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# SEMIPARAMETRIC ESTIMATION OF STRUCTURAL FUNCTIONS IN NONSEPARABLE TRIANGULAR MODELS

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ABSTRACT. This paper introduces two classes of semiparametric triangular systems with nonadditively separable unobserved heterogeneity. They are based on distribution and quantile regression modeling of the reduced-form conditional distributions of the endogenous variables. We show that these models are flexible and identify the average, distribution and quantile structural functions using a control function approach that does not require a large support condition. We propose a computationally attractive three-stage procedure to estimate the structural functions where the first two stages consist of quantile or distribution regressions. We provide asymptotic theory and uniform inference methods for each stage. In particular, we derive functional central limit theorems and bootstrap functional central limit theorems for the distribution regression estimators of the structural functions. We illustrate the implementation and applicability of our methods with numerical simulations and an empirical application to demand analysis.

KEYWORDS: Structural functions, nonseparable models, control function, quantile and distribution regression, semiparametric estimation, uniform inference.

## 1. INTRODUCTION

Models with nonadditively separable disturbances provide an important vehicle for incorporating heterogenous effects. However, accounting for endogenous treatments in such a setting can be challenging. One methodology which has been successfully employed in a wide range of models with endogeneity is the use of control functions (see, for surveys, Imbens and Wooldridge 2009, Wooldridge 2015 and Blundell, Newey

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and Vella 2017). The underlying logic of this approach is to account for the endogeneity by including an appropriate control function in the conditioning variables. This paper proposes some relatively simple control function procedures to estimate objects of interest in a triangular model with nonseparable disturbances. Our approach to circumventing the inherent difficulties in nonparametric estimation associated with the curse of dimensionality is to build our models upon a semiparametric specification. Our goal is to provide models and methods that are essentially parametric but still allow for nonseparable disturbances. These models can be interpreted as “baseline” models on which series approximations can be built by adding additional terms.

We consider two kinds of baseline models, quantile regression and distribution regression. These models allow the use of convenient and widely available methods to estimate objects of interest including average and quantile structural/treatment effects. A main feature of the baseline models is that interaction terms included would not usually be present as leading terms in estimation. These included terms are products of a transformation of the control function with the endogenous treatment. Their presence is meant to allow for heterogeneity in the coefficient of the endogenous variable. Such heterogenous coefficient linear models are of interest in many settings and provide a natural starting point for more general models that allow for nonlinear effects of the endogenous treatments.

We use these baseline models to construct estimators of the average, distribution and quantile structural functions based on parametric quantile and distribution regressions. We also show how these baseline models can be expanded to include higher order terms. The estimation procedure consists of three stages. First, we estimate the control function via quantile regression (QR) or distribution regression (DR) of the endogenous treatment on the exogenous covariates and exclusion restrictions. Second, we estimate the reduced form distribution of the outcome conditional on the treatment, covariates and estimated control function using DR or QR. Third, we construct estimators of the structural functions applying suitable functionals to the reduced form estimator from the second stage. We derive asymptotic theory for the estimators based on DR in all the stages using a trimming device that avoids tail estimation in the construction of the control function. We perform Monte Carlo experiments and give an empirical application based on the estimation of Engel curves.

Our results for the average structural function in the linear random coefficients model are similar to Garen (1984). Florens, Heckman, Meghir, Vytlacil (2008) give identification and estimation results for a restricted model with random coefficients for

powers of the endogenous treatment. Blundell and Powell (2003, 2004) introduce the average structural function, and Imbens and Newey (2009) give general models and results for a variety of objects of interest and control functions, including quantile structural functions. This work also complements the literature on local identification and estimation of triangular nonseparable models, as in Chesher (2003), Ma and Koenker (2006), and Jun (2009), and on global construction of structural functions (Stouli, 2012). Chernozhukov, Fernandez-Val and Kowalski (2015) developed a related two-stage quantile regression estimator for triangular nonseparable models but do not consider estimation of structural functions.

This paper makes four main contributions to the existing literature. First, we establish identification of structural functions in both classes of baseline models, providing conditions that do not impose large support requirements on the exclusion restriction. Second, we derive a functional central limit theorem and a bootstrap functional central limit theorem for the two-stage DR estimators in the second stage. These results are uniform over compact regions of values of the outcome. To the best of our knowledge, this result is new. Chernozhukov, Fernandez-Val and Kowalski (2015) derived similar results for two-stage quantile regression estimators but their results are pointwise over quantile indexes. Our analysis builds on Chernozhukov, Fernandez-Val, and Galichon (2010) and Chernozhukov, Fernandez-Val, and Melly (2013), which established the properties of the DR estimators that we use in the first stage. The theory of the two-stage estimator, however, does not follow from these results using standard techniques due to the dimensionality and entropy properties of the first stage DR estimators. We follow the proof strategy proposed by Chernozhukov, Fernandez-Val and Kowalski (2015) to deal with these issues. Third, we derive functional central limit theorems and bootstrap functional central limit theorems for plug-in estimators of functionals of the distribution of the outcome conditional on the treatment, covariates and control function via functional delta method. These functionals include all the structural functions of interest. We build on the results of Chernozhukov, Fernandez-Val, and Melly (2013), which established the properties of related counterfactual distribution and quantile functionals. We also use a linear functional for the average structural function which had not been previously considered. Fourth, we show that this linear operator that relates the average of a random variable with its distribution is Hadamard differentiable.

The rest of the paper is organized as follows. Section 2 describes the baseline models and objects of interest. Section 3 presents the estimation and inference methods.

Section 4 gives asymptotic theory. Section 5 reports the results of the empirical application to Engel curves and simulations calibrated to the application. Implementation algorithms and proofs of the main result are given in the Appendix. The online Appendix Chernozhukov et al. (2017) contains supplemental material.

## 2. MODELLING FRAMEWORK

We begin with a brief review of the triangular nonseparable model and some inherent objects of interest. Let  $Y$  denote an outcome variable of interest that can be continuous, discrete or mixed continuous-discrete,  $X$  a continuous endogenous treatment,  $Z$  a vector of exogenous variables,  $\varepsilon$  a structural disturbance vector of unknown dimension, and  $V$  a scalar reduced form disturbance. The model is

$$\begin{aligned} Y &= g(X, \varepsilon), \\ X &= h(Z, \tilde{V}), \quad (\varepsilon, \tilde{V}) \text{ indep of } Z, \end{aligned}$$

where  $v \mapsto h(z, v)$  is a one-to-one function for each  $z$ . This model implies that  $\varepsilon$  and  $X$  are independent conditional on  $\tilde{V}$  and that  $\tilde{V}$  is a one-to-one function of  $V = F_X(X | Z)$ , the cumulative distribution function (CDF) of  $X$  conditional on  $Z$  evaluated at the observed variables. Thus,  $V$  is a control function.

Objects of interest in this model include the average structural function (ASF),  $\mu(x)$ , and quantile structural function (QSF),  $Q(\tau, x)$ , where

$$\mu(x) = \int g(x, \varepsilon) F_\varepsilon(d\varepsilon), \quad Q(\tau, x) = \tau^{\text{th}} \text{ quantile of } g(x, \varepsilon).$$

Here  $\mu(\tilde{x}) - \mu(\bar{x})$  is like an average treatment effect and  $Q(\tau, \tilde{x}) - Q(\tau, \bar{x})$  is like a quantile treatment effect from the treatment effects literature. If the support of  $V$  conditional on  $X = x$  is the same as the marginal support of  $V$  then these objects are nonparametrically identified by

$$\mu(x) = \int E[Y | X = x, V] F_V(dV),$$

and

$$Q(\tau, x) = G^{\leftarrow}(\tau, x), \quad G(y, x) = \int F_Y(y | X = x, V) F_V(dV),$$

where  $G(y, x)$  is the Distribution Structural Function (DSF), and  $G^{\leftarrow}(\tau, x)$  denotes the left-inverse of  $y \mapsto G(y, x)$ , i.e.  $G^{\leftarrow}(\tau, x) := \inf\{y \in \mathbb{R} : G(y, x) \geq \tau\}$ .

It is straightforward to extend this approach to allow for covariates in the model by further conditioning on or integrating over them. Suppose that  $Z_1 \subset Z$  is included in the structural equation, which is now  $g(X, Z_1, \varepsilon)$ . Under the assumption that  $\varepsilon$  and  $V$  are jointly independent of  $Z$ , then  $\varepsilon$  will be independent of  $X$  and  $Z_1$  conditional on  $V$ . Conditional on covariates and unconditional average structural functions are identified by

$$\mu(x, z_1) = \int E[Y | X = x, Z_1 = z_1, V] F_V(dV),$$

and

$$\mu(x) = \int E[Y | X = x, Z_1, V] F_{Z_1}(dZ_1) F_V(dV).$$

Similarly, conditional on covariates and unconditional quantile and distribution structural functions are identified by

$$Q(\tau, x, z_1) = G^{\leftarrow}(\tau, x, z_1), \quad G(y, x, z_1) = \int F_Y(y | X = x, Z_1 = z_1, V) F_V(dV),$$

and

$$Q(\tau, x) = G^{\leftarrow}(\tau, x), \quad G(y, x) = \int F_Y(y | X = x, Z_1, V) F_{Z_1}(dZ_1) F_V(dV),$$

respectively.

With covariates the curse of dimensionality makes it difficult to estimate the control function  $V = F_X(X | Z)$ , the conditional mean  $E[Y | X, Z_1, V]$ , and the conditional CDF  $F_Y(Y | X, Z_1, V)$ . This difficulty motivates our specification of baseline parametric models in what follows. These baseline models provide good starting points for nonparametric estimation and may be of interest in their own right.

**2.1. Quantile Regression Baseline.** We start with a simplified specification with one endogenous treatment  $X$ , one exclusion restriction  $Z$ , and a continuous outcome  $Y$ . We show below how additional excluded variables and covariates can be included.

The baseline first stage is the quantile regression model

$$X = Q_X(V | Z) = \pi_1(V) + \pi_2(V)Z, \quad V | Z \sim U(0, 1).$$

Note that  $v \mapsto \pi_1(v)$  and  $v \mapsto \pi_2(v)$  are infinite dimensional parameters (functions). We can recover the control function  $V$  from  $V = F_X(X | Z) = Q_X^{-1}(X | Z)$  or equivalently from

$$V = F_X(X | Z) = \int_0^1 1\{\pi_1(v) + \pi_2(v)Z \leq X\} dv.$$

This generalized inverse representation of the CDF is convenient for estimation because it does not require the conditional quantile function to be strictly increasing to be well-defined. Model parameters can be estimated using Koenker and Bassett quantile regression (Koenker and Bassett, 1978).

The baseline second stage has a reduced form:

$$Y = Q_Y(U | X, V), \quad U | X, V \sim U(0, 1),$$

$$Q_Y(U | X, V) = \beta_1(U) + \beta_2(U)X + \beta_3(U)\Phi^{-1}(V) + \beta_4(U)X\Phi^{-1}(V),$$

where  $\Phi^{-1}$  is the standard normal inverse CDF. This transformation is included to expand the support of  $V$  and to encompass the normal system of equations as a special case. An example of a structural model with this reduced form is the random coefficient model

$$Y = g(X, \varepsilon) = \varepsilon_1 + \varepsilon_2 X,$$

with the restrictions

$$\varepsilon_j = Q_{\varepsilon_j}(U | X, V) = \theta_j(U) + \gamma_j(U)\Phi^{-1}(V), \quad U | X, V \sim U(0, 1), \quad j \in \{1, 2\}.$$

These restrictions include the control function assumption  $\varepsilon_j \perp\!\!\!\perp X | V$  and a joint functional form restriction, where the unobservable  $U$  is the same for  $\varepsilon_1$  and  $\varepsilon_2$ . Substituting in the second stage equation,

$$Y = \theta_1(U) + \theta_2(U)X + \gamma_1(U)\Phi^{-1}(V) + \gamma_2(U)\Phi^{-1}(V)X, \quad U | X, V \sim U(0, 1),$$

which has the form of (2.1). All model parameters can be estimated by QR of  $Y$  on  $(1, X, \Phi^{-1}(V), \Phi^{-1}(V)X)$ .

The specification (2.1) is a baseline, or starting point, for a more general series approximation to the quantiles of  $Y$  conditional on  $X$  and  $V$  based on including additional functions of  $X$  and  $\Phi^{-1}(V)$ . The baseline is unusual as it includes the interaction term  $\Phi^{-1}(V)X$ ; it is more usual to take the starting point to be  $(1, \Phi^{-1}(V), X)$ , which is linear in the regressors  $X$  and  $\Phi^{-1}(V)$ . The inclusion of the interaction term is motivated by allowing the coefficient of  $X$  to vary with individuals, so that  $\Phi^{-1}(V)$  then interacts  $X$  in the conditional distribution of  $\varepsilon_2$  given the control functions.

The ASF of the baseline specification is:

$$\mu(x) = \int_0^1 E[Y | X = x, V = v] dv = \beta_1 + \beta_2 x,$$

where the second equality follows by  $\int_0^1 \Phi^{-1}(v)dv = 0$  and

$$E[Y | X, V] = \int_0^1 Q_Y(u | X, V)du = \beta_1 + \beta_2 X + \beta_3 \Phi^{-1}(V) + \beta_4 X \Phi^{-1}(V)$$

with  $\beta_j := \int_0^1 \beta_j(u)du$ ,  $j \in \{1, \dots, 4\}$ . The QSF does not appear to have a closed form expression. It is the solution to

$$\begin{aligned} Q(\tau, x) &= G^{\leftarrow}(\tau, x), \\ G(y, x) &= \int_0^1 \int_0^1 1\{\beta_1(u) + \beta_2(u)x + \beta_3(u)\Phi^{-1}(v) + \beta_4(u)\Phi^{-1}(v)x \leq y\}dudv. \end{aligned}$$

A special case of the QR baseline is a heteroskedastic normal system of equations. We use this specification in the numerical simulations of Section 5.

**2.2. Distribution Regression Baseline.** We start again with a simplified specification with one endogenous treatment  $X$  and one excluded  $Z$ , but now the outcome  $Y$  can be continuous, discrete or mixed.

Let  $\Gamma$  denote a strictly increasing continuous CDF such as the standard normal or logistic CDF. The first stage equation is the distribution regression model

$$\eta = \pi_1(X) + \pi_2(X)Z, \quad \eta | Z \sim \Gamma,$$

which corresponds to the specification of the control variable  $V$  as

$$(2.1) \quad V = F_X(X | Z) = \Gamma(\pi_1(X) + \pi_2(X)Z).$$

While the first stage QR model specifies the conditional quantile function of  $X$  given  $Z$  to be linear in  $Z$ , the DR model (2.1) specifies the conditional distribution of  $X$  given  $Z$  to be generalized linear in  $Z$ , i.e. linear after applying the link function  $\Gamma$ .

The second stage baseline has a reduced form:

$$(2.2) \quad F_Y(Y | X, V) = \Gamma(\beta_1(Y) + \beta_2(Y)X + \beta_3(Y)\Phi^{-1}(V) + \beta_4(Y)\Phi^{-1}(V)X).$$

When  $Y$  is continuous, an example of a structural model that has reduced form (2.2) is the latent random coefficient model

$$(2.3) \quad \xi = \varepsilon_1 + \varepsilon_2 \Phi^{-1}(V), \quad \xi | X, V \sim \Gamma,$$

with the restrictions

$$\varepsilon_j = \theta_j(Y) + \gamma_j(Y)X, \quad j \in \{1, 2\},$$



such that the mapping  $y \mapsto \theta_j(y) + \gamma_j(y)x$  is strictly increasing, and the following conditional independence property is satisfied:

$$(2.4) \quad F_{\varepsilon_j}(\varepsilon_j | V) = F_{\varepsilon_j}(\varepsilon_j | X, V), \quad j \in \{1, 2\}.$$

Substituting the expression for  $\varepsilon_1$  and  $\varepsilon_2$  in (2.3) yields

$$\xi = \theta_1(Y) + \gamma_1(Y)X + \theta_2(Y)\Phi^{-1}(V) + \gamma_2(Y)\Phi^{-1}(V)X,$$

which has a reduced form for the distribution of  $Y$  conditional on  $(X, V)$  as in (2.2).

All the parameters of this model (2.2) can be estimated by DR. As in the quantile baseline, the specification (2.2) can be used as starting point for a more general series approximation to the distribution of  $Y$  conditional on  $X$  and  $V$  based on including additional functions of  $X$  and  $\Phi^{-1}(V)$ .

For the DR baseline, the QSF is the solution to

$$Q(\tau, x) = G^{\leftarrow}(\tau, x), \quad G(y, x) = \int_0^1 \Gamma(\beta_1(y) + \beta_2(y)x + \beta_3(y)\Phi^{-1}(v) + \beta_4(y)\Phi^{-1}(v)x) dv.$$

Compared to the QR baseline model, the ASF cannot be obtained as a linear projection but it can be conveniently expressed as a linear functional of  $G(y, x)$ . Let  $\mathcal{Y}$  denote the support of  $Y$ ,  $\mathcal{Y}^+ = \mathcal{Y} \cap [0, \infty)$  and  $\mathcal{Y}^- = \mathcal{Y} \cap (-\infty, 0)$ . The ASF can be characterized as

$$(2.5) \quad \mu(x) = \int_0^1 E[Y | X = x, V = v] dv = \int_{\mathcal{Y}^+} [1 - G(y, x)] \nu(dy) - \int_{\mathcal{Y}^-} G(y, x) \nu(dy),$$

where  $\nu$  is either the counting measure when  $\mathcal{Y}$  is countable or the Lebesgue measure otherwise, and we exploit the linear relationship between the expected value and the distribution of a random variable. This characterization simplifies both the computation and theoretical treatment of the DR-based estimator for the ASF. It also applies to the QR specification upon using the corresponding expression for  $G(y, x)$ .

Section 5 provides an example of a special case of the DR model.

