# Peer Effects in Endogenous Networks

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**Abstract**

This paper presents a simple model of strategic network formation with local complementarities in effort levels and positive local externalities. Equilibrium networks display - other than the complete and the empty network - a core-periphery structure, which is commonly observed in empirical studies. Ex-ante homogenous agents may obtain very different ex-post outcomes. These findings are relevant for a wide range of social and economic phenomena, such as educational attainment, criminal activity, labor market participation and R&D expenditures of firms.

**Key Words:** Network formation, peer effects, strategic complements, positive externalities. **JEL Codes:** D62, D85.

## 1 Introduction

Peer effects and social structure play an important role in determining individual behavior and aggregate outcomes in many social and economic settings. This has been documented by a large body of empirical work, which finds peer effects and network position crucial for decisions concerning educational attainment, criminal activity, labor market participation and R&D expenditures of firms. In these settings an agent's optimal action and payoff is thought to depend directly on the action or payoff of others (peer effects), while the relevant reference group is determined by the network of relationships between agents (social structure). This stands in contrast to markets, where individuals interact through an anonymous process of price formation.

This paper presents a model of strategic network formation in the presence of peer effects. In accordance with empirical studies, peer effects are modeled as local positive externalities

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and strategic complementarities.[1] The setup is fairly simple. Agents simultaneously choose a non-negative effort level and create links at a cost. We solve for two specifications of the model. First, two-sided link formation, where linking cost are shared equally. Here, we use Pairwise Nash equilibrium as equilibrium concept, since reflects the bilateral nature of creating a link (and sharing the cost). Second, one-sided link formation, where linking cost are borne unilaterally. This specification allows us to employ Nash equilibrium.[2] The meaning of a link is that agents benefit directly from efforts exerted by their neighbors (local positive externalities). We assume payoffs such that the value function is convex. That is, when best responding, own payoffs are convex in the sum of effort levels of direct neighbors. For both, one-sided and two-sided link formation, we show that equilibrium network structures are of only three different types: empty, core-periphery and complete.[3] We define a complete core-periphery network as a network where all agents in the periphery are connected to the core. For the case of linear-quadratic payoff functions, we provide necessary and sufficient conditions for the existence of a star network in the two-sided specification and for a periphery-sponsored core-periphery network in the one-sided specification. These structures are of particular interest, since they are frequently observed in empirical work.

Two related papers in the empirical networks literature are Calvó-Armengol, Pattacchini and Zenou (2005 and 2009). The authors use a detailed data set on friendship networks in U.S. high schools (AddHealth) to test a structural model with linear-quadratic payoffs on a fixed network. This allows for measurement of peer effects in education and delinquent behavior, respectively. In both studies Calvó-Armengol, Pattacchini and Zenou find a positive relationship between grades and delinquency rates on the one hand and centrality on the other hand. Network position turns out to be a key determinant for an individual's effort level. This emphasizes the importance of social structure for peer influences, as opposed to average in-group effects. In both papers local spillovers and strategic complementarities are observed, much in line with the assumptions of my model. Further note that the linear-quadratic payoff function used in their model is a special case of the class of payoff functions considered here.

A recent paper by König, Tessone, and Zenou (2012) addresses link formation for the linear-quadratic specification. However, their link formation process is very different. The setup is dynamic and in each time period players play a two-stage game. In the first stage,

[1] See Hoxby (2000), Sacerdote (2001) for a treatment of peer effects in education, Glaeser, Sacerdote and Scheinkman (1996), Case and Katz (1991) and Ludwig et al (2001) for criminal and delinquent behaviour, Topa (2001) and Conley and Topa (2001) for labor markets and Cohen and Levinthal (1989, 1990) and Levin and Reiss (1988) for R&D expenditure of firms.

[2] Pairwise Nash equilibrium was first discussed in Jackson and Wolinsky (1996). For applications see, for example, Goyal and Joshi (2003) and Belleflamme and Bloch (2004). The one-sided specification follows Bala and Goyal (2000).

[3] In a core-periphery network the set of agents can be partitioned into two sets, the core and the periphery, such that all pairs of agents in the core are connected and no pair of agents in the periphery is connected.

agents choose their effort levels on a fixed network, while in the second stage a randomly selected player may create a new link in the current network, at no cost. Links decay over time, with more valuable links decaying at a slower rate. König, Tessone, and Zenou (2012) then introduce noise into the model and derive the stochastically stable networks. Interestingly, these are shown to be nested split graphs, which subsume the core-periphery structures obtained here.[4] The approach undertaken by Galeotti and Goyal (2010) is similar to mine. They also solve a simultaneous move game, where agents choose a non-negative, continuous effort level and link formation is one-sided. Externalities are also positive and local, but different from the paper presented here, Galeotti and Goyal (2010) assume strategic substitutes. The only strict equilibria in Galeotti and Goyal (2010) are (complete) core-periphery networks, a network architecture for which we provide necessary and sufficient conditions in the linear-quadratic case. This is interesting from a theoretical point of view, as it shows that in a model with continuous effort levels, core-periphery networks are not a feature of strategic substitutes alone, but may also arise under strategic complementarities. The model presented by Baetz (2012) is also closely related. The setup is as in the one-sided link formation specification of my paper, but instead of convex value functions, Baetz assumes concave value functions. A complete characterization is not yet obtainable, but it can be shown that biregular bipartite graphs and core-periphery networks may be sustained in equilibrium. Finally, Ballester, Calvó-Armengol and Zenou (2006) exhibits some similarities with my work. Again the presence of a link allows agents to benefit from each other's effort levels and payoffs are linear-quadratic. Different from my paper, however, Ballester, Calvó-Armengol and Zenou (2006) assume not only local strategic complementarities, but also allow for global strategic substitutes. The authors then link equilibrium actions to Bonacich centrality on a fixed network.

