf{g}})$ by their effort levels in $\bar{\mathbf{g}}$. Assume that there are m distinct effort levels in $N_i(\bar{\mathbf{g}})$. Denote with $N_i^1(\bar{\mathbf{g}})$ the set of agents with the lowest effort levels in $N_i(\bar{\mathbf{g}})$, $N_i^2(\bar{\mathbf{g}})$ is the set of agents with the second lowest effort levels in $N_i(\bar{\mathbf{g}})$, and so forth, until $N_i^m(\bar{\mathbf{g}})$, the set of agent with the highest effort levels in $N_i(\bar{\mathbf{g}})$. From the above we know that any deletion best response, $\mathbf{g}'_i$, is such that either $D_i(\mathbf{g}'_i, \bar{\mathbf{g}}) = \emptyset$, or $D_i(\mathbf{g}'_i, \bar{\mathbf{g}}) = \cup_{j=1}^k N_i^j(\bar{\mathbf{g}})$ for some integer k with $1 \leq k \leq m$. If there are multiple optimal deletion deviation strategies, then $\mathbf{g}'^m_i$ must be such that k is maximal, which we denote with k^m. Since there is a finite number of agents in $N_i(\bar{\mathbf{g}})$, the number of partitions of $N_i(\bar{\mathbf{g}})$ is also finite and therefore such a k^m exists. Note that for all deletion best responses with $k^m = k$, the sets $D_i(\mathbf{g}'_i, \bar{\mathbf{g}})$ and $N_i(\bar{\mathbf{g}}')$ are the same. The optimal, minimal deletion deviation strategy is then such that $\mathbf{g}'^m_{i,j} = 0 \; \forall j \notin N_i(\bar{\mathbf{g}}')$. *Q.E.D.*

Lemma 12: *Assume γ is sufficiently low, $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ and agents play their Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}})$ and $\mathbf{x}(\hat{\bar{\mathbf{g}}})$, then an agent i's minimal deletion best response in $\bar{\mathbf{g}}$, $\mathbf{g}'^m_i$, and agent i's minimal optimal deletion deviation in $\hat{\bar{\mathbf{g}}}$, $\hat{\mathbf{g}}'^m_i$, are such that $N_i(\bar{\mathbf{g}}') \subseteq N_i(\hat{\bar{\mathbf{g}}}')$.*