**2.3. Identification.** The most general specifications that we consider include several exclusion restrictions, covariates and transformations of the regressors in both stages. For  $d_{z_1} := \dim(Z_1)$  and  $r_1(Z_1) := r_{11}(Z_{11}) \otimes \cdots \otimes r_{1L}(Z_{1d_{z_1}})$ , let

$$R := r(Z) \text{ and } W := w(X, Z_1, V) := p(X) \otimes r_1(Z_1) \otimes q(V)$$

denote the sets of regressors in the first and second stages, where  $r$ ,  $r_1$ ,  $p$  and  $q$  are vectors of transformations such as powers, b-splines and interactions, and

$\otimes$  denotes the Kronecker product. The simplest case is when  $r(Z) = (1, Z)'$ ,  $r_1(Z_1) = (1, Z_1)'$ ,  $p(X) = (1, X)'$  and  $q(V) = (1, \Phi^{-1}(V))'$ , so that  $w(X, Z_1, V) = (1, \Phi^{-1}(V), X, X\Phi^{-1}(V), Z_1, Z_1\Phi^{-1}(V), XZ_1, XZ_1\Phi^{-1}(V))'$ . The following assumption gathers the baseline specifications for the first and second stages.

**Assumption 1.** *[Baseline Models] The outcome  $Y$  has a conditional density function  $y \mapsto f_Y(y | X, Z_1, V)$  with respect to some measure that is a.s. bounded away from zero uniformly in  $\mathcal{Y}$ ; and (a)  $X$  conditional on  $Z$  follows the QR model*

$$X = Q_X(V | Z) = R'\pi(V), \quad V | Z \sim U(0, 1),$$

and  $Y$  conditional on  $(X, Z_1, V)$  follows the QR model

$$Y = Q_Y(U | X, Z_1, V) = W'\beta(U), \quad V = F_X(X | Z), \quad U | X, Z_1, V \sim U(0, 1);$$

or (b)  $X$  conditional on  $Z$  follows the DR model

$$V = \Lambda(R'\pi(X)), \quad V | Z \sim U(0, 1),$$

and  $Y$  conditional on  $(X, Z_1, V)$  follows the DR model,

$$U = \Gamma(W'\beta(Y)), \quad V = F_X(X | Z), \quad U | X, Z_1, V \sim U(0, 1),$$

where  $\Gamma$  is either the standard normal or logistic CDF.

The structural functions of the baseline models involve quantile and distribution regressions on the same set of regressors. A sufficient condition for identification of the coefficients of these regressions is that the second moment matrix of those regressors is nonsingular. The regressors have a Kronecker product form  $p(X) \otimes r_1(Z_1) \otimes q(V)$ . The second moment matrix for these regressors will be nonsingular if the joint distribution dominates a distribution where  $X$ ,  $Z_1$  and  $V$  are independent and the second moment matrices of  $X$ ,  $Z_1$  and  $V$  are positive definite. Define the product probability measure  $\varsigma(z_1) := \times_{l=1}^{d_{z_1}} \varsigma_l(z_{1l})$ .

**Assumption 2.** *The joint probability distribution of  $X$ ,  $Z_1$  and  $V$  dominates a product probability measure  $\mu(x) \times \varsigma(z_1) \times \rho(v)$  such that  $E_\mu[p(X)p(X)']$ ,  $E_{\varsigma_l}[r_{1l}(Z_{1l})r_{1l}(Z_{1l})']$ ,  $l = 1, \dots, d_{z_1}$ , and  $E_\rho[q(V)q(V)']$  are positive definite.*

When  $p(X) = (1, X)'$ ,  $r_{1l}(Z_{1l}) = (1, Z_{1l})'$ ,  $l = 1, \dots, d_{z_1}$ , and  $q(V) = (1, \Phi^{-1}(V))'$ , Assumption 2 simplifies to the requirement that the joint distribution of  $X$ ,  $Z_1$  and  $V$  be dominating one such that  $\text{Var}_\mu(X) > 0$ ,  $\text{Var}_{\varsigma_l}(Z_{1l}) > 0$ ,  $l = 1, \dots, d_{z_1}$ , and  $\text{Var}_\rho(\Phi^{-1}(V)) > 0$ . For general specifications where the regressors are higher order

power series, it is sufficient for Assumption 2 that the joint distribution of  $X$ ,  $Z_1$  and  $V$  be dominating one that has density bounded away from zero on a hypercube. That will mean that the joint distribution dominates a uniform distribution on that hypercube, and for a uniform distribution on a hypercube  $E[w(X, Z_1, V)w(X, Z_1, V)']$  is nonsingular.

**Lemma 1.** *If Assumption 2 holds, then  $E[w(X, Z_1, V)w(X, Z_1, V)']$  is nonsingular.*

Assumptions 1-2 are sufficient conditions for the map  $y \mapsto F_Y(y | x, z_1, v)$  to be well-defined for all  $(x, z_1, v)$ , and therefore for identification of the structural functions.

**Theorem 1.** *If Assumptions 1 and 2 hold, then the DSF, QSF and ASF are identified.*

Given the semiparametric specifications in Assumption 1, identification of structural functions does not require any restriction on the support of  $Z$ , and the full support assumption of Imbens and Newey (2009) need not be satisfied. Theorem 1 thus illustrates the identifying power of semiparametric restrictions and the trade-off between these restrictions and the full support condition for identification of structural functions in nonseparable triangular models.

### 3. ESTIMATION AND INFERENCE METHODS

The QR and DR baselines of the previous section lead to three-stage analog estimation and inference methods for the DSF, QSF and ASF. The first stage estimates the control function  $V = F_X(X | Z)$ . The second stage estimates the conditional distribution function  $F_Y(y | X, Z_1, V)$ , replacing  $V$  by the estimator from the first stage. The third stage obtains estimators of the structural functions, which are functionals of the first and second stages building blocks. We provide a detailed description of the implementation of each step for both QR and DR methods. We also describe a weighted bootstrap procedure to perform uniform inference on all structural functions considered. Detailed implementation algorithms are given in Appendix A.

We assume that we observe a sample of  $n$  independent and identically distributed realizations  $\{(Y_i, X_i, Z_i)\}_{i=1}^n$  of the random vector  $(Y, X, Z)$ , and that  $\dim(X) = 1$ . Calligraphic letters such as  $\mathcal{Y}$  and  $\mathcal{X}$  denote the supports of  $Y$  and  $X$ ; and  $\mathcal{Y}\mathcal{X}$  denotes the joint support of  $(Y, X)$ . The description of all the stages includes individual weights  $e_i$  which are set to 1 for the estimators, or drawn from a distribution that satisfies Assumption 3 in Section 4 for the weighted bootstrap version of the estimators.

**3.1. First Stage: Estimation of Control Function.** The first stage estimates the  $n$  target values of the control function,  $V_i = F_X(X_i | Z_i)$ ,  $i = 1, \dots, n$ . We estimate the conditional distribution of  $X$  in a trimmed support  $\overline{\mathcal{X}}$  that excludes extreme values. The purpose of the trimming is to avoid the far tails. We consider a fixed trimming rule, which greatly simplifies the derivation of the asymptotic properties. In our numerical and empirical examples we find that the results are not sensitive to the trimming rule and the choice of  $\overline{\mathcal{X}}$  as the observed support of  $X$ , i.e. no trimming, works well. We use bars to denote trimmed supports with respect to  $X$ , e.g.,  $\overline{\mathcal{XZ}} = \{(x, z) \in \mathcal{XZ} : x \in \overline{\mathcal{X}}\}$ . A subscript in a set denotes a finite grid covering the set, where the subscript is the number of grid points. Unless otherwise specified, the points of the grid are sample quantiles of the corresponding variable at equidistant probabilities in  $[0, 1]$ . For example,  $\mathcal{X}_5$  denotes a grid of 5 points covering  $\mathcal{X}$  located at the 0, 1/4, 1/2, 3/4 and 1 sample quantiles of  $X$ .

Denoting the usual check function by  $\rho_v(z) = (v - 1(z < 0))z$ , the first stage in the QR baseline is

$$(3.1) \quad \widehat{F}_X^e(x | z) = \epsilon + \int_{\epsilon}^{1-\epsilon} 1\{r'\widehat{\pi}^e(v) \leq x\} dv, \quad r = r(z), \quad (x, z) \in \overline{\mathcal{XZ}},$$

$$(3.2) \quad \widehat{\pi}^e(v) \in \arg \min_{\pi \in \mathbb{R}^{\dim(R)}} \sum_{i=1}^n e_i \rho_v(X_i - R_i' \pi),$$

for some small constant  $\epsilon > 0$ . The adjustment in the limits of the integral in (3.1) avoids tail estimation of quantiles.<sup>1</sup> The first stage in the DR baseline is,

$$(3.3) \quad \widehat{F}_X^e(x | z) = \Gamma(r'\widehat{\pi}^e(x)), \quad r = r(z), \quad (x, z) \in \overline{\mathcal{XZ}},$$

$$(3.4) \quad \widehat{\pi}^e(x) \in \arg \min_{\pi \in \mathbb{R}^{\dim(R)}} \sum_{i=1}^n e_i [1(X_i \leq x) \log \Gamma(R_i' \pi) + 1(X_i > x) \log (1 - \Gamma(R_i' \pi))].$$

When  $e_i = 1$  for all  $i = 1, \dots, n$ , expressions (3.1)-(3.2) and (3.3)-(3.4) define  $\widehat{F}_X$ , the QR and DR estimators of  $F_X$ . For  $(X_i, Z_i) \in \overline{\mathcal{XZ}}$ , the estimator and weighted bootstrap version of the control function are then  $\widehat{V}_i = \widehat{F}_X(X_i | Z_i)$  and  $\widehat{V}_i^e = \widehat{F}_X^e(X_i | Z_i)$ , respectively, and we set  $\widehat{V}_i = \widehat{V}_i^e = 0$  otherwise.

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<sup>1</sup>Chernozhukov, Fernandez-Val and Melly (2013) provide conditions under which this adjustment does not introduce bias.

*Remark 1.* For DR, the estimation of  $\pi(x)$  at each  $x = X_i$  can be computationally expensive. Substantial gains in computational speed is achieved by first estimating  $\pi(x)$  in a grid  $\overline{\mathcal{X}}_M$ , and then obtaining  $\widehat{\pi}(x)$  at each  $x = X_i$  by interpolation.

**3.2. Second Stage: Estimation of  $F_Y(\cdot | X, Z_1, V)$ .** With the estimated control function in hand, the second building block required for the estimation of structural functions is an estimate of the reduced form CDF of  $Y$  given  $(X, Z_1, V)$ . The baseline models provide direct estimation procedures based on QR and DR.

Let  $T := 1(X \in \overline{\mathcal{X}})$  be a trimming indicator, which is formally defined in Assumption 4 of Section 4. The estimator of  $F_Y$  in the QR baseline is

$$(3.5) \quad \widehat{F}_Y^e(y | x, z_1, v) = \epsilon + \int_{\epsilon}^{1-\epsilon} 1\{w(x, z_1, v)' \widehat{\beta}^e(u) \leq y\} du, \quad (y, x, z_1, v) \in \mathcal{Y} \overline{\mathcal{X}} \overline{\mathcal{Z}}_1 \overline{\mathcal{V}},$$

$$(3.6) \quad \widehat{\beta}^e(u) \in \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \sum_{i=1}^n e_i T_i \rho_u(Y_i - \widehat{W}_i^{e'} \beta), \quad \widehat{W}_i^e = w(X_i, Z_{1i}, \widehat{V}_i^e),$$

As for the first stage, the adjustment in the limits of the integral in (3.5) avoids tail estimation of quantiles. The estimator of  $F_Y$  in the DR baseline is

$$(3.7) \quad \widehat{F}_Y^e(y | x, z_1, v) = \Gamma(w(x, z_1, v)' \widehat{\beta}^e(y)), \quad (y, x, z_1, v) \in \mathcal{Y} \overline{\mathcal{X}} \overline{\mathcal{Z}}_1 \overline{\mathcal{V}},$$

$$(3.8) \quad \widehat{\beta}^e(y) \in \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \sum_{i=1}^n e_i T_i \left[ 1(Y_i \leq y) \log \Gamma(\widehat{W}_i^{e'} \beta) + 1(Y_i > y) \log \left( 1 - \Gamma(\widehat{W}_i^{e'} \beta) \right) \right].$$

When  $e_i = 1$  for all  $i = 1, \dots, n$ , expressions (3.5)-(3.6) and (3.7)-(3.8) define  $\widehat{F}_Y$ , the quantile and distribution regression estimators of  $F_Y$ , respectively.

**3.3. Third Stage: Estimation of Structural Functions.** Given the estimators  $(\{\widehat{V}_i\}_{i=1}^n, \widehat{F}_Y)$  and their bootstrap draws  $(\{\widehat{V}_i^e\}_{i=1}^n, \widehat{F}_Y^e)$ , we can form estimators of the structural functions as functionals of these building blocks.

The estimator and bootstrap draw of the DSF are

$$(3.9) \quad \widehat{G}(y, x) = \frac{1}{n_T} \sum_{i=1}^n \widehat{F}_Y(y | x, Z_{1i}, \widehat{V}_i) T_i,$$

where  $n_T = \sum_{i=1}^n T_i$ , and

$$(3.10) \quad \widehat{G}^e(y, x) = \frac{1}{n_T^e} \sum_{i=1}^n e_i \widehat{F}_Y^e(y | x, Z_{1i}, \widehat{V}_i^e) T_i,$$

where  $n_T^e = \sum_{i=1}^n e_i T_i$ . For the DR estimator,  $y \mapsto \widehat{G}(y, x)$  may not be monotonic. This can be addressed by applying the rearrangement method of Chernozhukov, Fernandez-Val and Galichon (2010).

Given the DSF estimate and bootstrap draw,  $\widehat{G}(y, x)$  and  $\widehat{G}^e(y, x)$ , the estimator and bootstrap draw of the QSF are

$$(3.11) \quad \widehat{Q}(\tau, x) = \int_{y^+} 1\{\widehat{G}(y, x) \leq \tau\} \nu(dy) - \int_{y^-} 1\{\widehat{G}(y, x) \geq \tau\} \nu(dy),$$

and

$$(3.12) \quad \widehat{Q}^e(\tau, x) = \int_{y^+} 1\{\widehat{G}^e(y, x) \leq \tau\} \nu(dy) - \int_{y^-} 1\{\widehat{G}^e(y, x) \geq \tau\} \nu(dy),$$

respectively. Finally, the estimator and bootstrap draw of the ASF are

$$(3.13) \quad \widehat{\mu}(x) = \int_{y^+} [1 - \widehat{G}(y, x)] \nu(dy) - \int_{y^-} \widehat{G}(y, x) \nu(dy),$$

and

$$(3.14) \quad \widehat{\mu}^e(x) = \int_{y^+} [1 - \widehat{G}^e(y, x)] \nu(dy) - \int_{y^-} \widehat{G}^e(y, x) \nu(dy),$$

respectively. When the set  $\mathcal{Y}$  is uncountable, we approximate the previous integrals by sums over a fine mesh of equidistant points  $\mathcal{Y}_S := \{\inf[y \in \mathcal{Y}] = y_1 < \dots < y_S = \sup[y \in \mathcal{Y}]\}$  with mesh width  $\delta$  such that  $\delta\sqrt{n} \rightarrow 0$ . For example, (3.12) and (3.14) are approximated by

$$(3.15) \quad \widehat{Q}_S^e(\tau, x) = \delta \sum_{s=1}^S \left[ 1(y_s \geq 0) - 1\{\widehat{G}^e(y_s, x) \geq \tau\} \right],$$

and

$$(3.16) \quad \widehat{\mu}_S^e(x) = \delta \sum_{s=1}^S \left[ 1(y_s \geq 0) - \widehat{G}^e(y_s, x) \right].$$

**3.4. Weighted Bootstrap Inference on Structural Functions.** We consider inference uniform over regions of values of  $(y, x, \tau)$ . We denote the region of interest as  $\mathcal{I}_G$  for the DSF,  $\mathcal{I}_Q$  for the QSF, and  $\mathcal{I}_\mu$  for the ASF. Examples include:

- (1) The DSF,  $y \mapsto \widehat{G}^e(y, x)$ , for fixed  $x$  and over  $y \in \widetilde{\mathcal{Y}} \subset \mathcal{Y}$ , by setting  $\mathcal{I}_G = \widetilde{\mathcal{Y}} \times \{x\}$ .
- (2) The QSF,  $x \mapsto \widehat{Q}^e(\tau, x)$  for fixed  $x$  and over  $\tau \in \widetilde{\mathcal{T}} \subset (0, 1)$ , by setting  $\mathcal{I}_Q = \widetilde{\mathcal{T}} \times \{x\}$ ,

(3) The ASF,  $\hat{\mu}^e(x)$ , over  $x \in \tilde{\mathcal{X}} \subset \bar{\mathcal{X}}$ , by setting  $\mathcal{I}_\mu = \tilde{\mathcal{X}}$ .

When the region of interest is not a finite set, we approximate it by a finite grid. All the details of the procedure we implement are summarized in Algorithm 1 in Appendix A.

The weighted bootstrap versions of the DSF, QSF and ASF estimators are obtained by rerunning the estimation procedure introduced in Section 3.3 with sampling weights drawn from a distribution that satisfies Assumption 3 in Section 4; see Algorithm 2 in Appendix A for details. They can then be used to perform uniform inference over the region of interest.

For instance, a  $(1 - \alpha)$ -confidence band for the DSF over the region  $\mathcal{I}_G$  can be constructed as

$$(3.17) \quad \left[ \hat{G}(y, x) \pm \hat{k}_G(1 - \alpha)\hat{\sigma}_G(y, x), (y, x) \in \mathcal{I}_G \right],$$

where  $\hat{\sigma}_G(y, x)$  is an estimator of  $\sigma_G(y, x)$ , the asymptotic standard deviation of  $\hat{G}(y, x)$ , such as the rescaled weighted bootstrap interquartile range

$$(3.18) \quad \hat{\sigma}_G(y, x) = \text{IQR} \left[ \hat{G}^e(y, x) \right] / 1.349,$$

and  $\hat{k}_G(1 - \alpha)$  denote a consistent estimator of the  $(1 - \alpha)$ -quantile of the maximal  $t$ -statistic

$$\|t_G(y, x)\|_{\mathcal{I}_G} = \sup_{(y, x) \in \mathcal{I}_G} \left| \frac{\hat{G}(y, x) - G(y, x)}{\sigma_G(y, x)} \right|,$$

such as the  $(1 - \alpha)$ -quantile of the bootstrap draw of the maximal  $t$ -statistic

$$(3.19) \quad \|t_G^e(y, x)\|_{\mathcal{I}_G} = \sup_{(y, x) \in \mathcal{I}_G} \left| \frac{\hat{G}^e(y, x) - \hat{G}(y, x)}{\hat{\sigma}_G(y, x)} \right|.$$

Confidence bands for the ASF can be constructed by a similar procedure, using the bootstrap draws of the ASF estimator. For the QSF, we can either use the same procedure based on the bootstrap draws of the QSF, or invert the confidence bands for the DSF following the generic method of Chernozhukov et al (2016). The first possibility works only when  $Y$  is continuous, whereas the second method is more generally applicable. We provide algorithms for the construction of the bands in Appendix A.