The rest of the paper is organized as follows: Section 2 describes the model and introduces the two-sided specification. Section 3 presents the analysis and Section 4 concludes. The proofs for the one-sided specification are relegated to the Appendix.

# 2 The Two-Sided Model

## 2.1 Model Description

Let $N = \{1, 2, ..., n\}$ with $n \geq 3$ be the set of players. Each agent $i$ chooses a personal effort level $x_i \in X$ and announces a set of agents to whom he wishes to be linked to, which we represent as a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n-1})$, with $g_{i,j} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of agent $i's$ strategies is

[4] A nested split graph is a graph such that, if the link between $i$ and $j$ exists and the degree of $k$ is at least as high as the degree of $j$, then the link between $i$ and $k$ also exists.

denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ then specifies the individual effort level for each player, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and the set of intended links, $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. A link between $i$ and $j$, denoted with $\bar{g}_{i,j} = 1$, is created if and only if *both* agents intend to create a link. That is, $\bar{g}_{i,j} = 1$ if $g_{i,j} = g_{j,i} = 1$, and $\bar{g}_{i,j} = 0$ otherwise. From $\mathbf{g}$ we thereby obtain the undirected graph $\bar{\mathbf{g}}$ with $\bar{g}_{i,j} = \bar{g}_{j,i}$. The presence of a link $\bar{g}_{i,j} = 1$ allows players to directly benefit from the effort level exerted by the respective other. Let $N_i(\mathbf{g}) = \{j \in N : g_{i,j} = 1\}$ be the set of agents to which agent $i$ extends a link and denote the corresponding cardinality with $\eta_i(\mathbf{g}) = |N_i(\mathbf{g})|$. Define the set of $i's$ neighbors in $\bar{\mathbf{g}}$ with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ and define $\eta_i(\bar{\mathbf{g}}) = |N_i(\bar{\mathbf{g}})|$. The aggregate effort level of agent $i's$ neighbors in $\bar{\mathbf{g}}$ is written as $y_i = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. We drop the subscript of $y_i$ when it is clear from the context. Given a network $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} + \bar{g}_{i,j}$ and $\bar{\mathbf{g}} - \bar{g}_{i,j}$ have the following interpretation. When $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} + \bar{g}_{i,j}$ adds the link $\bar{g}_{i,j} = 1$, while if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j} = \bar{\mathbf{g}}$. Similarly, if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} - \bar{g}_{i,j}$ deletes the link $\bar{g}_{i,j}$, while if $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j} = \bar{\mathbf{g}}$. The network is called empty and denoted with $\bar{\mathbf{g}}^e$, if $\bar{g}_{i,j} = 0 \; \forall i, j \in N$ and complete and denoted with $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1 \; \forall i, j \in N$.

Payoffs of player $i$ under strategy profile $\mathbf{s} = (\mathbf{x},\mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi(x_i, y_i) - \eta_i(\mathbf{g})k,$$

where $k$ denotes the cost of extending a link. Gross payoffs $\pi(x_i, y_i)$ are a function of own effort, $x_i$, and the sum of effort levels of direct neighbors, $y_i = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. We assume strict positive externalities and strict strategic complementarities in effort levels, so that $\frac{\partial \pi(x,y)}{\partial y} > 0$ and $\frac{\partial^2 \pi(x,y)}{\partial x \partial y} > 0$. Further assume that $\frac{\partial^2 \pi(x,y)}{\partial^2 x} < 0$. The latter assumption, together with the convexity of $X$, guarantees a unique maximizer, which is denoted by $\bar{x}(y)$. We also assume $\bar{x}(y) > 0$.[5] From $\frac{\partial^2 \pi(x,y)}{\partial x \partial y} > 0$ we know that $\frac{\partial \bar{x}(y)}{\partial y} > 0$. Best response functions are assumed to be either linear or concave, so that $\frac{\partial^2 \bar{x}(y)}{\partial^2 y} = 0$ or $\frac{\partial^2 \bar{x}(y)}{\partial^2 y} < 0$. Denote the value function with $v(y) = \pi(\bar{x}(y), y)$ and assume that $\frac{\partial^2 v(y)}{\partial^2 y} > 0$. In order to guarantee existence, we further assume that there exists a value of $y$ such that $\frac{\partial \bar{x}(y)}{\partial y} < \frac{1}{n-1}$.

One can easily check that $\pi(x_i, y_i) = x_i - \frac{\beta}{2} x_i^2 + \lambda x_i \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$ fulfills the above conditions. The class of payoff functions described above therefore includes the linear-quadratic payoff function in Ballester, Pattachini and Zenou (2005 and 2009). The payoff function in Galeotti and Goyal (2010) is given by $\pi(x_i, y_i) = f(x_i + \sum_{j \in N_i(\bar{\mathbf{g}})} x_j) - c(x_i)$, where $f$ is assumed to be concave and $c$ is linear. By making appropriate assumptions on $f$ and $c$, we can generate a model of positive externalities and strategic complements that fits our setup.[6] Link formation in Galeotti and Goyal (2010) is one-sided and we cover this case in

[5] This assumption guarantees that there does not always exist a Pairwise Nash equilibrium that is empty (for any linking cost).