*Proof of **Lemma 12.*** Assume that $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ holds and that γ is sufficiently small. Note first that the above statement holds trivially if either $\eta_i(\hat{\bar{\mathbf{g}}}) = 0$ or $\eta_i(\bar{\mathbf{g}}) = 0$. We therefore assume that $\eta_i(\hat{\bar{\mathbf{g}}}) \geq 1$ and $\eta_i(\bar{\mathbf{g}}) \geq 1$. Note further that the statement also holds trivially if $\eta_i(\bar{\mathbf{g}}') = 0$ and we therefore need to consider only cases when $\eta_i(\bar{\mathbf{g}}') \neq 0$. Denote with $\bar{\mathbf{g}}'^m$ the network that is obtained from agent i's minimal deletion best response, $\mathbf{g}'^m_i$, in $\bar{\mathbf{g}}$, while $\hat{\bar{\mathbf{g}}}'^m$ is obtained from agent i's minimal deletion best response, $\hat{\mathbf{g}}'^m_i$, in $\hat{\bar{\mathbf{g}}}$. Pick a ranking of agents in the set of agent i's neighbors *after* proposed deviation in $\bar{\mathbf{g}}$, $N_i(\bar{\mathbf{g}}'^m)$, such that $x_1(\bar{\mathbf{g}}'^m) \geq x_2(\bar{\mathbf{g}}'^m) \geq ... \geq x_{\eta_i(\bar{\mathbf{g}}'^m)}(\bar{\mathbf{g}}'^m)$, where we use the subscript to refer to the position in the ranking rather than an agent's label in the set N. Denote this ranking with $r(N_i(\bar{\mathbf{g}}'^m))$. Similarly, pick a ranking of agents in the set of agent i's neighbors prior to proposed deviation in $\hat{\bar{\mathbf{g}}}$, $N_i(\hat{\bar{\mathbf{g}}})$, such that $x_1(\hat{\bar{\mathbf{g}}}) \geq x_2(\hat{\bar{\mathbf{g}}}) \geq ... \geq x_{\eta_i(\hat{\bar{\mathbf{g}}})}(\hat{\bar{\mathbf{g}}})$. Denote this ranking with $r(N_i(\hat{\bar{\mathbf{g}}}))$. Note that since $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ holds and $\bar{\mathbf{g}}'^m$ is a deletion best response, $N_i(\bar{\mathbf{g}}'^m)) \subseteq N_i(\hat{\bar{\mathbf{g}}})$ also holds. Pick an agent $j \in N_i(\bar{\mathbf{g}}'^m)$ such that $x_k(\hat{\bar{\mathbf{g}}}) \geq x_j(\hat{\bar{\mathbf{g}}}) \; \forall k \in N_i(\bar{\mathbf{g}}'^m)$, i.e. pick an agent j in $N_i(\bar{\mathbf{g}}'^m)$, such that j's effort level in the network $\hat{\bar{\mathbf{g}}}$ is weakly smaller than the effort level of any other agent in $N_i(\bar{\mathbf{g}}'^m)$ in the network $\hat{\bar{\mathbf{g}}}$. Out of all agents in the ranking $r(N_i(\hat{\bar{\mathbf{g}}}))$ with effort level equal to $x_j(\hat{\bar{\mathbf{g}}})$, pick the agent with the largest subscript. Denote this agent with $x_t(\hat{\bar{\mathbf{g}}})$ and define a truncated ranking of $r(N_i(\hat{\bar{\mathbf{g}}}))$, denoted with $r_t(N_i(\hat{\bar{\mathbf{g}}}))$, such that $x_1(\hat{\bar{\mathbf{g}}}) \geq x_2(\hat{\bar{\mathbf{g}}}) \geq ... \geq x_{t-1}(\hat{\bar{\mathbf{g}}}) \geq x_t(\hat{\bar{\mathbf{g}}})$. Note that, since $x_j(\hat{\bar{\mathbf{g}}})$ was chosen such that all agents in $N_i(\bar{\mathbf{g}}'^m)$ display weakly higher effort levels in $\hat{\bar{\mathbf{g}}}$ and since $x_t(\hat{\bar{\mathbf{g}}})$ is the agent with the highest subscript in $r(N_i(\hat{\bar{\mathbf{g}}}))$ such that $x_t(\hat{\bar{\mathbf{g}}}) = x_j(\hat{\bar{\mathbf{g}}})$, all agents in $N_i(\bar{\mathbf{g}}'^m)$ are included in the ranking $r_t(N_i(\hat{\bar{\mathbf{g}}}))$. Note that, since $\mathbf{g}'^m_i$ is agent i's minimal deletion best response in $\bar{\mathbf{g}}$, agent i does not find it profitable to delete any further links in $\bar{\mathbf{g}}'^m$. From Lemma 11 we know that we only need to consider deviations $\mathbf{g}'_i$ such that $\forall k, l : k \in N_i(\bar{\mathbf{g}}')$ and $l \in D_i(\mathbf{g}'_i, \bar{\mathbf{g}})$, $x_k(\bar{\mathbf{g}}') > x_l(\bar{\mathbf{g}}')$ holds. That is, for $\mathbf{g}'^m_i$ to be a minimal deletion best response in $\bar{\mathbf{g}}$, we know that the following conditions, summarized in (A), need to hold

$$\text{(A)} \quad \frac{v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m)) - v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\bar{\mathbf{g}}'^m)-j}(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m))}{k+1} > \kappa$$

for all $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\bar{\mathbf{g}}') - 1$. To see this, note that if

$$\frac{v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m)) - v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\bar{\mathbf{g}}'^m)-j}(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m))}{k+1} < \kappa$$

holds, then agent i can increase deviation payoffs by deleting links to some subset of agents in $N_i(\bar{\mathbf{g}}'^m)$ and $\mathbf{g}_i'^m$ is therefore not optimal. If

$$\frac{v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m)) - v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\bar{\mathbf{g}}'^m)-j}(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m))}{k+1} = \kappa,$$

then there exists a deviation, $\tilde{\mathbf{g}}_i'$, that yields the same deviation payoffs as $\mathbf{g}_i'^m$, but $\tilde{\mathbf{g}}_i' \subset \mathbf{g}_i'^m$ holds and $\mathbf{g}_i'^m$ is not minimal.