#### 4. ASYMPTOTIC THEORY

We derive asymptotic theory for the estimators of the ASF, DSF and QSF where both the first and second stages are based on DR. The theory for the estimators based on QR can be derived using similar arguments.

In what follows, we shall use the following notation. We let the random vector  $A = (Y, X, Z, W, V)$  live on some probability space  $(\Omega_0, \mathcal{F}_0, P)$ . Thus, the probability measure  $P$  determines the law of  $A$  or any of its elements. We also let  $A_1, \dots, A_n$ , i.i.d. copies of  $A$ , live on the complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which contains the infinite product of  $(\Omega_0, \mathcal{F}_0, P)$ . Moreover, this probability space can be suitably enriched to carry also the random weights that appear in the weighted bootstrap. The distinction between the two laws  $P$  and  $\mathbb{P}$  is helpful to simplify the notation in the proofs and in the analysis. Unless explicitly mentioned, all functions appearing in the statements are assumed to be measurable.

We now state formally the assumptions. The first assumption is about sampling and the bootstrap weights.

**Assumption 3.** *[Sampling and Bootstrap Weights] (a) Sampling: the data  $\{Y_i, X_i, Z_i\}_{i=1}^n$  are a sample of size  $n$  of independent and identically distributed observations from the random vector  $(Y, X, Z)$ . (b) Bootstrap weights:  $(e_1, \dots, e_n)$  are i.i.d. draws from a random variable  $e \geq 0$ , with  $E_P[e] = 1$ ,  $\text{Var}_P[e] = 1$ , and  $E_P|e|^{2+\delta} < \infty$  for some  $\delta > 0$ ; live on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ; and are independent of the data  $\{Y_i, X_i, Z_i\}_{i=1}^n$  for all  $n$ .*

The second assumption is about the first stage where we estimate the control function  $(x, z) \mapsto \vartheta_0(x, z)$  defined as

$$\vartheta_0(x, z) := F_X(x \mid z),$$

with trimmed support  $\bar{\mathcal{V}} = \{\vartheta_0(x, z) : (x, z) \in \bar{\mathcal{X}}\bar{\mathcal{Z}}\}$ . We assume a logistic DR model for the conditional distribution of  $X$  in the trimmed support  $\bar{\mathcal{X}}$ .

**Assumption 4.** *[First Stage] (a) Trimming: we consider a trimming rule defined by the tail indicator*

$$T = 1(X \in \bar{\mathcal{X}}),$$

where  $\bar{\mathcal{X}} = [\underline{x}, \bar{x}]$  for some  $-\infty < \underline{x} < \bar{x} < \infty$ , such that  $P(T = 1) > 0$ . (b) Model: the distribution of  $X$  conditional on  $Z$  follows Assumption 1(b) with  $\Gamma = \Lambda$  in the trimmed support, where  $\Lambda$  is the logit link function; the coefficients  $x \mapsto \pi_0(x)$  are three



times continuously differentiable with uniformly bounded derivatives;  $\overline{\mathcal{R}}$  is compact; and the minimum eigenvalue of  $\mathbb{E}_P[\Lambda(R'\pi_0(x))[1 - \Lambda(R'\pi_0(x))]RR']$  is bounded away from zero uniformly over  $x \in \overline{\mathcal{X}}$ .

For  $x \in \overline{\mathcal{X}}$ , let

$$\widehat{\pi}^e(x) \in \arg \min_{\pi \in \mathbb{R}^{\dim(R)}} \frac{1}{n} \sum_{i=1}^n e_i \{1(X_i \leq x) \log \Lambda(R'_i \pi) + 1(X_i > x) \log [1 - \Lambda(R'_i \pi)]\},$$

and set

$$\vartheta_0(x, r) = \Lambda(r'\pi_0(x)); \quad \widehat{\vartheta}^e(x, r) = \Lambda(r'\widehat{\pi}^e(x)),$$

if  $(x, r) \in \overline{\mathcal{X}\mathcal{R}}$ , and  $\vartheta_0(x, r) = \widehat{\vartheta}^e(x, r) = 0$  otherwise.

Theorem 4 of Chernozhukov, Fernandez-Val and Kowalski (2015) established the asymptotic properties of the DR estimator of the control function. We repeat the result here as a lemma for completeness and to introduce notation that will be used in the results below. Let  $T(x) := 1(x \in \overline{\mathcal{X}})$ ,  $\|f\|_{T, \infty} := \sup_{a \in \mathcal{A}} |T(x)f(a)|$  for any function  $f : \mathcal{A} \mapsto \mathbb{R}$ , and  $\lambda = \Lambda(1 - \Lambda)$ , the density of the logistic distribution.

**Lemma 2.** *[First Stage] Suppose that Assumptions 3 and 4 hold. Then, (1)*

$$\begin{aligned} \sqrt{n}(\widehat{\vartheta}^e(x, r) - \vartheta_0(x, r)) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \ell(A_i, x, r) + o_{\mathbb{P}}(1) \rightsquigarrow \Delta^e(x, r) \text{ in } \ell^\infty(\overline{\mathcal{X}\mathcal{R}}), \\ \ell(A, x, r) &:= \lambda(r'\pi_0(x)) [1\{X \leq x\} - \Lambda(R'\pi_0(x))] \times \\ &\quad \times r' \mathbb{E}_P \{ \Lambda(R'\pi_0(x)) [1 - \Lambda(R'\pi_0(x))] RR' \}^{-1} R, \\ \mathbb{E}_P[\ell(A, x, r)] &= 0, \mathbb{E}_P[T\ell(A, X, R)^2] < \infty, \end{aligned}$$

where  $(x, r) \mapsto \Delta^e(x, r)$  is a Gaussian process with uniformly continuous sample paths and covariance function given by  $\mathbb{E}_P[\ell(A, x, r)\ell(A, \tilde{x}, \tilde{r})']$ . (2) There exists  $\widetilde{\vartheta}^e : \overline{\mathcal{X}\mathcal{R}} \mapsto [0, 1]$  that obeys the same first order representation uniformly over  $\overline{\mathcal{X}\mathcal{R}}$ , is close to  $\widehat{\vartheta}^e$  in the sense that  $\|\widetilde{\vartheta}^e - \widehat{\vartheta}^e\|_{T, \infty} = o_{\mathbb{P}}(1/\sqrt{n})$  and, with probability approaching one, belongs to a bounded function class  $\Upsilon$  such that

$$\log N(\epsilon, \Upsilon, \|\cdot\|_{T, \infty}) \lesssim \epsilon^{-1/2}, \quad 0 < \epsilon < 1.$$

The next assumptions are about the second stage. We assume a logistic DR model for the conditional distribution of  $Y$  given  $(X, Z_1, V)$ , impose compactness and smoothness conditions, and provide sufficient conditions for identification of the parameters. Compactness is imposed over the trimmed supports and can be relaxed at the cost of more complicated and cumbersome proofs. The smoothness conditions are fairly

tight. The assumptions on  $\mathcal{Y}$  cover continuous, discrete and mixed outcomes in the second stage. We denote partial derivatives as  $\partial_x f(x, y) := \partial f(x, y) / \partial x$ .

**Assumption 5.** [Second Stage] (a) *Model:* the distribution of  $Y$  conditional on  $(X, Z_1, V)$  follows Assumption 1(b) with  $\Gamma = \Lambda$ . (b) *Compactness and smoothness:* the set  $\overline{\mathcal{XZ}}\mathcal{W}$  is compact; the set  $\mathcal{Y}$  is either a compact interval in  $\mathbb{R}$  or a finite subset of  $\mathbb{R}$ ;  $X$  has a continuous conditional density function  $x \mapsto f_X(x | z)$  that is bounded above by a constant uniformly in  $z \in \overline{\mathcal{Z}}$ ; if  $\mathcal{Y}$  is an interval, then  $Y$  has a conditional density function  $y \mapsto f_Y(y | x, z)$  that is uniformly continuous in  $y \in \mathcal{Y}$  uniformly in  $(x, z) \in \overline{\mathcal{XZ}}$ , and bounded above by a constant uniformly in  $(x, z) \in \overline{\mathcal{XZ}}$ ; the derivative vector  $\partial_v w(x, z_1, v)$  exists and its components are uniformly continuous in  $v \in \overline{\mathcal{V}}$  uniformly in  $(x, z_1) \in \overline{\mathcal{XZ}_1}$ , and are bounded in absolute value by a constant, uniformly in  $(x, w, v) \in \overline{\mathcal{XZ}_1\mathcal{V}}$ ; and for all  $y \in \mathcal{Y}$ ,  $\beta_0(y) \in \mathcal{B}$ , where  $\mathcal{B}$  is a compact subset of  $\mathbb{R}^{\dim(W)}$ . (c) *Identification and nondegeneracy:* Assumption 2 holds conditional on  $T = 1$ , and the matrix  $C(y, v) := \text{Cov}_P[f_y(A) + g_y(A), f_v(A) + g_v(A)]$  is finite and is of full rank uniformly in  $y, v \in \mathcal{Y}$ , where

$$f_y(A) := \{\Lambda(W'\beta_0(y)) - 1(Y \leq y)\}WT,$$

and, for  $\dot{W} = \partial_v w(X, Z_1, v)|_{v=V}$ ,

$$g_y(A) := E_P[\{[\Lambda(W'\beta_0(y)) - 1(Y \leq y)]\dot{W} + \lambda(W'\beta_0(y))\dot{W}'\beta_0(y)W\}T\ell(a, X, R)]|_{a=A}.$$

For  $y \in \mathcal{Y}$ , let

$$\widehat{\beta}(y) = \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \frac{1}{n} \sum_{i=1}^n T_i \rho_y(Y_i, \beta' \widehat{W}_i), \quad \widehat{W}_i = w(X_i, Z_{1i}, \widehat{V}_i), \quad \widehat{V}_i = \widehat{\vartheta}(X_i, R_i),$$

where

$$\rho_y(Y, B) := -\{1(Y \leq y) \log \Lambda(B) + 1(Y > y) \log[1 - \Lambda(B)]\},$$

and  $\widehat{\vartheta}$  is the estimator of the control function in the unweighted sample; and

$$\widehat{\beta}^e(y) = \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \frac{1}{n} \sum_{i=1}^n e_i T_i \rho_y(Y_i, \beta' \widehat{W}_i^e), \quad \widehat{W}_i^e = w(X_i, Z_{1i}, \widehat{V}_i^e), \quad \widehat{V}_i^e = \widehat{\vartheta}^e(X_i, R_i),$$

where  $\widehat{\vartheta}^e$  is the estimator of the control function in the weighted sample.

The following lemma establishes a functional central limit theorem and a functional central limit theorem for the bootstrap for the estimator of the DR coefficients in the second stage. Let  $d_w := \dim(W)$ , and  $\ell^\infty(\mathcal{Y})$  be the set of all uniformly bounded real

functions on  $\mathcal{Y}$ . We use  $\rightsquigarrow_{\mathbb{P}}$  to denote bootstrap consistency, i.e. weak convergence conditional on the data in probability, which is formally defined in Appendix C.1.

**Lemma 3.** *[FCLT and Bootstrap FCLT for  $\widehat{\beta}(y)$ ] Under Assumptions 1–5, in  $\ell^\infty(\mathcal{Y})^{d_w}$ ,*

$$\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) \rightsquigarrow J(y)^{-1}G(y), \quad \text{and} \quad \sqrt{n}(\widehat{\beta}^e(y) - \widehat{\beta}(y)) \rightsquigarrow_{\mathbb{P}} J(y)^{-1}G(y),$$

where  $y \mapsto G(y)$  is a  $d_w$ -dimensional zero-mean Gaussian process with uniformly continuous sample paths and covariance function

$$\mathbb{E}_P[G(y)G(v)'] = C(y, v), \quad y, v \in \mathcal{Y}.$$

We consider now the estimators of the main quantities of interest – the structural functions. Let  $W_x := w(x, Z_1, V)$ ,  $\widehat{W}_x := w(x, Z_1, \widehat{V})$ , and  $\widehat{W}_x^e := w(x, Z_1, \widehat{V}^e)$ . The DR estimator and bootstrap draw of the DSF in the trimmed support,  $G_T(y, x) = \mathbb{E}_P\{\Lambda[\beta_0(y)'W_x] \mid T = 1\}$ , are  $\widehat{G}(y, x) = \sum_{i=1}^n \Lambda[\widehat{\beta}(y)'\widehat{W}_{xi}]T_i/n_T$ , and  $\widehat{G}^e(y, x) = \sum_{i=1}^n e_i \Lambda[\widehat{\beta}^e(y)'\widehat{W}_{xi}^e]T_i/n_T^e$ . Let  $p_T := P(T = 1)$ . The next result gives large sample theory for these estimators.

**Theorem 2** (FCLT and Bootstrap FCLT for DSF). *Under Assumptions 1–5, in  $\ell(\mathcal{Y}\overline{\mathcal{X}})$ ,*

$$\sqrt{np_T}(\widehat{G}(y, x) - G_T(y, x)) \rightsquigarrow Z(y, x) \quad \text{and} \quad \sqrt{np_T}(\widehat{G}^e(y, x) - \widehat{G}(y, x)) \rightsquigarrow_{\mathbb{P}} Z(y, x),$$

where  $(y, x) \mapsto Z(y, x)$  is a zero-mean Gaussian process with covariance function

$$\text{Cov}_P[\Lambda[W'_x\beta_0(y)] + h_{y,x}(A), \Lambda[W'_u\beta_0(v)] + h_{v,u}(A) \mid T = 1],$$

with

$$h_{y,x}(A) = \mathbb{E}_P\{\lambda[W'_x\beta_0(y)]W_x T\}^{-1}[f_y(A) + g_y(A)] + \mathbb{E}_P\{\lambda[W'_x\beta_0(y)]\dot{W}'_x\beta_0(y)T\ell(a, X, R)\}\Big|_{a=A}.$$

When  $Y$  is continuous and  $y \mapsto G_T(y, x)$  is strictly increasing, we can also characterize the asymptotic distribution of  $\widehat{Q}(\tau, x)$ , the estimator of the QSF in the trimmed support. Let  $g_T(y, x)$  be the density of  $y \mapsto G_T(y, x)$ ,  $\overline{\mathcal{T}} := \{\tau \in (0, 1) : Q(\tau, x) \in \mathcal{Y}, g_T(Q(\tau, x), x) > \epsilon, x \in \overline{\mathcal{X}}\}$  for fixed  $\epsilon > 0$ , and  $Q_T(\tau, x)$  the QSF in the trimmed support  $\overline{\mathcal{T}\mathcal{X}}$  defined as

$$Q_T(\tau, x) = \int_{\mathcal{Y}^+} 1\{G_T(y, x) \leq \tau\}dy - \int_{\mathcal{Y}^-} 1\{G_T(y, x) \geq \tau\}dy.$$

The estimator and its bootstrap draw given in (3.11)-(3.12) follow the functional central limit theorem:

**Theorem 3** (FCLT and Bootstrap FCLT for QSF). *Assume that  $y \mapsto G_T(y, x)$  is strictly increasing in  $\bar{\mathcal{Y}}$  and  $(y, x) \mapsto G_T(y, x)$  is continuously differentiable in  $\bar{\mathcal{Y}}\bar{\mathcal{X}}$ . Under Assumptions 1–5, in  $\ell^\infty(\bar{\mathcal{T}}\bar{\mathcal{X}})$ ,*

$$\sqrt{np_T}(\widehat{Q}(\tau, x) - Q_T(\tau, x)) \rightsquigarrow -\frac{Z(Q(\tau, x), x)}{g_T(Q(\tau, x), x)} \text{ and}$$

$$\sqrt{np_T}(\widehat{Q}^e(\tau, x) - \widehat{Q}(\tau, x)) \rightsquigarrow_{\mathbb{P}} -\frac{Z(Q(\tau, x), x)}{g_T(Q(\tau, x), x)},$$

where  $(y, x) \mapsto Z(y, x)$  is the same Gaussian process as in Theorem 2.

Finally, we consider the ASF in the trimmed support

$$\mu_T(x) = \int_{\mathcal{Y}^+} [1 - G_T(y, x)]\nu(dy) - \int_{\mathcal{Y}^-} G_T(y, x)\nu(dy).$$

The estimator and its bootstrap draw given in (3.13)-(3.14) follow the functional central limit theorem:

**Theorem 4** (FCLT and Bootstrap FCLT for ASF). *Under Assumptions 1–5, in  $\ell^\infty(\bar{\mathcal{X}})$ ,*

$$\sqrt{np_T}(\widehat{\mu}(x) - \mu_T(x)) \rightsquigarrow -\int_{\mathcal{Y}} Z(y, x)\nu(dy) \text{ and}$$

$$\sqrt{np_T}(\widehat{\mu}^e(x) - \widehat{\mu}(x)) \rightsquigarrow_{\mathbb{P}} -\int_{\mathcal{Y}} Z(y, x)\nu(dy),$$

where  $(y, x) \mapsto Z(y, x)$  is the same Gaussian process as in Theorem 2.