[6] Arguably the simplest such specification is to assume $\pi(x_i, y_i) = (x_i + \sum_{j \in N_i(\bar{\mathbf{g}})} x_j)^2 - x_i^3$.

the second part of the paper. First, however, we present the two-sided model and define pairwise Nash equilibrium ($PNE$).

A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is a pairwise Nash equilibrium *iff*

(i) $\mathbf{s}$ is a Nash Equilibrium, and

(ii) for all $\bar{g}_{i,j} = 0$, if $\Pi_i(x'_i, x'_j, \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{i,j}) > \Pi_i(\mathbf{s})$, then $\Pi_j(x'_i, x'_j, \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{ij}) < \Pi_j(\mathbf{s})$, $\forall x'_i, x'_j \in X$.

Note that a network is pairwise Nash stable if it is both, a Nash equilibrium and pairwise stable. Note also that due to the convexity of the value function, pairwise Nash stable and pairwise stable networks coincide.[7]

## 2.2 Analysis

We start the analysis by providing a proof for the existence and uniqueness of a Nash equilibrium on a fixed network. Part of the proof relies on a result provided by Kennan (2001). As in Kennan's paper, a vector $\mathbf{b}$ is larger than a vector $\mathbf{a}$, if and only if $b_i > a_i$ $\forall i \in N$.

**Proposition 1:** *For any fixed network, $\bar{\mathbf{g}}$, there exists a unique $NE$ in effort levels, $\mathbf{x}^*(\bar{\mathbf{g}})$.*

***Proof.*** We discern two cases. First, assume linear best response functions, such that $\bar{x}_i(\sum_{j \in N_i(\bar{\mathbf{g}})} x_j) = \frac{\lambda}{\beta} \sum_{j \in N_i(\mathbf{g})} x_j + \frac{1}{\beta}$. This allows us to use the existence result provided by Ballester, Calvó-Armengol and Zenou (2006). A $NE$ exists and is unique for $\beta > \lambda \mu_1(\mathbf{g})$, where $\mu_1(\mathbf{g})$ is the largest eigenvalue of the adjacency matrix of $\bar{\mathbf{g}}$. The largest eigenvector for a graph lies between the following bounds $max\{d_{avg}(\bar{\mathbf{g}}), \sqrt{d_{max}(\bar{\mathbf{g}})}\} \leq \mu_1(\bar{\mathbf{g}}) \leq d_{max}(\bar{\mathbf{g}})$,[8] where $d_{max}(\bar{\mathbf{g}})$ is the maximum number of degree and $d_{avg}(\bar{\mathbf{g}})$ the average degree in network $\bar{\mathbf{g}}$. Note that then the largest eigenvector for a graph with $n$ agents is at most $n-1$ (and maximal and equal to $n-1$ in the complete network, $\bar{\mathbf{g}}^c$). Therefore, a sufficient condition for the existence of a unique $NE$ is that the slope of the best response function, $\frac{\lambda}{\beta} < \frac{1}{n-1}$. Second, assume strictly concave best response functions. Define the function $f_{\bar{\mathbf{g}}} : X^n \to X^n$ as

$$f_{\bar{\mathbf{g}}}(\mathbf{x}) = \begin{pmatrix} \bar{x}(\sum_{j \in N_1(\bar{\mathbf{g}})} x_j) \\ \vdots \\ \bar{x}(\sum_{j \in N_n(\bar{\mathbf{g}})} x_j) \end{pmatrix}.$$

The best response function $\bar{x}(y)$ is assumed to be strictly concave. From strategic complementarities we know that $\bar{x}(y)$ is strictly increasing and therefore $f$ is increasing and

[7]For the relationship between pairwise Nash stability and Nash stability, see Calvó-Armengol and İlkılıç (2009).

[8]See L. Lovasz, Geometric Representations of Graphs (2009).

strictly concave. We can now apply a result provided by Kennan (2001), which is restated here. Suppose $f$ is an increasing and strictly concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that $f(\mathbf{0}) \geq \mathbf{0}$, $f(\mathbf{a}) > \mathbf{a}$ for some positive vector $\mathbf{a}$, and $f(\mathbf{b}) < \mathbf{b}$ for some vector $\mathbf{b} > \mathbf{a}$. Then $f$ has a unique positive fixed point. Recall that $\bar{x}(0) > 0$ and therefore $f(\mathbf{0}) > \mathbf{0}$. To see that there exists a vector $\mathbf{a}$ such that $f(\mathbf{a}) > \mathbf{a}$, choose $\mathbf{a} = (\varepsilon_1, ..., \varepsilon_n)$ such that $\varepsilon_i = \varepsilon < \frac{1}{n-1}\bar{x}(0)$ $\forall i \in N$. The effort level of an agents with $\eta_i(\bar{\mathbf{g}})$ neighbors is then given by $\eta_i(\bar{\mathbf{g}})\varepsilon$. Since $\bar{x}$ is strictly increasing $\bar{x}(\eta_i(\bar{\mathbf{g}})\varepsilon) > \bar{x}(0) > \eta_i(\bar{\mathbf{g}})\varepsilon$. To show that $f(\mathbf{b}) < \mathbf{b}$, choose a vector $\mathbf{b} = (b_1, ..., b_n)$ with $b = b_i$ $\forall i \in N$. The condition can be written as $\bar{x}(\eta_i(\bar{\mathbf{g}})b) < b$, which holds if $b$ is sufficiently large, due to the assumption that $\frac{\partial \bar{x}(y)}{\partial y} < \frac{1}{n-1}$ for some value of $y$. To show that $\mathbf{b} > \mathbf{a}$, note that we can choose $\varepsilon$ (and therefore $\mathbf{a}$) arbitrarily close to *zero* for $\bar{x}(\eta_i(\bar{\mathbf{g}})\varepsilon) > \bar{x}(0) > \eta_i(\bar{\mathbf{g}})\varepsilon$ to hold. *Q.E.D.*

The following Lemma shows that agents in a complete component exert the same effort levels. This result will be useful for the equilibrium characterization.