Next we first show that this implies that a deviation by agent i in $\hat{\bar{\mathbf{g}}}$, $\hat{\bar{\mathbf{g}}}_i'$, such that agent i keeps his links with the first $\eta_i(\bar{\mathbf{g}}'^m)$ agents in the ranking $r(N_i(\hat{\bar{\mathbf{g}}}))$ in $\hat{\bar{\mathbf{g}}}'$, but deletes all other links, yields strictly higher payoffs than a deviation where any further links are deleted. The appropriate conditions, summarized in (B), such that agent i does not find it profitable to delete any further links, are given by

$$\text{(B)}\quad \frac{v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}})) - v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\hat{\bar{\mathbf{g}}}) - \sum_{j=0}^{k} x_{\eta_i(\bar{\mathbf{g}}'^m)-j}(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}}))}{k+1} > \kappa$$

for all $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\bar{\mathbf{g}}') - 1$. To see that (A) implies (B), note first that, since $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ and γ is sufficiently small and $\eta_j(\bar{\mathbf{g}}'^m) \geq 1 \ \forall j \in N_i(\bar{\mathbf{g}}'^m)$, we know from Lemma 9 that all agents in $N_i(\bar{\mathbf{g}}'^m)$ display strictly lower effort levels in $\bar{\mathbf{g}}$ than in $\hat{\bar{\mathbf{g}}}$. That is, for the first $\eta_i(\bar{\mathbf{g}}'^m)$ agents in the ranking $r_t(N_i(\hat{\bar{\mathbf{g}}}))$, we know that agents with the same rank (i.e. the same subscript) as in $r(N_i(\bar{\mathbf{g}}'^m))$, display strictly larger effort levels. Note next that for $\gamma = 0$, v is strictly convex in the first argument and $v(y, z)$ can be treated as a function of the first argument in this case. Therefore, a deviation strategy, which deletes all agents up until the $\eta_i(\bar{\mathbf{g}}'^m)$-th agent in the ranking $r_t(N_i(\hat{\bar{\mathbf{g}}}))$, yields strictly larger deviation payoffs, than deleting any further agents. Since the conditions must hold strictly, we know from Lemma 7 that they also hold for γ sufficiently small. Note that if $t = \eta_i(\bar{\mathbf{g}}'^m)$, then $N_i(\bar{\mathbf{g}}'^m) \subseteq N_i(\hat{\bar{\mathbf{g}}}'^m)$. Assume next that $t > \eta_i(\bar{\mathbf{g}}'^m)$ holds. Consider the condition for $k = 0$ in (A), which reads

$$v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m)) - v(\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m) - x_{\eta_i(\bar{\mathbf{g}}'^m)}(\bar{\mathbf{g}}'^m), z_i(\bar{\mathbf{g}}'^m)) > \kappa.$$