## 5. NUMERICAL ILLUSTRATIONS

**5.1. Empirical Application: Engel Curves for Food and Leisure Expenditure.** In this section we apply our methods to the estimation of a semiparametric nonseparable triangular model for Engel curves. We focus on the structural relationship between household's total expenditure and household's demand for two goods: food and leisure. We take the outcome  $Y$  to be the expenditure share on either food or leisure, and  $X$  the logarithm of total expenditure. Following Blundell, Chen and Kristensen (2007) we use as an exclusion restriction the logarithm of gross earnings of

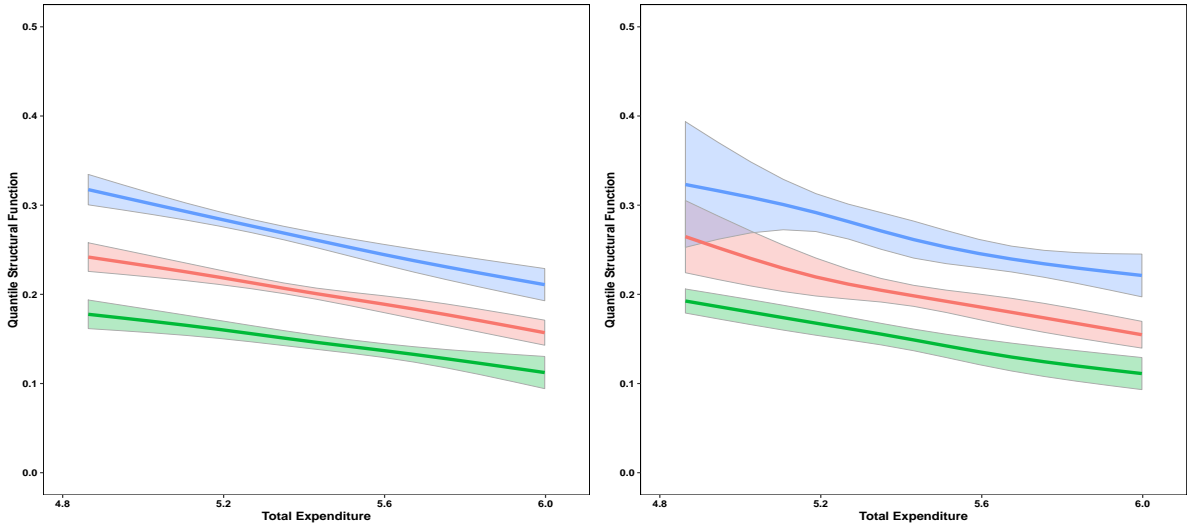
the head of household. We also include an additional binary covariate  $Z_1$  accounting for the presence of children in the household.

There is an extensive literature on Engel curve estimation (e.g., see Lewbel (2006) for a review), and the use of nonseparable triangular models for the identification and estimation of Engel curves has been considered in the recent literature. Blundell, Chen and Kristensen (2007) estimate semi-nonparametrically Engel curves for several categories of expenditure, Imbens and Newey (2009) estimate the QSF nonparametrically for food and leisure, and Chernozhukov, Fernandez-Val and Kowalski (2015) estimate Engel curves for alcohol accounting for censoring. For comparison purposes we use the same dataset as these papers, the 1995 U.K. Family Expenditure Survey. We restrict the sample to 1,655 married or cohabiting couples with two or fewer children, in which the head of the household is employed and between the ages of 20 and 55 years. For this sample we estimate the DSF, QSF and ASF for both goods. Unlike Imbens and Newey (2009) we also account for the presence of children in the household and we impose semiparametric restrictions through our baseline models. In contrast to Chernozhukov, Fernandez-Val and Kowalski (2015), we do not impose separability between the control function and other regressors, and we estimate the structural functions.

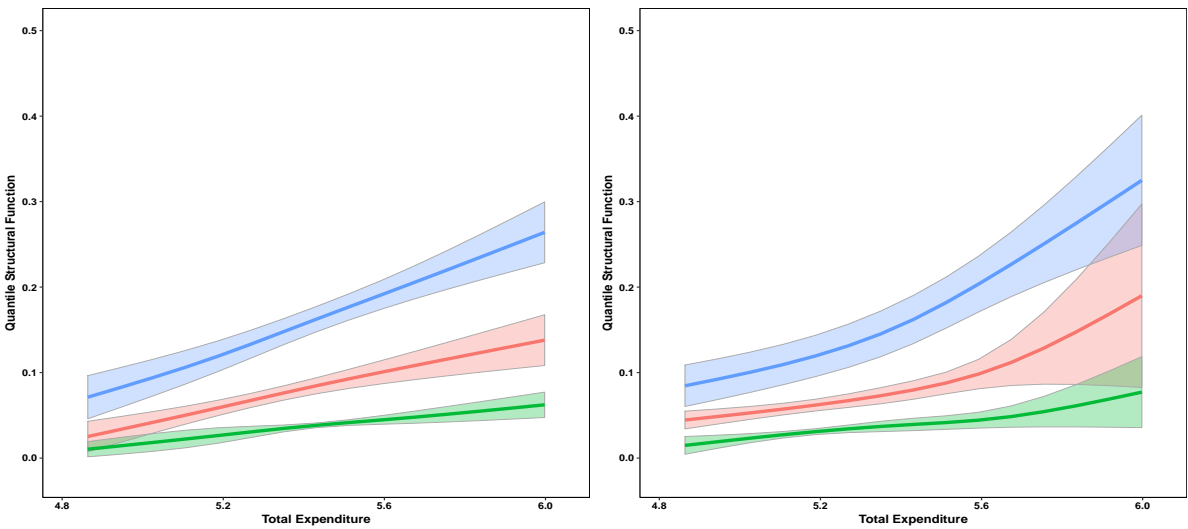
All structural functions are estimated by both QR and DR methods, following exactly the description of the implementation presented in Section 3 with the specifications  $r(Z) = (1, Z)'$ ,  $r_1(Z_1) = (1, Z_1)'$ ,  $p(X) = (1, X)'$ , and  $q(V) = (1, \Phi^{-1}(V))'$ . We set  $M = 599$  and  $\epsilon = 0.01$  in Algorithm 1, approximate the integrals using  $S = 599$  points, and run  $B = 199$  bootstrap replications in Algorithm 2 for both methods. The regions of interest are  $\tilde{\mathcal{X}} = [\hat{Q}_X(0.1), \hat{Q}_X(0.9)]$  and  $\tilde{\mathcal{Y}} = [\hat{Q}_Y(0.1), \hat{Q}_Y(0.9)]$ , where  $\hat{Q}_X(u)$  and  $\hat{Q}_Y(u)$  are the sample  $u$ -quantiles of  $X$  and  $Y$ . We approximate  $\tilde{\mathcal{X}}$  by a grid  $\tilde{\mathcal{X}}_K$  with  $K = 3, 5$ , and  $\tilde{\mathcal{Y}}$  by a grid  $\tilde{\mathcal{Y}}_{15}$ . We estimate the structural functions and perform uniform inference over the following regions:

- (1) For the QSF,  $\hat{Q}(\tau, x)$ , we take  $\tilde{\mathcal{T}} = \{0.25, 0.5, 0.75\}$ , and then set:  $\mathcal{I}_Q = \tilde{\mathcal{T}}\tilde{\mathcal{X}}_5$ .
- (2) For the DSF,  $\hat{G}(y, x)$ , we set:  $\mathcal{I}_G = \tilde{\mathcal{Y}}_{15}\tilde{\mathcal{X}}_3$ .
- (3) For the ASF,  $\hat{\mu}(x)$ , we set:  $\mathcal{I}_\mu = \tilde{\mathcal{X}}_5$ .

We implement the DR estimator using the logit link function. Since the estimated DSF may be non-monotonic in  $y$ , we apply rearrangement to  $y \mapsto \hat{G}(y, x)$  at each value of  $x$  in  $\mathcal{I}_G$ . None of the methods uses trimming, that is we set  $T = 1$  a.s.



(A) Food.



(B) Leisure.

FIGURE 5.1. QSF. Quantile (left) and distribution regression (right).

Figures 5.1-5.3 show the QSF, ASF and DSF for both goods<sup>2</sup>. For each structural function, we report weighted bootstrap 90%-confidence bands that are uniform over the corresponding region specified above. Our empirical results illustrate that QR and DR specifications are able to capture different features of structural functions, and are therefore complementary. For food, both estimation methods deliver very similar

<sup>2</sup>For graphical representation the QSF and ASF are interpolated by splines over  $\bar{\mathcal{X}}$  and the DSF over  $\bar{\mathcal{Y}}$ .

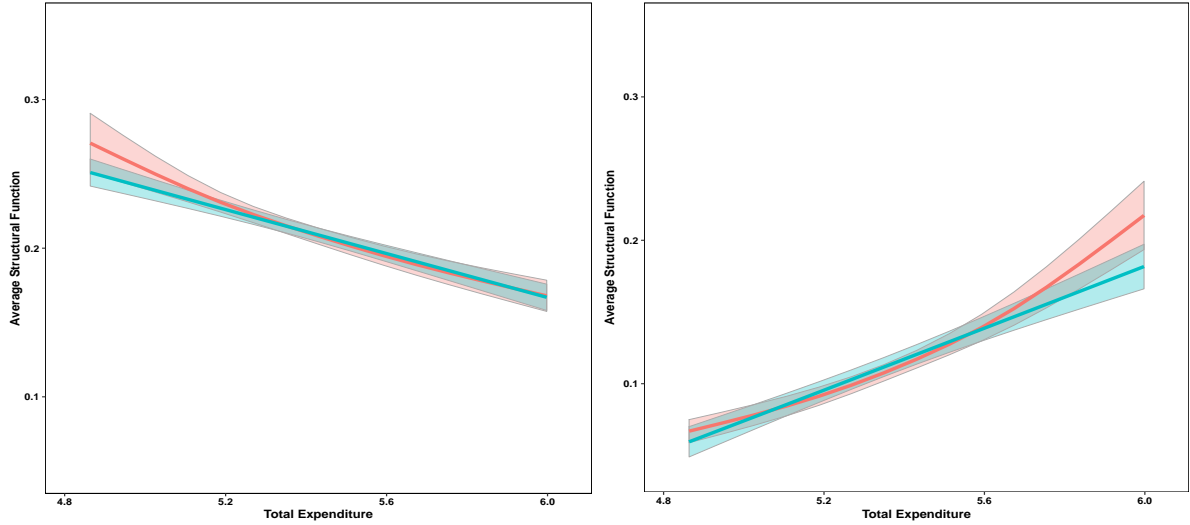
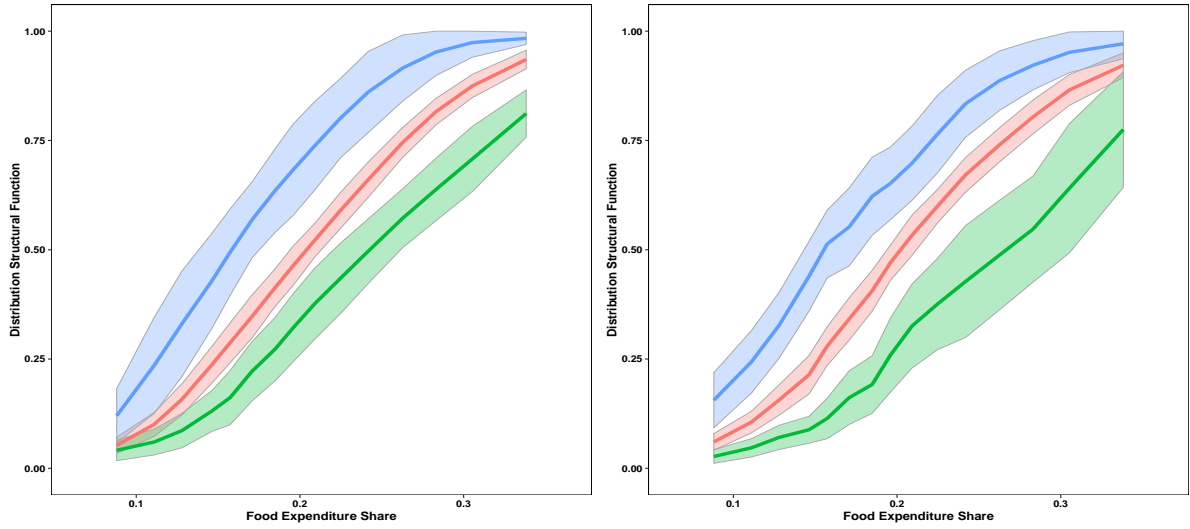


FIGURE 5.2. ASF for food (left) and leisure (right). Quantile (blue) and distribution regression (red).

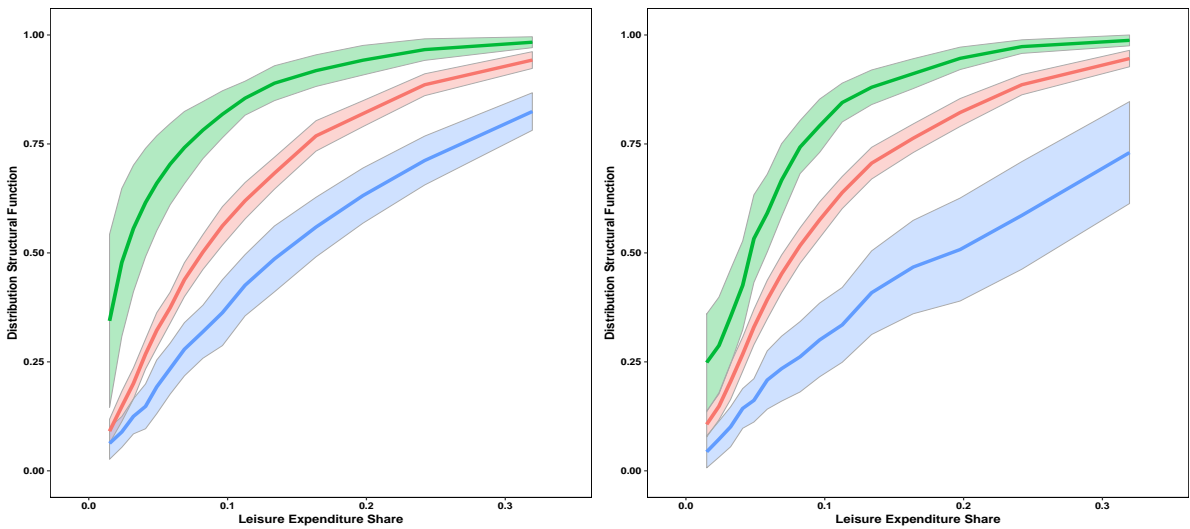
of the QSF, close to being linear, although linearity is not imposed in the estimation procedure. For leisure, the QSF and ASF estimated by DR are able to capture some nonlinearity which is absent from those obtained by QR. For QR, this reflects the specified linear structure of the ASF which also constrains the shape of the QSF. In addition, some degree of heteroskedasticity appears to be a feature of the structural model for both goods, although much more markedly for leisure, so our methods are well-suited for this problem. Increased dispersion across quantile levels in Figure 5.1 is reflected by the increasing spread across probability levels between the two extreme DSF estimates in Figure 5.3. Finally, our semiparametric specifications are able to capture the asymmetry across leisure expenditure shares, an important feature of the data highlighted in Imbens and Newey (2009).

In the Supplementary Material we perform a thorough sensitivity analysis which further shows that our empirical results are robust to the modeling, estimation and integration choices. Overall, for this dataset, the main features of food and leisure Engel curves are well captured by our semiparametric specifications.

**5.2. Numerical Simulations.** To assess the performance of our estimators we implement Monte Carlo experiments based on three different designs, calibrated to the leisure empirical application. The first two experiments are based on Gaussian location-scale and DR triangular models, designed to reflect the respective strengths



(A) Food.



(B) Leisure.

FIGURE 5.3. DSF. Quantile (left) and distribution regression (right).

of the QR and DR estimators. The third experiment is a location triangular model, for which both estimators are consistent for the corresponding structural functions.

**Design QR.** Our first design is the linear location-scale shift system of equations

$$\begin{aligned}
 X &= \pi_{11} + \pi_{21}Z + (\pi_{12} + \pi_{22}Z)\eta, \\
 Y &= \theta_{11} + \theta_{21}X + (\theta_{12} + \theta_{22}X)\varepsilon.
 \end{aligned}$$



The ASF and QSF of this model are linear,

$$\mu(x) = \theta_{11} + \theta_{21}x, \quad Q(\tau, x) = \theta_{11} + \theta_{21}x + (\theta_{12} + \theta_{22}x)\Phi^{-1}(\tau).$$

**Design DR.** Our second design is the nonlinear location-scale shift system of equations

$$\begin{aligned} X &= -\left(\frac{\pi_{11} + \pi_{12}Z}{\pi_{21} + \pi_{22}Z}\right) + \left(\frac{1}{\pi_{21} + \pi_{22}Z}\right)\eta, \\ Y &= -\left(\frac{\theta_{11} + \theta_{12}x}{\theta_{21} + \theta_{22}x}\right) + \left(\frac{1}{\theta_{21} + \theta_{22}x}\right)\varepsilon. \end{aligned}$$

The ASF and QSF of this model are nonlinear,

$$\mu(x) = -\left(\frac{\theta_{11} + \theta_{12}x}{\theta_{21} + \theta_{22}x}\right), \quad Q(\tau, x) = -\left(\frac{\theta_{11} + \theta_{12}x}{\theta_{21} + \theta_{22}x}\right) + \left(\frac{1}{\theta_{21} + \theta_{22}x}\right)\Phi^{-1}(\tau).$$

**Design LOC.** Our third design is the linear location shift system of equations

$$\begin{aligned} X &= \pi_{11} + \pi_{21}Z + \sigma_{\eta}\eta, \\ Y &= \theta_{11} + \theta_{21}X + \sigma_{\varepsilon}\varepsilon, \end{aligned}$$

for which the QR and DR models are correctly specified. The ASF and QSF of this model are

$$\mu(x) = \theta_{11} + \theta_{21}x, \quad Q(\tau, x) = \theta_{11} + \theta_{21}x + \sigma_{\varepsilon}\Phi^{-1}(\tau).$$

For all three experiments, the sample size is set to  $n = 1655$ , the number of observations in the empirical application, and 500 simulations are performed. For the regions of interest, we use the same  $\mathcal{T}_3$  and  $\bar{\mathcal{X}}_5$  as in the empirical application. We let  $(\eta, \varepsilon)$  be jointly normal scalar random variables with zero means, unit variances and correlation  $\rho$ , and assess the performance of our estimators under two different levels of endogeneity by setting  $\rho = -0.2$ , for low endogeneity, and  $\rho = -0.9$ , for extreme endogeneity. Accordingly, the DR estimator is implemented with the probit link function. For brevity, in the main text we only report simulation results for the ASF which reflect the main features of our simulations for the QSF as well. A detailed discussion of the calibration of these models and simulation results for the QSF are given in the Supplemental Material.