**Lemma 1:** *NE effort levels are equal for all players in a complete component.*

***Proof.*** Assume to the contrary that there exists a pair of players $k$ and $l$, such that $x_k^* \neq x_l^*$, and, without loss of generality, that $x_k^* > x_l^*$. Note that in a complete component $N_k(\bar{\mathbf{g}}) \setminus \{l\} = N_l(\bar{\mathbf{g}}) \setminus \{k\}$. But then, if $x_k^* > x_l^*$, the sum of effort levels of $l$'s neighbors, $\sum_{j \in N_l(\bar{\mathbf{g}})} x_j^*$, is larger than the sum of effort levels of $k's$ neighbors, $\sum_{j \in N_k(\bar{\mathbf{g}})} x_j^*$. We have reached a contradiction, as from strict strategic complementarities it follows that $\sum_{j \in N_l(\bar{\mathbf{g}})} x_j^* > \sum_{j \in N_k(\bar{\mathbf{g}})} x_j^*$ implies $x_l^* > x_k^*$. *Q.E.D.*

In Lemma 2 we show that effort levels are maximal in the complete network. We use this result to prove Proposition 3.

**Lemma 2:** *NE effort levels are maximal in the complete network.*

***Proof.*** Denote the Nash equilibrium effort level in the complete network, $\bar{\mathbf{g}}^{\mathrm{c}}$, with $\mathbf{x}^{\mathrm{c}*}$, where $x^{c*} = x_i^*(\bar{\mathbf{g}}^c)$ $\forall i \in N$ from Lemma 1. Start by deleting a link $\bar{g}_{i,j}$ from $\bar{\mathbf{g}}^c$ and consider each player's best response to $\mathbf{x}^{\mathrm{c}*}$in $\bar{\mathbf{g}}^c - \bar{g}_{i,j}$. Agent $i's$ initial best response will be lower in $\bar{\mathbf{g}}^c - \bar{g}_{i,j}$ than in $\bar{\mathbf{g}}^c$, as $\sum_{j \in N_i(\bar{\mathbf{g}}^c - \bar{g}_{i,j})} x_j^* < \sum_{j \in N_i(\bar{\mathbf{g}}^{\mathbf{c}})} x_j^*$. Iterating on best responses, any agent $l$ with $\bar{g}_{i,l}^* = 1$ will decrease his effort level, and those sustaining links with $l$ will decrease their effort levels in turn, and so forth. The effort level of each agent is a decreasing sequence of real numbers, which is bounded below by $\bar{x}(0)$. We have therefore established convergence to a new equilibrium in $\bar{\mathbf{g}}^c - \bar{g}_{i,j}$ with $x_l^*(\bar{\mathbf{g}}^c - \bar{g}_{i,j}) < x^*(\bar{\mathbf{g}}^c)$ $\forall l \in N$. Note that any network $\bar{\mathbf{g}} \neq \bar{\mathbf{g}}^c$ can be obtained from $\bar{\mathbf{g}}^c$ by deleting a sequence of links. Effort levels are weakly decreasing at each step (strictly for any agent that is in the component from which a link is deleted) and therefore effort levels are maximal in the complete network. *Q.E.D.*

Next, we define two cost threshold cost, $k^1$and $k^2$. The first threshold, $k^1$, is equal to the gross marginal payoffs when a pair of agents creates a link in the empty network. Note that under Pairwise equilibrium we allow both agents creating the new link to adjust their effort levels. The second threshold, $k^2$, is defined as the average gross marginal payoffs of linking to $(n-1)$ agents in the complete network. Proposition 2 shows that for linking cost smaller than $k^1$, the unique pairwise Nash equilibrium is the complete network, while for linking cost larger than or equal to $k^1$, there exists a pairwise Nash equilibrium such that the network is empty. Proposition 3 shows that for linking cost larger than $k^2$, the unique pairwise Nash equilibrium is the empty network, while for linking cost smaller or equal to $k^2$, there exists a pairwise Nash equilibrium such that the network is empty.

**Definition 1:** $k^1 = v_i(x_j^*(\bar{\mathbf{g}}^e + \bar{g}_{i,j})) - v_i(0)$ and $k^2 = v((n-1)x^*(\bar{\mathbf{g}}^c)) - v(0)$

**Proposition 2:** *If $k < k^1$, then the unique PNE is the complete network. If $k \geq k^1$, then there exists a PNE such that the network is empty.*

***Proof.*** From Proposition 1 we know that there exists a unique equilibrium in a network where the only link is between $i$ and $j$, $\bar{g}_{i,j}^* = 1$. Since $i$ and $j$ form a complete component, $x^* = x_i^* = x_j^*$ (from Lemma 1) and the corresponding gross payoffs are given by $v(x_j^*(\bar{\mathbf{g}}^e + \bar{g}_{i,j})) = \pi_i^*(x_i^*, x_j^*)$. If $k < v_i(x_j^*(\bar{\mathbf{g}}^e + \bar{g}_{i,j})) - v_i(0) = k^1$, then a pair of agents $i$ and $j$ finds it profitable to create the link $\bar{g}_{i,j}^*$. Note that this is the least profitable link in any network, due to the convexity of the value function $v$ and strict strategic complementarities in effort levels. Therefore, every pair of agents must be connected for any $k < k^1$ and the unique $PNE$ is the complete network. If, on the other hand $k \geq v_i(x_j^*(\bar{\mathbf{g}}^e + \bar{g}_{i,j})) - v_i(0) = k^1$, then no pair of agents can profitably deviate in the empty network. Therefore, for $k \geq k^1$ a $PNE$ exists such that the network is empty. $Q.E.D.$