Note that $\sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\hat{\bar{\mathbf{g}}}) \geq \sum_{j=1}^{\eta_i(\bar{\mathbf{g}}'^m)} x_j(\bar{\mathbf{g}}'^m)$ and $x_l(\bar{\mathbf{g}}'^m) \geq x_{\eta_i(\bar{\mathbf{g}}'^m)}(\bar{\mathbf{g}}'^m)$ for all agents with a subscript l larger than $\eta_i(\bar{\mathbf{g}}'^m)$ in $r_t(N_i(\hat{\bar{\mathbf{g}}}))$. Since $v(y, z)$ is convex in the first argument and payoffs are arbitrarily close to the case when $\gamma = 0$ for γ sufficiently small, we know that the marginal benefit of adding links in the sequence from the $\eta_i(\tilde{\bar{\mathbf{g}}}')$-th agent in $r_t(N_i(\hat{\bar{\mathbf{g}}}))$ to agent t is strictly positive for each agent. Therefore, keeping all of the first t links in the ranking $r_t(N_i(\hat{\bar{\mathbf{g}}}))$ yields strictly larger payoffs than deleting any subset of links in $r_t(N_i(\hat{\bar{\mathbf{g}}}))$. By Lemma 11 we then know that any optimal deletion best response must be such that $N_i(\bar{\mathbf{g}}'^m) \subseteq N_i(\hat{\bar{\mathbf{g}}}'^m)$ holds.*Q.E.D.*

Theorem 2: *Assume $\bar{\mathbf{g}} \neq \bar{\mathbf{g}}^e$ is a PNE network. If γ is sufficiently small, then a key-player policy always exists and it prescribes eliminating an agent with the highest number of links.*

Proof of **Theorem 2.** From Lemma 10 we know that, if no pair of agents j and k find it profitable to create a link $\bar{\mathbf{g}}$, then no pair of agents j and k find it profitable to create a link in $\bar{\mathbf{g}}^{-i} = \bar{\mathbf{g}}_0^{-i}$. Therefore, $\bar{\mathbf{g}}_0^{-i} = \bar{\mathbf{g}}_1^{-i}$. From Lemma 12 we know that a minimal optimal deviation strategy exists and is unique. If for all agents j in $\bar{\mathbf{g}}_1^{-i}$ the minimal optimal deviation strategy is such that no link is deleted, then $\overline{\mathbf{g}}_0^{-i} = \overline{\mathbf{g}}_1^{-i} = \overline{\mathbf{g}}_2^{-i}$ and the process converged. If there exists an agents j in $\overline{\mathbf{g}}_1^{-i}$ such that at least one link is deleted, then the network $\overline{\mathbf{g}}_2^{-i}$ is such that $\overline{\mathbf{g}}_2^{-i} \subset \overline{\mathbf{g}}_1^{-i}$. Note that then $\overline{\mathbf{g}}_2^{-i} \subset \overline{\mathbf{g}}_0^{-i}$ also holds and again from Lemma 10 we know that no pair of agents j and k finds it profitable to create a link in $\overline{\mathbf{g}}_2^{-i}$, so that $\overline{\mathbf{g}}_3^{-i} = \overline{\mathbf{g}}_2^{-i}$. If for all agents j in $\overline{\mathbf{g}}_3^{-i}$ the minimal deletion best response is such that no link is deleted, then $\overline{\mathbf{g}}_4^{-i} = \overline{\mathbf{g}}_3^{-i} = \overline{\mathbf{g}}_2^{-i}$ and the process converged. If there exists an agent j in in $\overline{\mathbf{g}}_3^{-i}$ with an minimal optimal deviation strategy such that at least one link is deleted, then $\overline{\mathbf{g}}_4^{-i} \subset \overline{\mathbf{g}}_3^{-i} \subset \overline{\mathbf{g}}_0^{-i}$. We can now use the above argument iteratively. That is, at each $\overline{\mathbf{g}}_l^{-i}$ with l even no link is created, while at each $\overline{\mathbf{g}}_l^{-i}$ with l odd links may or may not be deleted. If $\overline{\mathbf{g}}_l^{-i}$ with l odd is such that no link is deleted then the process converged. Since the number of links is bounded below by zero links, the process converges. Finally, from Lemma 6 we know that, if $\eta_k(\bar{\mathbf{g}}) > \eta_l(\bar{\mathbf{g}})$ then $\overline{\mathbf{g}}^{-l} \subset \overline{\mathbf{g}}^{-k}$. From Lemma 12 it then follows immediately that eliminating an agent with the highest number of links guarantees that at each iteration the network is minimal and therefore converges to a minimal network. From Ballester et al. (2006) we know that effort levels are then minimal. *Q.E.D.*

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