Table 1 reports a first set of results regarding the accuracy of ASF estimates by DR and QR. For comparison purposes, Table 1 also includes ASF estimates by ordinary least-squares (OLS), providing a benchmark with no correction for endogeneity. We report average estimation errors across simulations of QR and DR estimators, and

Design	QR			LOC			DR		
	$\rho = -0.2$								
	$L_1$	$L_2$	$L_\infty$	$L_1$	$L_2$	$L_\infty$	$L_1$	$L_2$	$L_\infty$
DR	6.9	8.1	15.2	5.2	6.7	8.3	5.9	7.4	10.2
QR	2.7	3.4	3.5	4.7	6.0	6.9	8.2	9.7	15.4
Ratio $\times 100$	251.1	237.2	426.8	110.3	111.9	121.2	72.4	76.6	66.2
OLS	10.6	10.9	22.4	14.9	15.4	26.0	15.4	16.0	33.0
	$\rho = -0.9$								
	$L_1$	$L_1$	$L_\infty$	$L_1$	$L_2$	$L_\infty$	$L_1$	$L_2$	$L_\infty$
DR	4.7	6.0	7.7	6.4	7.9	10.5	7.8	9.5	13.6
QR	3.8	4.5	9.4	4.9	6.0	7.3	9.2	10.5	24.4
Ratio $\times 100$	123.6	132.7	82.0	131.1	131.9	144.7	84.4	90.4	56.0
OLS	47.2	47.3	100.2	66.2	66.3	117.9	73.2	73.3	152.8

TABLE 1. Average  $L^p$  estimation errors of ASF  $\times 1000$  for the DR and QR estimators and their ratio  $\times 100$ , for  $p = 1, 2$  and  $\infty$ . Average  $L^p$  estimation errors of ASF  $\times 1000$  for OLS are included as a benchmark.

their ratio in percentage terms. Estimation errors are measured in  $L^p$  norms  $\|\cdot\|_p$ ,  $p = 1, 2$ , and  $\infty$ , where for a function  $f : \mathcal{X} \mapsto \mathbb{R}$ ,  $\|f\|_p = \left\{ \int_{\mathbb{R}} |f(s)|^p ds \right\}^{1/p}$ , and are then averaged over the 500 simulations.

For this design, DR and QR-based estimators both perform very well and significantly improve over the OLS benchmark, including for  $\rho = -0.2$ . As expected, the accuracy of the estimates obtained by each method dominates for the corresponding design. For the QR design, the ratio of average estimation errors ranges from 82 to 426.8. Interestingly, the relative accuracy of DR-based estimates for  $\rho = -0.9$  is close to the accuracy of QR estimates, with the ratio of average estimation errors ranging from 82 to 132.7, across norms; this feature is specific to the ASF and does not apply to the QSF. For the DR design, the ratio of average estimation errors ranges from 56 to 90.4. The larger reduction in average errors in  $L^\infty$  norm reflects the higher accuracy in estimation of extreme parts of the support where the ASF displays some curvature. Finally, for the LOC design, the performance of both methods is very similar for  $\rho = -0.2$ , and the QR-based estimator dominates more markedly for  $\rho = -0.9$ .

Overall, the simulations show that both DR- and QR-based estimation methods perform well for their respective designs, and yield substantial correction for endogeneity.

QR-based estimation dominates for both the QR and LOC designs, but the DR estimator is able to correct for endogeneity in data generating processes displaying nonlinearities in the structural functions. These simulation results illustrate further the complementarity of the two estimation methods introduced in this paper.

## APPENDIX A. IMPLEMENTATION ALGORITHMS

This section gathers the algorithms for the three-stage estimation procedure, weighted bootstrap, and the constructions of uniform bands for the structural functions.

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### Algorithm 1 *Three-Stage Estimation Procedure.*

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For  $i = 1, \dots, n$ , set  $e_i = 1$ .

**First Stage.** [Control function estimation]

- (1) (QR) For  $\epsilon$  in  $(0, 0.5)$  (e.g.,  $\epsilon = .01$ ) and a fine mesh of  $M$  values  $\{\epsilon = v_1 < \dots < v_M = 1 - \epsilon\}$ , estimate  $\{\hat{\pi}^e(v_m)\}_{m=1}^M$  by solving (3.2). Then set  $\hat{V}_i^e = \hat{F}_X^e(X_i | Z_i)$ ,  $i = 1, \dots, n$ , as in (3.1).
- (2) (DR) Estimate  $\{\hat{\pi}(X_i)\}_{i=1}^n$  by solving (3.4). Then set  $\hat{V}_i^e = \hat{F}_X^e(X_i | Z_i)$ ,  $i = 1, \dots, n$ , as in (3.3).

**Second Stage.** [Reduced-form CDF estimation]

- (1) (QR) (a) For  $\epsilon$  in  $(0, 0.5)$  (e.g.,  $\epsilon = .01$ ) and a fine mesh of  $M$  values  $\{\epsilon = u_1, \dots, u_M = 1 - \epsilon\}$ , estimate  $\{\hat{\beta}^e(u_m)\}_{m=1}^M$  by solving (3.6). (b) Obtain  $\hat{F}_Y^e(y | x, Z_{1i}, \hat{V}_i^e)$  as in (3.5)
- (2) (DR) (a) For each  $y_m \in \mathcal{Y}_M$ , estimate  $\{\hat{\beta}(y_m)\}_{m=1}^M$  by solving (3.8). (b) Obtain  $\hat{F}_Y^e(y | x, Z_{1i}, \hat{V}_i^e)$  as in (3.7).

**Third Stage.** [Structural functions estimation] Compute  $\hat{G}^e(y, x)$ ,  $\hat{Q}_S^e(\tau, x)$  and  $\hat{\mu}_S^e(x)$  using (3.10), (3.15) and (3.16).

---

*Remark 2.* The size of the grids  $M$  can differ across stages and methods. For our empirical application, we have found that the estimates are not very sensitive to  $M$ .

*Remark 3.* All the estimation steps can also be implemented keeping  $Z_1$ , or some component of  $Z_1$ , fixed as a conditioning variable. The estimated structural functions are then evaluated at values of the conditioning variable(s) of interest. Denoting the DSF estimator and bootstrap draw by  $\hat{G}(y, x, z_1) = \sum_{i=1}^n \hat{F}_Y^e(y | x, z_1, \hat{V}_i) T_i / n_T$  and  $\hat{G}^e(y, x, z_1) = \sum_{i=1}^n e_i \hat{F}_Y^e(y | x, z_1, \hat{V}_i^e) T_i / n_T^e$ , the corresponding QSF and ASF estimators and bootstrap draws obtain upon substituting  $\hat{G}(y, x, z_1)$  and  $\hat{G}^e(y, x, z_1)$  for  $\hat{G}(y, x)$  and  $\hat{G}^e(y, x)$  in (3.9)-(3.10).

*Remark 4.* For the QR specification, the estimator of the ASF in the second and third stages can be replaced by  $\widehat{\mu}(x) = w(x, \bar{Z}_1, 0)' \widehat{\beta}$ , where  $\bar{Z}_1 = \sum_{i=1}^n Z_{1i}/n$  and  $\widehat{\beta}$  the least squares estimator of the linear regression of  $Y$  on  $\widehat{W}_i^e$ . Our numerical implementation in the Supplementary Material shows that estimates thus obtained are very similar to those formed according to (3.16).

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**Algorithm 2 Weighted Bootstrap.**

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For  $b = 1, \dots, B$ , repeat the following steps:

**Step 0.** Draw  $e_b := \{e_{ib}\}_{i=1}^n$  i.i.d. from a random variable that satisfies Assumption 3 (e.g., the standard exponential distribution).

**Step 1.** Reestimate the control function  $\widehat{V}_{ib}^e = \widehat{F}_{X,b}^e(X_i | Z_i)$  in the weighted sample, according to (3.1)-(3.2) or (3.3)-(3.4).

**Step 2.** Reestimate the reduced form CDF  $\widehat{F}_{Y,b}^e$  in the weighted sample according to (3.5)-(3.6) or (3.7)-(3.8).

**Step 3.** For  $n_{Tb}^e = \sum_{i=1}^n e_{ib} T_i$ , compute

$$\widehat{G}_b^e(y, x) = \sum_{i=1}^n e_{ib} \widehat{F}_{Y,b}^e(y | x, Z_{1i}, \widehat{V}_{ib}^e) T_i / n_{Tb}^e,$$

$$\widehat{Q}_b^e(\tau, x) = \delta \sum_{s=1}^S \left[ 1(y_s \geq 0) - 1\{\widehat{G}_b^e(y_s, x) \geq \tau\} \right], \text{ and}$$

$$\widehat{\mu}_b^e(x) = \delta \sum_{s=1}^S \left[ 1(y_s \geq 0) - \widehat{G}_b^e(y_s, x) \right],$$


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**Algorithm 3 Uniform Inference for DSF and ASF.**

---

**Step 1.** Given  $B$  bootstrap draws  $\left\{ (\widehat{G}_b^e(y, x), \widehat{\mu}_b^e(x)) \right\}_{b=1}^B$ , compute the standard errors of  $\widehat{G}(y, x)$  and  $\widehat{\mu}(x)$  as

$$\widehat{\sigma}_G(y, x) = \text{IQR} \left[ \left\{ \widehat{G}_b^e(y, x) \right\}_{b=1}^B \right] / 1.349, \quad \widehat{\sigma}_\mu(x) = \text{IQR} \left[ \left\{ \widehat{\mu}_b^e(x) \right\}_{b=1}^B \right] / 1.349.$$

**Step 2.** For  $b = 1, \dots, B$ , compute the bootstrap draws of the maximal  $t$ -statistics for the DSF and ASF as

$$\|t_{G,b}^e(y, x)\|_{\mathcal{I}_G} = \sup_{(y,x) \in \mathcal{I}_G} \left| \frac{\widehat{G}_b^e(y, x) - \widehat{G}(y, x)}{\widehat{\sigma}_G(y, x)} \right|, \quad \|t_{\mu,b}^e(x)\|_{\mathcal{I}_\mu} = \sup_{x \in \mathcal{I}_\mu} \left| \frac{\widehat{\mu}_b^e(x) - \widehat{\mu}(x)}{\widehat{\sigma}_\mu(x)} \right|.$$

**Step 3.** Form  $(1 - \alpha)$ -confidence bands for the DSF and ASF as

$$\left\{ \widehat{G}(y, x) \pm \widehat{k}_G(1 - \alpha) \widehat{\sigma}_G(y, x) : (y, x) \in \mathcal{I}_G \right\}, \quad \left\{ \widehat{\mu}(x) \pm \widehat{k}_\mu(1 - \alpha) \widehat{\sigma}_\mu(x) : x \in \mathcal{I}_\mu \right\},$$

where  $\widehat{k}_G(1 - \alpha)$  is the sample  $(1 - \alpha)$ -quantile of  $\left\{ \|t_{G,b}^e(y, x)\|_{\mathcal{I}_G} : 1 \leq b \leq B \right\}$ , and  $\widehat{k}_\mu(1 - \alpha)$  is the sample  $(1 - \alpha)$ -quantile of  $\left\{ \|t_{\mu,b}^e(x)\|_{\mathcal{I}_\mu} : 1 \leq b \leq B \right\}$ .

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**Algorithm 4 Uniform Inference for QSF.**


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**Step 1.** Given  $B$  bootstrap draws  $\left\{(\widehat{G}_b^e(y, x), \widehat{Q}_b^e(\tau, x))\right\}_{b=1}^B$ , compute the standard errors of  $\widehat{G}(y, x)$  and  $\widehat{Q}(\tau, x)$  as

$$\widehat{\sigma}_G(y, x) = IQR \left[ \left\{ \widehat{G}_b^e(y, x) \right\}_{b=1}^B \right] / 1.349, \quad \widehat{\sigma}_Q(\tau, x) = IQR \left[ \left\{ \widehat{Q}_b^e(\tau, x) \right\}_{b=1}^B \right] / 1.349.$$

**Step 2.** For  $b = 1, \dots, B$ , compute the bootstrap draws of the maximal  $t$ -statistics for the DSF and ASF as

$$\|t_{G,b}^e(\tau, x)\|_{\mathcal{I}_G} = \sup_{(y,x) \in \mathcal{I}_G} \left| \frac{\widehat{G}_b^e(y, x) - \widehat{G}(y, x)}{\widehat{\sigma}_G(y, x)} \right|, \quad \|t_{Q,b}^e(\tau, x)\|_{\mathcal{I}_Q} = \sup_{(\tau,x) \in \mathcal{I}_Q} \left| \frac{\widehat{Q}_b^e(\tau, x) - \widehat{Q}(\tau, x)}{\widehat{\sigma}_Q(\tau, x)} \right|.$$

**Step 3.** If  $Y$  is continuous, form a  $(1 - \alpha)$ -confidence band for the QSF as

$$\left\{ \widehat{Q}(\tau, x) \pm \widehat{k}_Q(1 - \alpha)\widehat{\sigma}_Q(\tau, x) : (\tau, x) \in \mathcal{I}_Q \right\},$$

where  $\widehat{k}_Q(1 - \alpha)$  is the sample  $(1 - \alpha)$ -quantile of  $\left\{ \|t_{Q,b}^e(\tau, x)\|_{\mathcal{I}_Q} : 1 \leq b \leq B \right\}$ .

Otherwise, form a  $(1 - \alpha)$ -confidence band for the QSF as

$$\left\{ \left[ \widehat{G}_U^{\leftarrow}(\tau, x), \widehat{G}_L^{\leftarrow}(\tau, x) \right] : (\tau, x) \in \mathcal{I}_G^{\leftarrow} \right\},$$

where  $\mathcal{I}_G^{\leftarrow} = \{(\tau, x) : \widehat{G}_L(y, x) = \tau, (y, x) \in \mathcal{I}_G\} \cap \{(\tau, x) : \widehat{G}_U(y, x) = \tau, (y, x) \in \mathcal{I}_G\}$ ,

$$\widehat{G}_L(y, x) = \widehat{G}(y, x) - \widehat{k}_G(1 - \alpha)\widehat{\sigma}_G(y, x), \quad \widehat{G}_U(y, x) = \widehat{G}(y, x) + \widehat{k}_G(1 - \alpha)\widehat{\sigma}_G(y, x),$$

and  $\widehat{k}_G(1 - \alpha)$  is the sample  $(1 - \alpha)$ -quantile of  $\left\{ \|t_{G,b}^e(y, x)\|_{\mathcal{I}_G} : 1 \leq b \leq B \right\}$ .

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## APPENDIX B. IDENTIFICATION

**B.1. Proof of Lemma 1.** By Assumption 2  $E_\mu[p(X)p(X)']$ ,  $E_{\varsigma_l}[r_{1l}(Z_{1l})r_{1l}(Z_{1l})']$ ,  $l = 1, \dots, d_{z_1}$ , and  $E_\rho[q(V)q(V)']$  are positive definite. Also, with  $W = w(X, Z_1, V)$ , there is a positive constant  $C$  such that

$$\begin{aligned} E[w(X, Z_1, V)w(X, Z_1, V)'] &\geq C \int w(x, z_1, v)w(x, z_1, v)'[\mu(dx) \times \varsigma(dz_1) \times \rho(dv)] \\ &= C \int \{p(x)p(x)'\} \otimes \{r_{11}(z_{11})r_{11}(z_{11})'\} \otimes \cdots \\ &\quad \otimes \{r_{1d_{z_1}}(z_{1d_{z_1}})r_{1d_{z_1}}(z_{1d_{z_1}})'\} \otimes \{q(v)q(v)'\}[\mu(dx) \times \varsigma(dz_1) \times \rho(dv)] \\ &= CE_\mu[p(X)p(X)'] \otimes E_{\varsigma_1}[r_{11}(Z_{11})r_{11}(Z_{11})'] \otimes \cdots \\ &\quad \otimes E_{\varsigma_{d_{z_1}}}[r_{1d_{z_1}}(Z_{1d_{z_1}})r_{1d_{z_1}}(Z_{1d_{z_1}})'] \otimes E_\rho[q(V)q(V)']. \end{aligned}$$

where the inequality means no less than in the usual partial ordering for positive semi-definite matrices. The conclusion then follows by both the matrices following the last equality being positive definite.  $\square$

**B.2. Proof of Theorem 1.** Under Assumption 2, Lemma 1 implies that the QR coefficients  $\beta(U)$  and DR coefficients  $\beta(Y)$  are unique. For the QR specification, suppose there exists  $\tilde{\beta}(U)$  such that  $\beta(U)'w(X, Z_1, V) = \tilde{\beta}(U)'w(X, Z_1, V)$ . Then  $\{\beta(U) - \tilde{\beta}(U)\}'w(X, Z_1, V) = 0$ , and after applying iterated expectations, independence of  $U$  and  $(X, Z_1, V)$  implies

$$\begin{aligned} 0 &= E[(\beta(U) - \tilde{\beta}(U))' \{w(X, Z_1, V)w(X, Z_1, V)'\} (\beta(U) - \tilde{\beta}(U))] \\ &= E[(\beta(U) - \tilde{\beta}(U))' E[w(X, Z_1, V)w(X, Z_1, V)' | U] (\beta(U) - \tilde{\beta}(U))] \\ &\geq CE[|\beta(U) - \tilde{\beta}(U)|^2] \end{aligned}$$

for some positive constant  $C$ , by positive definiteness of  $E[w(X, Z_1, V)w(X, Z_1, V)']$ . Therefore, the map  $u \mapsto Q_Y(u | x, v)$  is well-defined for all  $(x, z_1, v) \in \mathcal{XZ}_1\mathcal{V}$  under Assumption 1(a). Strict monotonicity of  $u \mapsto Q_Y(u | x, z_1, v)$  for all  $(x, z_1, v) \in \mathcal{XZ}_1\mathcal{V}$  then implies that the inverse map  $y \mapsto F_Y(y | x, z_1, v) = Q_Y^{-1}(y | x, z_1, v)$  is well-defined for all  $(x, z_1, v) \in \mathcal{XZ}_1\mathcal{V}$ . For the DR specification, positive definiteness of  $E[w(X, Z_1, V)w(X, Z_1, V)']$  is also sufficient for uniqueness of DR coefficients by standard identification results for Logit and Probit models, e.g., see Example 1.2 in Newey and McFadden (1994). Therefore, the map  $y \mapsto F_Y(y | x, z_1, v)$  is well-defined for all  $(x, z_1, v) \in \mathcal{XZ}_1\mathcal{V}$  under Assumption 1(b). For both specifications the result now follows from the definitions of structural functions in Section 2.  $\square$

## APPENDIX C. ASYMPTOTIC THEORY

**C.1. Notation.** In what follows  $\vartheta$  denotes a generic value for the control function. It is convenient also to introduce some additional notation, which will be extensively used in the proofs. Let  $V_i(\vartheta) := \vartheta(X_i, Z_i)$ ,  $W_i(\vartheta) := w(X_i, Z_{1i}, V_i(\vartheta))$ , and  $\dot{W}_i(\vartheta) := \partial_v w(X_i, Z_{1i}, v)|_{v=V_i(\vartheta)}$ . When the previous functions are evaluated at the true values we use  $V_i = V_i(\vartheta_0)$ ,  $W_i = W_i(\vartheta_0)$ , and  $\dot{W}_i = \dot{W}_i(\vartheta_0)$ . Also, let  $\rho_y(u, v) := -1(u \leq y) \log \Lambda(v) - 1(u > y) \log \Lambda(-v)$ . Recall that  $A := (Y, X, Z, W, V)$ ,  $T(x) = 1(x \in \bar{\mathcal{X}})$ , and  $T = T(X)$ . For a function  $f : \mathcal{A} \mapsto \mathbb{R}$ , we use  $\|f\|_{T, \infty} = \sup_{a \in \mathcal{A}} |T(x)f(a)|$ ; for a  $K$ -vector of functions  $f : \mathcal{A} \mapsto \mathbb{R}^K$ , we use  $\|f\|_{T, \infty} = \sup_{a \in \mathcal{A}} \|T(x)f(a)\|_2$ . We make functions in  $\Upsilon$  as well as estimators  $\hat{\vartheta}$  to take values in  $[0, 1]$ , the support of the control function  $V$ . This allows us to simplify notation in what follows.