**Proposition 3:** *If $k > k^2$, then the unique PNE is the empty network. If $k \leq k^2$, then there exists a PNE such that the network is complete.*

***Proof.*** The relevant deviation to consider in a complete network is an agent deleting all his links. To see this, note that due to the convexity of $v$, $v(hx^{c*}) - v((h-1)x^{c*}) < v((n-1)x^{c*}) - v((n-2)x^{c*})$ for all $0 < h < n-1$. That is, marginal payoffs are increasing and an agent will want to delete all of his links, if any. Therefore, the maximum linking cost that can be sustained in the complete network are given by $v((n-1)x^{c*}) - v(0) = k^2$. Next we show that if $k = k^2$, then there exists no $PNE$ other than the complete network or the empty network. Assume that the most profitable deviation of an agent $i$ in network $\bar{\mathbf{g}} \neq \bar{\mathbf{g}}^c$ consists of deleting $h$ of his $\eta_i(\bar{\mathbf{g}}) =\mid N_i(\bar{\mathbf{g}}) \mid$ links. Note that $n-1 \geq \eta_i(\bar{\mathbf{g}}) \geq h$. Denote the network after proposed deviation with $\bar{\mathbf{g}}'$ and the set of agents whose links are deleted by the deviating agent $i$ with $H = \{j : \bar{g}_{i,j} = 1 \wedge \bar{g}'_{i,j} = 0\}$. We can then compare average payoffs per link in the complete network $\bar{\mathbf{g}}^{c*}$ with payoffs in $\bar{\mathbf{g}} \neq \bar{\mathbf{g}}^c$ and write

$$\frac{v((n-1)x^{c*})-v(0)}{n-1} \geq \frac{v(\eta_i(\bar{\mathbf{g}})x^{c*})-v(\eta_i(\bar{\mathbf{g}})x^{c*}-hx^{c*})}{\eta_i(\bar{\mathbf{g}})-h} > \frac{v(\sum_{j\in N_i(\bar{\mathbf{g}})} x_j^*)-v(\sum_{j\in N_i(\bar{\mathbf{g}})} x_j^*-\sum_{j\in H} x_j^*)}{\eta_i(\bar{\mathbf{g}})-h}.$$

The first inequality follows from the convexity of $v$, $n-1 \geq \eta_i(\bar{\mathbf{g}})$ and $h \geq 0$. The second inequality follows from the convexity of $v$ and $\eta_i(\bar{\mathbf{g}})x^{c*} - hx^{c*} > \sum_{j\in N_i(\mathbf{g})} x_j^* - \sum_{j\in H} x_j^*$ and $hx^{c*} > \sum_{j\in H} x_j^*$ (for the last two inequalities, recall that effort levels are maximal in the complete network). Therefore, for linking cost $k > k^2$, no links can be sustained and the unique $PNE$ is the empty network. For $k \leq k^2$ a $PNE$ exists such that the network is complete, since no agent finds it profitable to delete his links from $\frac{v((n-1)x^{c*})-v(0)}{n-1} = k^2$. *Q.E.D.*

Lemma 3 shows that $k^1 < k^2$. We have therefore shown that, for linking cost smaller than $k^1$, the unique $PNE$ is the complete network, while for linking cost larger than $k^2$ the unique $PNE$ network is the empty network. For linking cost $k \in [k^1, k^2]$ the complete and the empty network are a $PNE$.

**Lemma 3:** $k^1 < k^2$.

***Proof.*** $k^2 - k^1 = v((n-1)x^*(\bar{\mathbf{g}}^c)) - v_i(x_j^*(\bar{\mathbf{g}}^e + \bar{g}_{i,j}))$. From Lemma 2 we know that $x^*(\mathbf{g}^c) > x_j^*(\mathbf{g}^e + \bar{g}_{i,j})$ and since $v$ is increasing, $k^2 - k^1 > 0$. *Q.E.D.*

Next, we formally define a $core-periphery\ network$ as a network, such that the set of agents can be partitioned into two sets, where all pairs of agents within the first set (the core) are connected and no pair of agents within the second set (the periphery) is connected. Note that this definition does not state anything about links between pairs of agents where one agent is in the core and the other is in the periphery. A $complete\ core-periphery\ network$ is defined as a core-periphery network, in which all agents in the core are linked to all agents in the periphery. Note that the *star network* is a special case of a complete core-periphery network.

**Definition 2:** A network $\mathbf{g}$ is a $core-periphery\ network$ if the set of agents $N$ can be partitioned into two sets $C(\bar{\mathbf{g}})$ (the core) and $P(\bar{\mathbf{g}})$ (the periphery), such that $\bar{g}_{i,j} = 1\ \forall i,j \in C(\bar{\mathbf{g}})$ and $\bar{g}_{i,j} = 0\ \forall i,j \in P(\bar{\mathbf{g}})$. A $complete\ core-periphery\ network$ is a $core-periphery\ network$ such that $\bar{g}_{i,j} = 1\ \forall i \in C(\bar{\mathbf{g}})$ and $\forall j \in P(\bar{\mathbf{g}})$. A *star* is a a $complete\ core-periphery\ network$ such that $\mid C(\bar{\mathbf{g}}) \mid = 1$.