We adopt the standard notation in the empirical process literature (see, e.g., van der Vaart, 1998),

$$\mathbb{E}_n[f] = \mathbb{E}_n[f(A)] = n^{-1} \sum_{i=1}^n f(A_i),$$

and

$$\mathbb{G}_n[f] = \mathbb{G}_n[f(A)] = n^{-1/2} \sum_{i=1}^n (f(A_i) - \mathbb{E}_P[f(A)]).$$

When the function  $\hat{f}$  is estimated, the notation should be interpreted as:

$$\mathbb{G}_n[\hat{f}] = \mathbb{G}_n[f] |_{f=\hat{f}} \quad \text{and} \quad \mathbb{E}_P[\hat{f}] = \mathbb{E}_P[f] |_{f=\hat{f}}.$$

We also use the concepts of covering entropy and bracketing entropy in the proofs. The covering entropy  $\log N(\epsilon, \mathcal{F}, \|\cdot\|)$  is the logarithm of the minimal number of  $\|\cdot\|$ -balls of radius  $\epsilon$  needed to cover the set of functions  $\mathcal{F}$ . The bracketing entropy  $\log N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|)$  is the logarithm of the minimal number of  $\epsilon$ -brackets in  $\|\cdot\|$  needed to cover the set of functions  $\mathcal{F}$ . An  $\epsilon$ -bracket  $[\ell, u]$  in  $\|\cdot\|$  is the set of functions  $f$  with  $\ell \leq f \leq u$  and  $\|u - \ell\| < \epsilon$ .

For a sequence of random functions  $y \mapsto f_n(y)$  and a deterministic sequence  $a_n$ , we use  $f_n(y) = \bar{o}_{\mathbb{P}}(a_n)$  and  $f_n(y) = \bar{O}_{\mathbb{P}}(a_n)$  to denote uniform in  $y \in \mathcal{Y}$  orders in probability, i.e.  $\sup_{y \in \mathcal{Y}} f_n(y) = o_{\mathbb{P}}(a_n)$  and  $\sup_{y \in \mathcal{Y}} f_n(y) = O_{\mathbb{P}}(a_n)$ , respectively. The uniform in  $y \in \mathcal{Y}$  deterministic orders  $\bar{o}(a_n)$  and  $\bar{O}(a_n)$  are defined analogously suppressing the  $\mathbb{P}$  subscripts.

We follow the notation and definitions in van der Vaart and Wellner (1996) of bootstrap consistency. Let  $D_n$  denote the data vector and  $E_n$  be the vector of bootstrap weights. Consider the random element  $Z_n^e = Z_n(D_n, E_n)$  in a normed space  $\mathbb{Z}$ . We say that the bootstrap law of  $Z_n^e$  consistently estimates the law of some tight random element  $Z$  and write  $Z_n^e \rightsquigarrow_{\mathbb{P}} Z$  in  $\mathbb{Z}$  if

$$(C.1) \quad \sup_{h \in \text{BL}_1(\mathbb{Z})} |\mathbb{E}_P^e h(Z_n^e) - \mathbb{E}_P h(Z)| \rightarrow_{\mathbb{P}^*} 0,$$

where  $\text{BL}_1(\mathbb{Z})$  denotes the space of functions with Lipschitz norm at most 1,  $\mathbb{E}_P^e$  denotes the conditional expectation with respect to  $E_n$  given the data  $D_n$ , and  $\rightarrow_{\mathbb{P}^*}$  denotes convergence in (outer) probability.

**C.2. Proof of Lemma 3.** We only consider the case where  $\mathcal{Y}$  is a compact interval of  $\mathbb{R}$ . The case where  $\mathcal{Y}$  is finite is simpler and follows similarly.

C.2.1. *Auxiliary Lemmas.* We start with 2 results on stochastic equicontinuity and a local expansion for the second stage estimators that will be used in the proof of Lemma 3.

**Lemma 4.** *[Stochastic equicontinuity] Let  $e \geq 0$  be a positive random variable with  $E_P[e] = 1$ ,  $\text{Var}_P[e] = 1$ , and  $E_P|e|^{2+\delta} < \infty$  for some  $\delta > 0$ , that is independent of  $(Y, X, Z, W, V)$ , including as a special case  $e = 1$ , and set, for  $A = (e, Y, X, Z, W, V)$ ,*

$$f_y(A, \vartheta, \beta) := e \cdot [\Lambda(W(\vartheta)' \beta) - 1(Y \leq y)] \cdot W(\vartheta) \cdot T.$$

Under Assumptions 3–5 the following relations are true.

(a) Consider the set of functions

$$\mathcal{F} = \{f_y(A, \vartheta, \beta)' \alpha : (\vartheta, \beta, y) \in \Upsilon_0 \times \mathcal{B} \times \mathcal{Y}, \alpha \in \mathbb{R}^{\dim(W)}, \|\alpha\|_2 \leq 1\},$$

where  $\mathcal{Y}$  is a compact subset of  $\mathbb{R}$ ,  $\mathcal{B}$  is a compact set under the  $\|\cdot\|_2$  metric containing  $\beta_0(y)$  for all  $y \in \mathcal{Y}$ ,  $\Upsilon_0$  is the intersection of  $\Upsilon$ , defined in Lemma 2, with a neighborhood of  $\vartheta_0$  under the  $\|\cdot\|_{T,\infty}$  metric. This class is  $P$ -Donsker with a square integrable envelope of the form  $e$  times a constant.

(b) Moreover, if  $(\vartheta, \beta(y)) \rightarrow (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$ , then

$$\sup_{y \in \mathcal{Y}} \|f_y(A, \vartheta, \beta(y)) - f_y(A, \vartheta_0, \beta_0(y))\|_{P,2} \rightarrow 0.$$

(c) Hence for any  $(\tilde{\vartheta}, \tilde{\beta}(y)) \rightarrow_{\mathbb{P}} (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$  such that  $\tilde{\vartheta} \in \Upsilon_0$ ,

$$\sup_{y \in \mathcal{Y}} \|\mathbb{G}_n f_y(A, \tilde{\vartheta}, \tilde{\beta}(y)) - \mathbb{G}_n f_y(A, \vartheta_0, \beta_0(y))\|_2 \rightarrow_{\mathbb{P}} 0.$$

(d) For any  $(\hat{\vartheta}, \tilde{\beta}(y)) \rightarrow_{\mathbb{P}} (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$ , so that

$$\|\hat{\vartheta} - \tilde{\vartheta}\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n}), \text{ where } \tilde{\vartheta} \in \Upsilon_0,$$

we have that

$$\sup_{y \in \mathcal{Y}} \|\mathbb{G}_n f_y(A, \hat{\vartheta}, \tilde{\beta}(y)) - \mathbb{G}_n f_y(A, \vartheta_0, \beta_0(y))\|_2 \rightarrow_{\mathbb{P}} 0.$$

**Proof of Lemma 4.** The proof is divided in subproofs of each of the claims.

Proof of Claim (a). The proof proceeds in several steps.



Step 1. Here we bound the bracketing entropy for

$$\mathcal{I}_1 = \{[\Lambda(W(\vartheta)'\beta) - 1(Y \leq y)]T : \beta \in \mathcal{B}, \vartheta \in \Upsilon_0, y \in \mathcal{Y}\}.$$

For this purpose consider a mesh  $\{\vartheta_k\}$  over  $\Upsilon_0$  of  $\|\cdot\|_{T,\infty}$  width  $\delta$ , a mesh  $\{\beta_l\}$  over  $\mathcal{B}$  of  $\|\cdot\|_2$  width  $\delta$ , and a mesh  $\{y_j\}$  over  $\mathcal{Y}$  of  $\|\cdot\|_2$  width  $\delta$ . A generic bracket over  $\mathcal{I}_1$  takes the form

$$[i_1^0, i_1^1] = [\{\Lambda(W(\vartheta_k)'\beta_l - \kappa\delta) - 1(Y \leq y_j - \delta)\}T, \{\Lambda(W(\vartheta_k)'\beta_l + \kappa\delta) - 1(Y \leq y_j + \delta)\}T],$$

where  $\kappa = L_W \max_{\beta \in \mathcal{B}} \|\beta\|_2 + L_W$ , and  $L_W := \|\partial_v w\|_{T,\infty} \vee \|w\|_{T,\infty}$ .

Note that this is a valid bracket for all elements of  $\mathcal{I}_1$  because for any  $\vartheta$  located within  $\delta$  from  $\vartheta_k$  and any  $\beta$  located within  $\delta$  from  $\beta_l$ ,

$$\begin{aligned} |W(\vartheta)'\beta - W(\vartheta_k)'\beta_l|T &\leq |(W(\vartheta) - W(\vartheta_k))'\beta|T + |W(\vartheta_k)'(\beta - \beta_l)|T \\ (C.2) \qquad \qquad \qquad &\leq L_W \delta \max_{\beta \in \mathcal{B}} \|\beta\|_2 + L_W \delta \leq \kappa \delta, \end{aligned}$$

and the  $\|\cdot\|_{P,2}$ -size of this bracket is given by

$$\begin{aligned} \|i_1^0 - i_1^1\|_{P,2} &\leq \sqrt{\mathbf{E}_P[P\{Y \in [y \pm \delta] \mid X, Z\}T]} \\ &\quad + \sqrt{\mathbf{E}_P[\{\Lambda(W(\vartheta_k)'\beta_l + \kappa\delta) - \Lambda(W(\vartheta_k)'\beta_l - \kappa\delta)\}^2 T]} \\ &\leq \sqrt{\|f_Y(\cdot \mid \cdot)\|_{T,\infty} 2\delta + \kappa\delta/2}, \end{aligned}$$

because  $\|\lambda(\cdot)\|_{T,\infty} \leq 1/4$ , where  $\lambda = \Lambda(1 - \Lambda)$  is the derivative of  $\Lambda$ .

Hence, counting the number of brackets induced by the mesh created above, we arrive at the following relationship between the bracketing entropy of  $\mathcal{I}_1$  and the covering entropies of  $\Upsilon_0$ ,  $\mathcal{B}$ , and  $\mathcal{Y}$ ,

$$\begin{aligned} \log N_{[]}(\epsilon, \mathcal{I}_1, \|\cdot\|_{P,2}) &\lesssim \log N(\epsilon^2, \Upsilon_0, \|\cdot\|_{T,\infty}) + \log N(\epsilon^2, \mathcal{B}, \|\cdot\|_2) + \log N(\epsilon^2, \mathcal{Y}, \|\cdot\|_2) \\ &\lesssim 1/(\epsilon^2 \log^4 \epsilon) + \log(1/\epsilon) + \log(1/\epsilon), \end{aligned}$$

and so  $\mathcal{I}_1$  is  $P$ -Donsker with a constant envelope.

Step 2. Similarly to Step 1, it follows that

$$\mathcal{I}_2 = \{W(\vartheta)'\alpha T : \vartheta \in \Upsilon_0, \alpha \in \mathbb{R}^{\dim(W)}, \|\alpha\|_2 \leq 1\}$$

also obeys a similar bracketing entropy bound

$$\log N_{[]}(\epsilon, \mathcal{I}_2, \|\cdot\|_{P,2}) \lesssim 1/(\epsilon^2 \log^4 \epsilon) + \log(1/\epsilon)$$

with a generic bracket taking the form  $[i_2^0, i_2^1] = [\{W(\vartheta_k)' \beta_l - \kappa \delta\}T, \{W(\vartheta_k)' \beta_l + \kappa \delta\}T]$ . Hence, this class is also  $P$ -Donsker with a constant envelope.

Step 3. In this step we verify the claim (a). Note that  $\mathcal{F} = e \cdot \mathcal{I}_1 \cdot \mathcal{I}_2$ . This class has a square-integrable envelope under  $P$ . The class  $\mathcal{F}$  is  $P$ -Donsker by the following argument. Note that the product  $\mathcal{I}_1 \cdot \mathcal{I}_2$  of uniformly bounded classes is  $P$ -Donsker, e.g., by Theorem 2.10.6 of van der Vaart and Wellner (1996). Under the stated assumption the final product of the random variable  $e$  with the  $P$ -Donsker class remains to be  $P$ -Donsker by the Multiplier Donsker Theorem, namely Theorem 2.9.2 in van der Vaart and Wellner (1996).

Proof of Claim (b). The claim follows by the Dominated Convergence Theorem, since any  $f \in \mathcal{F}$  is dominated by a square-integrable envelope under  $P$ , and, uniformly in  $y \in \mathcal{Y}$ ,  $\Lambda[W(\vartheta)' \beta(y)]T \rightarrow \Lambda[W' \beta_0(y)]T$  and  $|W(\vartheta)' \beta(y)T - W' \beta_0(y)T| \rightarrow 0$  in view of the relation such as (C.2).

Proof of Claim (c). This claim follows from the asymptotic equicontinuity of the empirical process  $(\mathbb{G}_n[f_y], f_y \in \mathcal{F})$  under the  $L_2(P)$  metric, and hence also with respect to the  $\|\cdot\|_{T, \infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$  in view of Claim (b).

Proof of Claim (d). It is convenient to set  $\widehat{f}_y := f_y(A, \widehat{\vartheta}, \widehat{\beta}(y))$  and  $\widetilde{f}_y := f_y(A, \widetilde{\vartheta}, \widetilde{\beta}(y))$ . Note that

$$\begin{aligned} \max_{1 \leq j \leq \dim W} |\mathbb{G}_n[\widehat{f}_y - \widetilde{f}_y]|_j &\leq \max_{1 \leq j \leq \dim W} |\sqrt{n} \mathbb{E}_n[\widehat{f}_y - \widetilde{f}_y]|_j + \max_{1 \leq j \leq \dim W} |\sqrt{n} \mathbb{E}_P[\widehat{f}_y - \widetilde{f}_y]|_j \\ &\lesssim \sqrt{n} \mathbb{E}_n[\widehat{\zeta}] + \sqrt{n} \mathbb{E}_P[\widehat{\zeta}] \lesssim \mathbb{G}_n[\widehat{\zeta}] + 2\sqrt{n} \mathbb{E}_P[\widehat{\zeta}], \end{aligned}$$

where  $|f_y|_j$  denotes the  $j$ th element of an application of absolute value to each element of the vector  $f_y$ , and  $\widehat{\zeta}$  is defined by the following relationship, which holds with probability approaching one uniformly in  $y \in \mathcal{Y}$ ,

$$\begin{aligned} \max_{1 \leq j \leq \dim W} |\widehat{f}_y - \widetilde{f}_y|_j &\lesssim |e| \cdot \{\|W(\widehat{\vartheta}) - W(\widetilde{\vartheta})\|_2 + |\Lambda[W(\widehat{\vartheta})' \widetilde{\beta}(y)] - \Lambda[W(\widetilde{\vartheta})' \widetilde{\beta}(y)]|\} \cdot T \\ &\lesssim \widehat{\zeta} := e \cdot \kappa \Delta_n, \end{aligned}$$

where  $\kappa = L_W \max_{\beta \in \mathcal{B}} \|\beta\|_2 + L_W$ ,  $L_W = \|\partial_v w\|_{T, \infty} \vee \|w\|_{T, \infty}$ , and  $\Delta_n = o(1/\sqrt{n})$  is a deterministic sequence such that

$$\Delta_n \geq \|\widehat{\vartheta} - \widetilde{\vartheta}\|_{T, \infty}.$$

By part (c) the result follows from

$$\mathbb{G}_n[\widehat{\zeta}] = \bar{o}_{\mathbb{P}}(1), \quad \sqrt{n} \mathbb{E}_P[\widehat{\zeta}] = \bar{o}_{\mathbb{P}}(1).$$

Indeed,

$$\|e \cdot \kappa \Delta_n\|_{P,2} = \bar{o}(1) \Rightarrow \mathbb{G}_n[\widehat{\zeta}] = \bar{o}_{\mathbb{P}}(1),$$

and

$$\|e \cdot \kappa \Delta_n\|_{P,1} \leq \mathbb{E}_P |e| \cdot \kappa \Delta_n = \bar{o}(1/\sqrt{n}) \Rightarrow \mathbb{E}_P |\widehat{\zeta}| = \bar{o}_{\mathbb{P}}(1/\sqrt{n}),$$

since  $\Delta_n = o(1/\sqrt{n})$ .