In the following we provide three Lemmas which are useful for establishing our first main result in Proposition 4. Proposition 4 shows that any network that is not complete, empty, or core-periphery is not a $PNE$. In Lemma 4 we prove that in any $PNE$, if an agent $i$ is linked to agent $l$, then agent $i$ must also be linked to any agent $k$ with higher or equal effort level than agent $l$. This is a direct consequence of the convexity of the value function. Lemma 5 then shows that in any $PNE$, agents with same effort levels must be connected to the same set of agents, while in Lemma 6 we prove that the neighborhoods of agents with lower effort levels are contained in the neighborhoods of agents with higher effort levels.

**Lemma 4:** *If $\bar{g}^*_{i,l} = 1$, then $\bar{g}^*_{i,k} = 1$ for all agents $k$ with $x^*_k \geq x^*_l$.*

***Proof.*** For $\bar{g}^*_{i,l} = 1$ to be part of a $PNE$, it must be that agent $i$ and agent $j$ can not profitably deviate by deleting the link. For agent $i$ this condition reads $v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. From the convexity of the value function it then follows that linking to any agent $k$ with $x^*_k \geq x^*_l$ is also profitable for agent $i$. To see this, note that $v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j + x^*_k) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j) > v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. For agent $l$ to not be able to profitably deviate by deleting his link with agent $i$, we need that $v(\sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j - x^*_i) \geq k$. Note next that, for $x^*_k \geq x^*_l$to hold we must have $\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j \geq \sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j$, which follows directly from strict strategic complementarities. Therefore, $v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j + x^*_i) - v(\sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j) > v(\sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_l(\bar{\mathbf{g}}^*)} x^*_j - x_i) \geq k$. We have therefore shown that if $\bar{g}^*_{i,l} = 1$, then agent $i$ finds it profitable to link to any agent $k$ with $x^*_k \geq x^*_l$, while any agent $k$ finds it profitable to link to agent $i$ and therefore $\bar{g}^*_{i,k} = 1$ for all agents $k$ with $x^*_k \geq x^*_l$. *Q.E.D.*

**Lemma 5:** *In any $PNE$, $x^*_i = x^*_k \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.*

***Proof.*** First, $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\} \Rightarrow x^*_i = x^*_k$. If $\bar{g}^*_{i,k} = 0$, then $i$ and $k$ access the same effort level, i.e. $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j = y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j$ and therefore $x^*_i = x^*_k$. Assume next that $\bar{g}^*_{i,k} = 1$ and, without loss of generality, that $x^*_i > x^*_k$. But then $k$ accesses a higher effort level than $i$, $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j < y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j$, and we have reached a contradiction. Second, $x^*_i = x^*_k \Rightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$. Assume to the contrary that $x^*_i = x^*_k$ and $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$. Note that for $x^*_i = x^*_k$, effort levels accessed must be equal by strict strategic complementarities, so that $y_i = y_k$. There must therefore exist an agent $l$, such that $l \in N_k(\bar{\mathbf{g}}^*)$ and $l \notin N_i(\bar{\mathbf{g}}^*)$. For the link $\bar{g}^*_{k,l} = 1$ to be in place in $\bar{\mathbf{g}}^*$ we must have that $v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. From $y_i = y_k$ and the convexity of the value function we then reach a contradiction since $v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j + x_l) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j) > v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. *Q.E.D.*

**Lemma 6:** *In any $PNE$, $x^*_i \leq x^*_k \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.*

***Proof.*** First, $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\} \Rightarrow x^*_i \leq x^*_k$. If $\bar{g}^*_{i,k} = 0$, then $k$ accesses a weakly higher effort level, i.e. $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j \leq y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j$ and therefore $x^*_i \leq x^*_k$. Assume next that $\bar{g}^*_{i,k} = 1$ and, without loss of generality that $x^*_i > x^*_k$. But then $k$ accesses a strictly higher effort level than $i$, $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j < y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x^*_j$, and we have reached a contradiction. Second, $x^*_i \leq x^*_k \Rightarrow N_i(\mathbf{g}^*) \setminus \{k\} \subseteq N_k(\mathbf{g}^*) \setminus \{i\}$. Assume to the contrary that $x^*_i \leq x^*_k$ and there exists an agent $l$ such that $l \in N_i(\bar{\mathbf{g}}^*)$ and $l \notin N_k(\bar{\mathbf{g}}^*)$. For the link

$\bar{g}^*_{i,l} = 1$ to be in place in $\bar{\mathbf{g}}^*$ we must have that $v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. But from $y_i \leq y_k$ and the convexity of the value function we reach a contradiction since $v(\sum_{j\in N_k(\bar{\mathbf{g}}^*)} x^*_j + x_l) - v(\sum_{j\in N_k(\bar{\mathbf{g}}^*)} x^*_j) > v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x^*_j) - v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x^*_j - x^*_l) \geq k$. *Q.E.D.*

We are now in the position to prove Proposition 4, which states that in any $PNE$, such that there exists a pair of agents with different effort levels, the network must display a core-periphery structure.

**Proposition 4:** *In any $PNE$ with a pair of agents $i$ and $j$, such that $x^*_i \neq x^*_j$, the network displays a core-periphery structure.*

***Proof.*** Rank agents by their effort levels in increasing order, such that $x^*_1 \leq x^*_2 \leq ... \leq x^*_{n-1} \leq x^*_n$. We know from Lemma 1 that the network is not complete, since there exists a pair of agents $i$ and $j$ such that $x^*_i \neq x^*_j$. The network is not empty, as agents have identical payoff functions and singleton agents display equal effort levels, $\bar{x}(0)$. First, we show that the two lowest ranked agents, agent 1 and agent 2, are not connected. Assume to the contrary that $\bar{g}^*_{1,2} = 1$. From Lemma 4 we know that $i$ must be connected to all agents, since $x^*_j \geq x^*_2$ $\forall j \geq 2$. Lemma 6 then implies that the network is complete, since $N_i(\bar{\mathbf{g}}^*) \setminus \{j\} \subseteq N_j(\bar{\mathbf{g}}^*) \setminus \{i\}$ holds for agents $j$ with $x^*_j \geq x^*_1$ $\forall j \geq 1$. But then $x^*_i = x^*_j$ $\forall i, j \in N$ by Lemma 1 and we have reached a contradiction. Since the network is neither empty nor complete, at least one link exists. Pick the agent $i$ with the lowest subscript who has a link. If $i$ has more than one link, pick the link to the agent with the lowest subscript $j$. We discern two cases. First, agent $i$ and $j$ are adjacent. As $i$ is the agent with the lowest subscript to sustain a link, all agents with lower subscripts have no links. All agents with a subscript higher or equal to $i$ are connected to each other. Again by Lemma 4 agent $i$ is connected to all agents with subscripts higher or equal than $j$ and by Lemma 6, $\bar{g}^*_{l,m} = 1$ $\forall l, m \geq i$. The periphery, $P(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k < i$, while the core, $C(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k \geq i$. Second, agent $i$ and $j$ are not adjacent. Note that since $\bar{g}^*_{i,j} = 1$ and $x^*_i \leq x^*_{j-1}$, we know by Lemma 6 that the link between $j-1$ and $j$, $\bar{g}^*_{j-1,j} = 1$, also exists. Next, check for the link $\bar{g}^*_{j-2,j-1}$. If $\bar{g}^*_{j-2,j-1} = 0$, then by Lemma 6 no agent with a subscript lower than $j-2$ is connected to $j-1$. Furthermore, no pair of agents with subscripts of lower or equal than $j-2$ is connected. Assume to the contrary that there exists a pair of nodes $l, m$ with $l \leq m < j-2$ and $\bar{g}^*_{l,m} = 1$. By Lemma 4 we must then have that $\bar{g}^*_{l,j-1} = 1$. This, however, contradicts Lemma 6, since $\bar{g}^*_{j-2,j-1} = 0$. The periphery, $P(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k < j$, while the core, $C(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k \geq j$. If $\bar{g}^*_{j-2,j-1} = 1$, check for the link $\bar{g}^*_{j-3,j-2}$. If $\bar{g}^*_{j-3,j-2} = 0$, then by above argument the periphery, $P(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k < j-1$, while the core, $C(\bar{\mathbf{g}}^*)$, consists of agents with subscripts $k \geq j-1$. If $\bar{g}^*_{j-3,j-2} = 1$, proceed in descending order until a pair of adjacent agents is found that is not connected and define the core and periphery accordingly. Note that such a link exists, since $i$ and $j$ were assumed to be not adjacent and therefore $\bar{g}^*_{i,i+1} = 0$. This concludes the proof. $Q.E.D.$

Proposition 5 provides an existence result for core-periphery networks. If $\frac{\delta\pi(x,y)}{\delta x \delta y}$ is sufficiently small, i.e. if strategic complementarities are not too strong, then a core-periphery network exists with at least three agents in the core for appropriately chosen $k$. Denote the cardinality of the core with $c(\bar{\mathbf{g}}^*) =\mid C(\bar{\mathbf{g}}^*) \mid$ and the cardinality of the periphery with $p(\bar{\mathbf{g}}^*) =\mid P(\bar{\mathbf{g}}^*) \mid$. For ease of notation we will write $c$ and $p$, respectively.

**Proposition 5:** *For* $\frac{\delta\pi(x,y)}{\delta x \delta y}$ *sufficiently small, there exist a linking cost* $k$*, such that a* $PNE$ *displays a* $core-periphery$ *network with* $\mid C(\bar{\mathbf{g}}^*) \mid \geq 3$.

***Proof.*** Partition the set of agents into the core, $C(\mathbf{g}^*)$, with $\bar{g}^*_{i,j} = 1$ $\forall i, j \in C(\mathbf{g}^*)$, and the periphery, $P(\mathbf{g}^*)$, with $\bar{g}^*_{i,j} = 0$ $\forall i, j \in P(\mathbf{g}^*)$. Further assume that $\bar{g}^*_{i,j} = 0$ $\forall i \in C(\mathbf{g}^*)$ and $\forall j \in P(\mathbf{g}^*)$. That is, we have a complete component, consisting of $C(\mathbf{g}^*)$, and a set of singletons, $P(\mathbf{g}^*)$. Denote the $PNE$ effort level of an agent in the core of size $c$ with $x^*_c$. Recall from Lemma 1 that agents in the core display the equal effort levels. Note that for $\frac{\delta\pi(x,y)}{\delta x \delta y}$ sufficiently small, effort levels are arbitrarily close to $\bar{x}(0)$. The condition for an agent in the core to not delete all his links (recall the argument from Proposition 3) is given by $\frac{v((c-1)x^*_c)-v(0)}{c-1} \geq k$. Note that for $\frac{\delta\pi(x,y)}{\delta x \delta y}$ sufficiently small this is arbitrarily close to $\frac{v((c-1)\bar{x}(0))-v(0)}{c-1}$. Denote with $x'_p$ and $x'_c$ the effort level in a deviation where a pair of agents $p \in P(\mathbf{g}^*)$ and $c \in C(\mathbf{g}^*)$ create a link. The condition for an agent in the periphery to not find it profitable to link to an agent in the core is given by $v(x'_c) - v(0) < k$. Again, for $\frac{\delta\pi(x,y)}{\delta x \delta y}$ sufficiently small, this is arbitrarily close to $v(\bar{x}(0)) - v(0)$. The condition can then be written as, for $\frac{\delta\pi(x,y)}{\delta x \delta y}$ sufficiently small, $\frac{v((c-1)\bar{x}(0))-v(0)}{c-1} > v(\bar{x}(0)) - v(0)$. The inequality follows from the convexity of $v$ and $c \geq 3$. We can therefore find a value of $k$ such that $\frac{v((c-1)x^*_{cl})-v(0)}{l-1} \simeq \frac{v((c-1)\bar{x}(0))-v(0)}{l-1} > k > v(\bar{x}(0)) - v(0) \simeq v(x'_c) - v(0)$. That is, $p$ will not find it profitable to link to $c$. $Q.E.D.$