**Lemma 5.** *[Local expansion] Under Assumptions 3–5, for*

$$\begin{aligned} \widehat{\delta}(y) &= \sqrt{n}(\widetilde{\beta}(y) - \beta_0(y)) = \bar{O}_{\mathbb{P}}(1); \\ \widehat{\Delta}(x, r) &= \sqrt{n}(\widehat{\vartheta}(x, r) - \vartheta_0(x, r)) = \sqrt{n} \mathbb{E}_n[\ell(A, x, r)] + o_{\mathbb{P}}(1) \text{ in } \ell^\infty(\overline{\mathcal{X}\mathcal{R}}), \\ \|\sqrt{n} \mathbb{E}_n[\ell(A, \cdot)]\|_{T,\infty} &= O_{\mathbb{P}}(1), \end{aligned}$$

we have that

$$\sqrt{n} \mathbb{E}_P \{[\Lambda[W(\widehat{\vartheta})' \widetilde{\beta}(y)] - 1(Y \leq y)]W(\widehat{\vartheta})T\} = J(y)\widehat{\delta}(y) + \sqrt{n} \mathbb{E}_n [g_y(A)] + \bar{o}_{\mathbb{P}}(1),$$

where

$$g_y(a) = \mathbb{E}_P \{[\Lambda(W' \beta_0(y)) - 1(Y \leq y)]\dot{W} + \lambda(W' \beta_0(y))W\dot{W}' \beta_0(y)\}T \ell(a, X, R).$$

**Proof of Lemma 5.**

Uniformly in  $\xi := (X, Z) \in \overline{\mathcal{X}\mathcal{Z}}$  and  $y \in \mathcal{Y}$ ,

$$\begin{aligned} &\sqrt{n} \mathbb{E}_P \{[\Lambda[W(\widehat{\vartheta})' \widetilde{\beta}(y)] - 1(Y \leq y) \mid X, Z]T\} \\ &= \sqrt{n} \mathbb{E}_P \{[\Lambda[W' \beta_0(y)] - 1(Y \leq y) \mid X, Z]T\} \\ &\quad + \lambda[W(\bar{\vartheta}_\xi)' \bar{\beta}_\xi(y)] \{W(\bar{\vartheta}_\xi)' \widehat{\delta}(y) + \dot{W}(\bar{\vartheta}_\xi)' \bar{\beta}_\xi \widehat{\Delta}(X, R)\}T \\ &= \sqrt{n} \mathbb{E}_P \{[\Lambda[W' \beta_0(y)] - 1(Y \leq y) \mid X, Z]T\} \\ &\quad + \lambda[W' \beta_0(y)] \{W' \widehat{\delta}(y) + \dot{W}' \beta_0(y) \widehat{\Delta}(X, R)\}T + R_\xi(y), \end{aligned}$$

and

$$\bar{R}(y) = \sup_{\{\xi \in \overline{\mathcal{X}\mathcal{Z}}\}} |R_\xi(y)| = \bar{o}_{\mathbb{P}}(1)$$

where  $\bar{\vartheta}_\xi$  is on the line connecting  $\vartheta_0$  and  $\widehat{\vartheta}$  and  $\bar{\beta}_\xi(y)$  is on the line connecting  $\beta_0(y)$  and  $\widetilde{\beta}(y)$ . The first equality follows by the mean value expansion. The second equality follows by uniform continuity of  $\lambda(\cdot)$ , uniform continuity of  $W(\cdot)$  and  $\dot{W}(\cdot)$ , and by  $\|\widehat{\vartheta} - \vartheta_0\|_{T,\infty} \rightarrow_{\mathbb{P}} 0$  and  $\sup_{y \in \mathcal{Y}} \|\widetilde{\beta}(y) - \beta_0(y)\|_2 \rightarrow_{\mathbb{P}} 0$ .

Since  $\lambda(\cdot)$  and the entries of  $W$  and  $\dot{W}$  are bounded,  $\widehat{\delta}(y) = \bar{O}_{\mathbb{P}}(1)$ , and  $\|\widehat{\Delta}\|_{T,\infty} = O_{\mathbb{P}}(1)$ , with probability approaching one uniformly in  $y \in \mathcal{Y}$ ,

$$\begin{aligned} & \sqrt{n} \mathbb{E}_P \{ \Lambda[W(\widehat{\vartheta})' \widetilde{\beta}(y)] - 1(Y \leq y) \} W(\widehat{\vartheta}) T = \mathbb{E}_P \{ \Lambda(W' \beta_0(y)) - 1(Y \leq y) \} \dot{W} T \widehat{\Delta}(X, R) \\ & + \mathbb{E}_P \{ \lambda[W' \beta_0(y)] W W' T \} \widehat{\delta}(y) + \mathbb{E}_P \{ \lambda[W' \beta_0(y)] W \dot{W}' \beta_0(y) T \widehat{\Delta}(X, R) \} + O_{\mathbb{P}}(\bar{R}(y)) \\ & = J(y) \widehat{\delta}(y) + \mathbb{E}_P \{ \{ \Lambda(W' \beta_0(y)) - 1(Y \leq y) \} \dot{W} + \lambda[W' \beta_0(y)] W \dot{W}' \beta_0(y) \} T \widehat{\Delta}(X, R) + o_{\mathbb{P}}(1). \end{aligned}$$

Substituting in  $\widehat{\Delta}(x, r) = \sqrt{n} \mathbb{E}_n[\ell(A, x, r)] + o_{\mathbb{P}}(1)$  and interchanging  $\mathbb{E}_P$  and  $\mathbb{E}_n$ , we obtain

$$\mathbb{E}_P \{ \{ \Lambda(W' \beta_0(y)) - 1(Y \leq y) \} \dot{W} + \lambda[W' \beta_0(y)] W \dot{W}' \beta_0(y) \} T \widehat{\Delta}(X, R) = \sqrt{n} \mathbb{E}_n[g_y(A)] + \bar{o}_{\mathbb{P}}(1),$$

since  $\{ \{ \Lambda(W' \beta_0(y)) - 1(Y \leq y) \} \dot{W} + \lambda[W' \beta_0(y)] W \dot{W}' \beta_0(y) \} T$  is bounded uniformly in  $y \in \mathcal{Y}$ . The claim of the lemma follows.  $\square$

*C.2.2. Proof of Lemma 3.* The proof is divided in two parts corresponding to the FCLT and bootstrap FCLT.

### Part 1: FCLT

In this part we show  $\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) \rightsquigarrow J(y)^{-1}G(y)$  in  $\ell^\infty(\mathcal{Y})^{d_w}$ .

Step 1. This step shows that  $\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) = \bar{O}_{\mathbb{P}}(1)$ .

Recall that

$$\widehat{\beta}(y) = \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \mathbb{E}_n[\rho_y(Y, W(\widehat{\vartheta})' \beta) T].$$

Due to convexity of the objective function, it suffices to show that for any  $\epsilon > 0$  there exists a finite positive constant  $B_\epsilon$  such that uniformly in  $y \in \mathcal{Y}$ ,

$$(C.3) \quad \liminf_{n \rightarrow \infty} \mathbb{P} \left( \inf_{\|\eta\|_2=1} \sqrt{n} \eta' \mathbb{E}_n \left[ \widehat{f}_{\eta, B_\epsilon, y} \right] > 0 \right) \geq 1 - \epsilon,$$

where

$$\widehat{f}_{\eta, B_\epsilon, y}(A) := \left\{ \Lambda[W(\widehat{\vartheta})'(\beta_0(y) + B_\epsilon \eta / \sqrt{n})] - 1(Y \leq y) \right\} W(\widehat{\vartheta}) T.$$

Let

$$f_y(A) := \{ \Lambda[W' \beta_0(y)] - 1(Y \leq y) \} W T.$$

Then uniformly in  $\|\eta\|_2 = 1$ ,

$$\begin{aligned}
\sqrt{n}\eta'\mathbb{E}_n[\widehat{f}_{\eta,B_\epsilon,y}] &= \eta'\mathbb{G}_n[\widehat{f}_{\eta,B_\epsilon,y}] + \sqrt{n}\eta'\mathbb{E}_P[\widehat{f}_{\eta,B_\epsilon,y}] \\
&=_{(1)} \eta'\mathbb{G}_n[f_y] + \bar{o}_{\mathbb{P}}(1) + \eta'\sqrt{n}\mathbb{E}_P[\widehat{f}_{\eta,B_\epsilon,y}] \\
&=_{(2)} \eta'\mathbb{G}_n[f_y] + \bar{o}_{\mathbb{P}}(1) + \eta'J(y)\eta B_\epsilon + \eta'\mathbb{G}_n[g_y] + \bar{o}_{\mathbb{P}}(1) \\
&=_{(3)} \bar{O}_{\mathbb{P}}(1) + \bar{o}_{\mathbb{P}}(1) + \eta'J(y)\eta B_\epsilon + \bar{O}_{\mathbb{P}}(1) + \bar{o}_{\mathbb{P}}(1),
\end{aligned}$$

where relations (1) and (2) follow by Lemma 4 and Lemma 5 with  $\tilde{\beta}(y) = \beta_0(y) + B_\epsilon\eta/\sqrt{n}$ , respectively, using that  $\|\widehat{\vartheta} - \tilde{\vartheta}\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n})$ ,  $\tilde{\vartheta} \in \Upsilon$ ,  $\|\tilde{\vartheta} - \vartheta_0\|_{T,\infty} = O_{\mathbb{P}}(1/\sqrt{n})$  and  $\|\beta_0(y) + B_\epsilon\eta/\sqrt{n} - \beta_0(y)\|_2 = \bar{O}(1/\sqrt{n})$ ; relation (3) holds because  $f_y$  and  $g_y$  are  $P$ -Donsker by step-2 below. Since uniformly in  $y \in \mathcal{Y}$ ,  $J(y)$  is positive definite, with minimal eigenvalue bounded away from zero, the inequality (C.3) follows by choosing  $B_\epsilon$  as a sufficiently large constant.

Step 2. In this step we show the main result. Let

$$\widehat{f}_y(A) := \left\{ \Lambda[W(\widehat{\vartheta})'\widehat{\beta}(y)] - 1(Y \leq y) \right\} W(\widehat{\vartheta})T.$$

From the first order conditions of the distribution regression problem,

$$\begin{aligned}
0 = \sqrt{n}\mathbb{E}_n[\widehat{f}_y] &= \mathbb{G}_n[\widehat{f}_y] + \sqrt{n}\mathbb{E}_P[\widehat{f}_y] \\
&=_{(1)} \mathbb{G}_n[f_y] + \bar{o}_{\mathbb{P}}(1) + \sqrt{n}\mathbb{E}_P[\widehat{f}_y] \\
&=_{(2)} \mathbb{G}_n[f_y] + \bar{o}_{\mathbb{P}}(1) + J(y)\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) + \mathbb{G}_n[g_y] + \bar{o}_{\mathbb{P}}(1),
\end{aligned}$$

where relations (1) and (2) follow by Lemma 4 and Lemma 5 with  $\tilde{\beta}(y) = \widehat{\beta}(y)$ , respectively, using that  $\|\widehat{\vartheta} - \tilde{\vartheta}\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n})$ ,  $\tilde{\vartheta} \in \Upsilon$ , and  $\|\tilde{\vartheta} - \vartheta\|_{T,\infty} = O_{\mathbb{P}}(1/\sqrt{n})$  by Lemma 2, and  $\|\widehat{\beta}(y) - \beta_0(y)\|_2 = \bar{O}_{\mathbb{P}}(1/\sqrt{n})$ .

Therefore by uniform invertibility of  $J(y)$  in  $y \in \mathcal{Y}$ ,

$$\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) = -J(y)^{-1}\mathbb{G}_n(f_y + g_y) + \bar{o}_{\mathbb{P}}(1).$$

The function  $f_y$  is  $P$ -Donsker by standard argument for distribution regression (e.g., step 3 in the proof of Theorem 5.2 of Chernozhukov, Fernandez-Val and Melly, 2013). Similarly,  $g_y$  is  $P$ -Donsker by Example 19.7 in van der Vaart (1998) because  $g_y \in \{h_y(A) : |h_y(A) - h_v(A)| \leq M(A)|y - v|; \mathbb{E}_P M(A)^2 < \infty; y, v \in \mathcal{Y}\}$ , since

$$|g_y - g_v| \leq L\mathbb{E}_P[T|\ell(a, X, R)|]_{a=A}|y - v|,$$

with  $L = 2L_W + L_W^2 \max_{\beta \in \mathcal{B}} \|\beta\|_2/4$ ,  $L_W := \|\partial_v w\|_{T,\infty} \vee \|w\|_{T,\infty}$ , and  $\mathbb{E}_P[T\ell(A, X, R)^2] < \infty$  by Lemma 2. Hence, by the Functional Central Limit Theorem

$$\mathbb{G}_n(f_y + g_y) \rightsquigarrow G(y) \text{ in } \ell^\infty(\mathcal{Y})^{d_w},$$

where  $y \mapsto G(y)$  is a zero mean Gaussian process with uniformly continuous sample paths and the covariance function  $C(y, v)$  specified in the lemma. Conclude that

$$\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) \rightsquigarrow J(y)^{-1}G(y) \text{ in } \ell^\infty(\mathcal{Y})^{d_w}.$$

□

## Part 2: Bootstrap FCLT

In this part we show  $\sqrt{n}(\widehat{\beta}^e(y) - \widehat{\beta}(y)) \rightsquigarrow_{\mathbb{P}} J(y)^{-1}G(y)$  in  $\ell^\infty(\mathcal{Y})^{d_w}$ .

Step 1. This step shows that  $\sqrt{n}(\widehat{\beta}^e(y) - \beta_0(y)) = \bar{O}_{\mathbb{P}}(1)$  under the unconditional probability  $\mathbb{P}$ .

Recall that

$$\widehat{\beta}^e(y) = \arg \min_{\beta \in \mathbb{R}^{\dim(W)}} \mathbb{E}_n[e\rho_y(Y, W(\widehat{\vartheta}^e)'\beta)T],$$

where  $e$  is the random variable used in the weighted bootstrap. Due to convexity of the objective function, it suffices to show that for any  $\epsilon > 0$  there exists a finite positive constant  $B_\epsilon$  such that uniformly in  $y \in \mathcal{Y}$ ,

$$(C.4) \quad \liminf_{n \rightarrow \infty} \mathbb{P} \left( \inf_{\|\eta\|_2=1} \sqrt{n}\eta' \mathbb{E}_n \left[ \widehat{f}_{\eta, B_\epsilon, y}^e \right] > 0 \right) \geq 1 - \epsilon,$$

where

$$\widehat{f}_{\eta, B_\epsilon, y}^e(A) := e \cdot \left\{ \Lambda[W(\widehat{\vartheta}^e)'(\beta_0(y) + B_\epsilon\eta/\sqrt{n})] - 1(Y \leq y) \right\} W(\widehat{\vartheta}^e)T.$$

Let

$$f_y^e(A) := e \cdot \left\{ \Lambda[W'\beta_0(y)] - 1(Y \leq y) \right\} WT.$$

Then uniformly in  $\|\eta\|_2 = 1$ ,

$$\begin{aligned} \sqrt{n}\eta' \mathbb{E}_n[\widehat{f}_{\eta, B_\epsilon, y}^e] &= \eta' \mathbb{G}_n[\widehat{f}_{\eta, B_\epsilon, y}^e] + \sqrt{n}\eta' \mathbb{E}_P[\widehat{f}_{\eta, B_\epsilon, y}^e] \\ &=_{(1)} \eta' \mathbb{G}_n[f_y^e] + \bar{o}_{\mathbb{P}}(1) + \eta' \sqrt{n} \mathbb{E}_P[\widehat{f}_{\eta, B_\epsilon, y}^e] \\ &=_{(2)} \eta' \mathbb{G}_n[f_y^e] + \bar{o}_{\mathbb{P}}(1) + \eta' J(y)\eta B_\epsilon + \eta' \mathbb{G}_n[g_y^e] + \bar{o}_{\mathbb{P}}(1) \\ &=_{(3)} \bar{O}_{\mathbb{P}}(1) + \bar{o}_{\mathbb{P}}(1) + \eta' J(y)\eta B_\epsilon + \bar{O}_{\mathbb{P}}(1) + \bar{o}_{\mathbb{P}}(1), \end{aligned}$$

where relations (1) and (2) follow by Lemma 4 and Lemma 5 with  $\widetilde{\beta}(y) = \beta_0(y) + B_\epsilon\eta/\sqrt{n}$ , respectively, using that  $\|\widehat{\vartheta}^e - \widetilde{\vartheta}^e\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n})$ ,  $\widetilde{\vartheta}^e \in \Upsilon$  and  $\|\widetilde{\vartheta}^e - \vartheta_0\|_{T,\infty} =$

$O_{\mathbb{P}}(1/\sqrt{n})$  by Lemma 2, and  $\|\beta_0(y) + B_\epsilon \eta/\sqrt{n} - \beta_0(y)\|_2 = \bar{O}(1/\sqrt{n})$ ; relation (3) holds because  $f_y^e = e \cdot f_y$  and  $g_y^e = e \cdot g_y$ , where  $f_y$  and  $g_y$  are  $P$ -Donsker by step-2 of the proof of Theorem 3 and  $E_P e^2 < \infty$ . Since uniformly in  $y \in \mathcal{Y}$ ,  $J(y)$  is positive definite, with minimal eigenvalue bounded away from zero, the inequality (C.4) follows by choosing  $B_\epsilon$  as a sufficiently large constant.

Step 2. In this step we show that  $\sqrt{n}(\widehat{\beta}^e(y) - \beta_0(y)) = -J(y)^{-1} \mathbb{G}_n(f_y^e + g_y^e) + \bar{o}_{\mathbb{P}}(1)$  under the unconditional probability  $\mathbb{P}$ .

Let

$$\widehat{f}_y^e(A) := e \cdot \{\Lambda[W(\widehat{\vartheta}^e)' \widehat{\beta}^e(y)] - 1(Y \leq y)\} W(\widehat{\vartheta}^e) T.$$

From the first order conditions of the distribution regression problem in the weighted sample, uniformly in  $y \in \mathcal{Y}$ ,

$$\begin{aligned} 0 = \sqrt{n} \mathbb{E}_n \left[ \widehat{f}_y^e \right] &= \mathbb{G}_n \left[ \widehat{f}_y^e \right] + \sqrt{n} E_P \left[ \widehat{f}_y^e \right] \\ &=_{(1)} \mathbb{G}_n[f_y^e] + \bar{o}_{\mathbb{P}}(1) + \sqrt{n} E_P \left[ \widehat{f}_y^e \right] \\ &=_{(2)} \mathbb{G}_n[f_y^e] + \bar{o}_{\mathbb{P}}(1) + J(y) \sqrt{n} (\widehat{\beta}^e(y) - \beta_0(y)) + \mathbb{G}_n[g_y^e] + \bar{o}_{\mathbb{P}}(1), \end{aligned}$$

where relations (1) and (2) follow by Lemma 4 and Lemma 5 with  $\widetilde{\beta}(y) = \widehat{\beta}^e(y)$ , respectively, using that  $\|\widehat{\vartheta}^e - \widetilde{\vartheta}^e\|_{T, \infty} = o_{\mathbb{P}}(1/\sqrt{n})$ ,  $\widetilde{\vartheta}^e \in \Upsilon$  and  $\|\widetilde{\vartheta}^e - \vartheta_0\|_{T, \infty} = O_{\mathbb{P}}(1/\sqrt{n})$  by Lemma 2, and  $\|\widehat{\beta}^e(y) - \beta_0(y)\|_2 = \bar{O}_{\mathbb{P}}(1/\sqrt{n})$ .

Therefore by uniform invertibility of  $J(y)$  in  $y \in \mathcal{Y}$ ,

$$\sqrt{n}(\widehat{\beta}^e(y) - \beta_0(y)) = -J(y)^{-1} \mathbb{G}_n(f_y^e + g_y^e) + \bar{o}_{\mathbb{P}}(1).$$

Step 3. In this final step we establish the behavior of  $\sqrt{n}(\widehat{\beta}^e(y) - \widehat{\beta}(y))$  under  $\mathbb{P}^e$ . Note that  $\mathbb{P}^e$  denotes the conditional probability measure, namely the probability measure induced by draws of  $e_1, \dots, e_n$  conditional on the data  $A_1, \dots, A_n$ . By Step 2 of the proof of Theorem 1 and Step 2 of this proof, we have that under  $\mathbb{P}$ :

$$\begin{aligned} \sqrt{n}(\widehat{\beta}^e(y) - \beta_0(y)) &= -J(y)^{-1} \mathbb{G}_n(f_y^e + g_y^e) + \bar{o}_{\mathbb{P}}(1), \\ \sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) &= -J(y)^{-1} \mathbb{G}_n(f_y + g_y) + \bar{o}_{\mathbb{P}}(1). \end{aligned}$$

Hence, under  $\mathbb{P}$

$$\begin{aligned} \sqrt{n}(\widehat{\beta}^e(y) - \widehat{\beta}(y)) &= -J(y)^{-1} \mathbb{G}_n(f_y^e - f_y + g_y^e - g_y) + r_n(y) \\ &= -J(y)^{-1} \mathbb{G}_n((e - 1)(f_y + g_y)) + r_n(y), \end{aligned}$$

where  $r_n(y) = \bar{o}_{\mathbb{P}}(1)$ . Note that it is also true that

$$r_n(y) = \bar{o}_{\mathbb{P}^e}(1) \text{ in } \mathbb{P}\text{-probability,}$$

where the latter statement means that for every  $\epsilon > 0$ ,  $\mathbb{P}^e(\|r_n(y)\|_2 > \epsilon) = \bar{o}_{\mathbb{P}}(1)$ . Indeed, this follows from Markov inequality and by

$$\mathbb{E}_{\mathbb{P}}[\mathbb{P}^e(\|r_n(y)\|_2 > \epsilon)] = \mathbb{P}(\|r_n(y)\|_2 > \epsilon) = \bar{o}(1),$$

where the latter holds by the Law of Iterated Expectations and  $r_n(y) = \bar{o}_{\mathbb{P}}(1)$ .

Note that  $f_y^e = e \cdot f_y$  and  $g_y^e = e \cdot g_y$ , where  $f_y$  and  $g_y$  are  $P$ -Donsker by step-2 of the proof of the first part and  $\mathbb{E}_P e^2 < \infty$ . Then, by the Conditional Multiplier Functional Central Limit Theorem, e.g., Theorem 2.9.6 in van der Vaart and Wellner (1996),

$$G_n^e(y) := \mathbb{G}_n((e-1)(f_y + g_y)) \rightsquigarrow_{\mathbb{P}} G(y) \text{ in } \ell^\infty(\mathcal{Y})^{dw}.$$

Conclude that

$$\sqrt{n}(\widehat{\beta}^e(y) - \widehat{\beta}(y)) \rightsquigarrow_{\mathbb{P}} J(y)^{-1}G(y) \text{ in } \ell^\infty(\mathcal{Y})^{dw}.$$

□

**C.3. Proof of Theorems 2–4.** In this section we use the notation  $W_x(\vartheta) = w(x, Z_1, V(\vartheta))$  such that  $W_x = w(x, Z_1, V(\vartheta_0))$ . Again we focus on the case where  $\mathcal{Y}$  is a compact interval of  $\mathbb{R}$ .

**C.3.1. Proof of Theorem 2.** The result follows by a similar argument to the proof of Lemma 3 using Lemmas 6 and 7 in place of Lemmas 4 and 5, and the delta method. For the sake of brevity, here we just outline the proof of the FCLT.

Let  $\psi_x(A, \vartheta, \beta) := \Lambda(W_x(\vartheta)' \beta) T$  such that  $G_T(y, x) = \mathbb{E}_P \psi_x(A, \vartheta_0, \beta_0(y)) / \mathbb{E}_P T$  and  $\widehat{G}(y, x) = \mathbb{E}_n \psi_x(A, \widehat{\vartheta}, \widehat{\beta}(y)) / \mathbb{E}_n T$ . Then, for  $\widehat{\psi}_{y,x} := \psi_x(A, \widehat{\vartheta}, \widehat{\beta}(y))$  and  $\psi_{y,x} := \psi_x(A, \vartheta_0, \beta_0(y))$ ,

$$\begin{aligned} \sqrt{n} \left[ \mathbb{E}_n \psi_x(A, \widehat{\vartheta}, \widehat{\beta}(y)) - \mathbb{E}_P \psi_x(A, \vartheta_0, \beta_0(y)) \right] &= \mathbb{G}_n \left[ \widehat{\psi}_{y,x} \right] + \sqrt{n} \mathbb{E}_P \left[ \widehat{\psi}_{y,x} - \psi_{y,x} \right] \\ &=_{(1)} \mathbb{G}_n[\psi_{y,x}] + \bar{o}_{\mathbb{P}}(1) + \sqrt{n} \mathbb{E}_P \left[ \widehat{\psi}_{y,x} - \psi_{y,x} \right] \\ &=_{(2)} \mathbb{G}_n[\psi_{y,x}] + \bar{o}_{\mathbb{P}}(1) + \mathbb{G}_n[h_{y,x}] + \bar{o}_{\mathbb{P}}(1), \end{aligned}$$

where relations (1) and (2) follow by Lemma 6 and Lemma 7 with  $\widetilde{\beta}(y) = \widehat{\beta}(y)$ , respectively, using that  $\|\widehat{\vartheta} - \widetilde{\vartheta}\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n})$ ,  $\widetilde{\vartheta} \in \Upsilon$ , and  $\|\widetilde{\vartheta} - \vartheta\|_{T,\infty} = O_{\mathbb{P}}(1/\sqrt{n})$  by Lemma 2, and  $\sqrt{n}(\widehat{\beta}(y) - \beta_0(y)) = -J(y)^{-1} \mathbb{G}_n(f_y + g_y) + \bar{o}_{\mathbb{P}}(1)$  from step 2 of the proof of Lemma 3.



The functions  $(y, x) \mapsto \psi_{y,x}$  and  $(y, x) \mapsto h_{y,x}$  are  $P$ -Donsker by Example 19.7 in van der Vaart (1998) because they are Lipschitz continuous on  $\mathcal{Y}\overline{\mathcal{X}}$ . Hence, by the Functional Central Limit Theorem

$$\mathbb{G}_n(\psi_{y,x} + h_{y,x}) \rightsquigarrow Z(y, x) \text{ in } \ell^\infty(\mathcal{Y}\overline{\mathcal{X}}),$$

where  $(y, x) \mapsto Z(y, x)$  is a zero mean Gaussian process with uniformly continuous sample paths and covariance function

$$\text{Cov}_P[\psi_{y,x} + h_{y,x}, \psi_{v,u} + h_{v,u}], \quad (y, x), (v, u) \in \mathcal{Y}\overline{\mathcal{X}}.$$

The result follows by the functional delta method applied to the ratio of  $\mathbb{E}_n\psi_x(A, \widehat{\vartheta}, \widehat{\beta}(y))$  and  $\mathbb{E}_nT$  using that

$$\begin{pmatrix} \mathbb{G}_n\psi_x(A, \widehat{\vartheta}, \widehat{\beta}(y)) \\ \mathbb{G}_nT \end{pmatrix} \rightsquigarrow \begin{pmatrix} Z(y, x) \\ Z_T \end{pmatrix},$$

where  $Z_T \sim N(0, p_T(1 - p_T))$ ,

$$\text{Cov}_P(Z(y, x), Z_T) = G_T(y, x)p_T(1 - p_T),$$

and

$$\begin{aligned} & \text{Cov}_P[\psi_{y,x} + h_{y,x}, \psi_{v,u} + h_{v,u} \mid T = 1] \\ &= \frac{\text{Cov}_P[\psi_{y,x} + h_{y,x}, \psi_{v,u} + h_{v,u}] - G_T(y, x)G_T(v, u)p_T(1 - p_T)}{p_T}. \end{aligned}$$

□

**Lemma 6.** *[Stochastic equicontinuity] Let  $e \geq 0$  be a positive random variable with  $\mathbb{E}_P[e] = 1$ ,  $\text{Var}_P[e] = 1$ , and  $\mathbb{E}_P|e|^{2+\delta} < \infty$  for some  $\delta > 0$ , that is independent of  $(Y, X, Z, W, V)$ , including as a special case  $e = 1$ , and set, for  $A = (e, Y, X, Z, W, V)$ ,*

$$\psi_x(A, \vartheta, \beta) := e \cdot \Lambda(W_x(\vartheta)' \beta) \cdot T.$$

*Under Assumptions 3–5, the following relations are true.*

(a) *Consider the set of functions*

$$\mathcal{F} := \{\psi_x(A, \vartheta, \beta) : (\vartheta, \beta, x) \in \Upsilon_0 \times \mathcal{B} \times \overline{\mathcal{X}}\},$$

*where  $\overline{\mathcal{X}}$  is a compact subset of  $\mathbb{R}$ ,  $\mathcal{B}$  is a compact set under the  $\|\cdot\|_2$  metric containing  $\beta_0(y)$  for all  $y \in \mathcal{Y}$ ,  $\Upsilon_0$  is the intersection of  $\Upsilon$ , defined in Lemma*

2, with a neighborhood of  $\vartheta_0$  under the  $\|\cdot\|_{T,\infty}$  metric. This class is  $P$ -Donsker with a square integrable envelope of the form  $e$  times a constant.

(b) Moreover, if  $(\vartheta, \beta(y)) \rightarrow (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$ , then

$$\sup_{(y,x) \in \mathcal{Y}\bar{\mathcal{X}}} \|\psi_x(A, \vartheta, \beta(y)) - \psi_x(A, \vartheta_0, \beta_0(y))\|_{P,2} \rightarrow 0.$$

(c) Hence for any  $(\tilde{\vartheta}, \tilde{\beta}(y)) \rightarrow_{\mathbb{P}} (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$  such that  $\tilde{\vartheta} \in \Upsilon_0$ ,

$$\sup_{(y,x) \in \mathcal{Y}\bar{\mathcal{X}}} \|\mathbb{G}_n \psi_x(A, \tilde{\vartheta}, \tilde{\beta}(y)) - \mathbb{G}_n \psi_x(A, \vartheta_0, \beta_0(y))\|_2 \rightarrow_{\mathbb{P}} 0.$$

(d) For any  $(\hat{\vartheta}, \tilde{\beta}(y)) \rightarrow_{\mathbb{P}} (\vartheta_0, \beta_0(y))$  in the  $\|\cdot\|_{T,\infty} \vee \|\cdot\|_2$  metric uniformly in  $y \in \mathcal{Y}$ , so that

$$\|\hat{\vartheta} - \tilde{\vartheta}\|_{T,\infty} = o_{\mathbb{P}}(1/\sqrt{n}), \text{ where } \tilde{\vartheta} \in \Upsilon_0,$$

we have that

$$\sup_{(y,x) \in \mathcal{Y}\bar{\mathcal{X}}} \|\mathbb{G}_n \psi_x(A, \hat{\vartheta}, \tilde{\beta}(y)) - \mathbb{G}_n \psi_x(A, \vartheta_0, \beta_0(y))\|_2 \rightarrow_{\mathbb{P}} 0.$$

**Proof of Lemma 6.** The proof is omitted because is similar to the proof of Lemma 4.  $\square$

**Lemma 7.** [Local expansion] Under Assumptions 3–5, for

$$\begin{aligned} \hat{\delta}(y) &= \sqrt{n}(\tilde{\beta}(y) - \beta_0(y)) = \bar{O}_{\mathbb{P}}(1); \\ \hat{\Delta}(x, r) &= \sqrt{n}(\hat{\vartheta}(x, r) - \vartheta_0(x, r)) = \sqrt{n} \mathbb{E}_n[\ell(A, x, r)] + o_{\mathbb{P}}(1) \text{ in } \ell^\infty(\bar{\mathcal{X}\mathcal{R}}), \\ \|\sqrt{n} \mathbb{E}_n[\ell(A, \cdot)]\|_{T,\infty} &= O_{\mathbb{P}}(1), \end{aligned}$$

we have that

$$\begin{aligned} \sqrt{n} \left\{ \mathbb{E}_P \Lambda[W_x(\hat{\vartheta})' \tilde{\beta}(y)]T - \mathbb{E}_P \Lambda[W_x' \beta_0(y)]T \right\} &= \mathbb{E}_P \{ \lambda[W_x' \beta_0(y)] W_x T \} \hat{\delta}(y) \\ &\quad + \mathbb{E}_P \{ \lambda[W_x' \beta_0(y)] \dot{W}_x' \beta_0(y) T \ell(a, X, R) \} \Big|_{a=A} + \bar{o}_{\mathbb{P}}(1), \end{aligned}$$

where  $\bar{o}_{\mathbb{P}}(1)$  denotes order in probability uniform in  $(y, x) \in \mathcal{Y}\bar{\mathcal{X}}$ .

**Proof of Lemma 7.** The proof is omitted because is similar to the proof of Lemma 5.  $\square$

C.3.2. *Proof of Theorem 3.* The result follows from Theorem 2 and the functional delta method, because the map  $\phi : H \mapsto \int_{\mathcal{Y}^+} 1(H(y, x) \leq \tau)dy - \int_{\mathcal{Y}^-} 1(H(y, x) \geq \tau)dy$  is Hadamard differentiable at  $H = G_T$  under the conditions of the theorem by Proposition 2 of Chernozhukov, Fernandez-Val and Galichon (2010) with derivative

$$\phi'_{G_T}(h) = -\frac{h(\phi(\cdot, x), x)}{g_T(\phi(\cdot, x), x)}.$$

C.3.3. *Proof of Theorem 4.* The result follows from Theorem 2 and the functional delta method, because the map  $\varphi : H \mapsto \int_{\mathcal{Y}} [1(y \geq 0) - H(y, x)]dy$  is Hadamard differentiable at  $H = G_T$  by Lemma 8 with derivative

$$\varphi'_{G_T}(h) = -\int_{\mathcal{Y}} h(y, x)\nu(dy).$$

**Lemma 8.** [*Hadamard Differentiability of ASF Map*] The ASF map  $\varphi : \ell^\infty(\mathcal{Y}\overline{\mathcal{X}}) \rightarrow \ell^\infty(\overline{\mathcal{X}})$  defined by

$$H \mapsto \varphi(H) := \int_{\mathcal{Y}} [1(y \geq 0) - H(y, x)]\nu(dy),$$

is Hadamard-differentiable at  $H = G$ , tangentially to the set of uniformly continuous functions on  $\mathcal{Y}\overline{\mathcal{X}}$ , with derivative map  $h \mapsto \varphi'_G(h)$  defined by

$$\varphi'_G(h) := -\int_{\mathcal{Y}} h(y, x)\nu(dy),$$

where the derivative is defined and is continuous on  $\ell^\infty(\mathcal{Y}\overline{\mathcal{X}})$ .

**Proof of Lemma 8.** Consider any sequence  $H^t \in \ell^\infty(\mathcal{Y}\overline{\mathcal{X}})$  such that for  $h^t := (H^t - G)/t$ ,  $h^t \rightarrow h$  in  $\ell^\infty(\mathcal{Y}\overline{\mathcal{X}})$  as  $t \searrow 0$ , where  $h$  is a uniformly continuous function on  $\mathcal{Y}\overline{\mathcal{X}}$ . We want to show that as  $t \searrow 0$ ,

$$\frac{\varphi(H^t) - \varphi(G)}{t} - \varphi'_G(h) \rightarrow 0 \text{ in } \ell^\infty(\mathcal{Y}\overline{\mathcal{X}}).$$

The result follows because by linearity of the map  $\varphi$

$$\frac{\varphi(H^t) - \varphi(G)}{t} = -\int_{\mathcal{Y}} h^t(y, x)\nu(dy) \rightarrow -\int_{\mathcal{Y}} h(y, x)\nu(dy) = \varphi'_G(h).$$

The derivative is well-defined over  $\ell^\infty(\mathcal{Y}\overline{\mathcal{X}})$  and continuous with respect to the sup-norm on  $\ell^\infty(\mathcal{Y}\overline{\mathcal{X}})$ .

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