The following Lemma shows that for $n = 3$, the only $PNE$ networks are the complete and the empty network. Note that with $n = 3$, the core can consist of at most two agents.

**Lemma 7:** *For* $n = 3$ *the only* $PNE$ *networks are the complete and the empty network.*

***Proof.*** There are two configurations to consider. First, the star and second, two connected agents. We start by showing that the star network is not a $PNE$ for $n = 3$. Assume to the contrary that $\bar{\mathbf{g}}^*$is a star network and $n = 3$. Denote the equilibrium effort levels with $x^*_c$ and $x^*_p$ for the center of the star and the two agents in the periphery, respectively. Note first that in a star $x^*_c > x^*_p$. We will show this for general $n$. Assume to the contrary that $x^*_p \geq x^*_c$. But then the agent in the center accesses an effort level of $(n-1)x^*_p$, while the periphery accesses $x_c^*$. From $x_p^* \geq x_c^*$it follows that $(n-1)x_p^* > x_c^*$ and therefore $x_c^*((n-1)x_p^*) > x_p^*(x_c^*)$. We have reached a contradiction. Next, consider a deviation where the two agents in the periphery create a link. Denote the effort level of the peripheral agents after proposed deviation with $x_p^{'}$. From strategic complementarities we know that $x_p^{'} > x_p^*$. The effort level accessed by each agent after the deviation is then $x_p^{'} + x_c^* > x_p^* + x_c^* > 2x_p^*$. The deviating agents access a higher effort level than the agent in the center, while each incurring the cost of two links. That is, if it is profitable for the center of the star to sustain his links, then it is profitable for the periphery to link to each other. More formally, $v(x_p^{'}+x_c^*)-v(x_c^*) > v(x_p^*+x_c^*)-v(x_c^*) > v(2x_p^*)-v(x_p^*) > k$. The inequalities then follow from $x_c^* > x_p^*$ and the convexity of the value function. Assume next, that $n = 3$ and $\bar{\mathbf{g}}^*$consists of two connected agents and the third agent is a singleton. Denote by $x_c^*$ the effort level of the two connected agents and consider a deviation where a new link is created. Note again that $x_p^{'} > x_c^*$. By an analogous reasoning as above, $v(x_p^{'} + x_c^*) - v(x_c^*) > v(x_c^*) - v(0) > k$ and $v(x_p^{'}) - v(0) > v(x_c^*) - v(0) > k$. This exhausts all possibilities and we can conclude that the only possible $PNE$ networks for $n = 3$ are the complete and the empty network. $Q.E.D.$

The results obtained so far were obtained for the general class of payoff functions defined in the model description. For Proposition 6 we assume special case of the linear-quadratic specification of Calvó-Armengol, Pattacchini and Zenou (2005 and 2009). Recall that this payoff function is given by $\pi(x_i, y_i) = x_i - \frac{\beta}{2}x_i^2 + \lambda x_i \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. In the following we obtain a necessary and sufficient condition for the existence of a $PNE$ with a star network. The conditions are then restated as sufficient conditions in Corollary 1: If $n \geq 6$ and either $\beta$, the own concavity parameter is sufficiently large, or $\lambda$, the parameter determining strategic complementarities, is sufficiently small, then there exists a linking cost $k$, such that star network is a $PNE$. Note further that the convexity of the value function in the linear-quadratic case is given by $\frac{\lambda^2}{\beta}$, so that one can also state Corollary 1 in terms of the convexity of the value function.

**Proposition 6:** *If best response functions are linear, there exists a linking cost $k$, such that a $PNE$ with a star network exists if and only if $\beta > (2+\sqrt{2})\lambda$ and one of the following conditions holds:*

- $1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3} \leq n < 1 + \frac{\beta^2}{\lambda^2}$ or
- $1 + \frac{\beta^2}{\lambda^2} < n \leq 1 + \frac{\beta^2}{\lambda^2} + \sqrt{\frac{\beta^2(\beta+\lambda)^2(\beta^2-4\beta\lambda+2\lambda^2)}{(\beta-\lambda)^2\lambda^4}}$.

***Proof.*** *See the Appendix.*

**Corollary 1:** *If best response functions are linear and $\lambda$ sufficiently small, or $\beta$ sufficiently large, then there exists a linking cost $k$, such that the star network is a $PNE$ for $n \geq 6$.*

***Proof.*** *See the Appendix.*

# 3 The One-Sided Model

## 3.1 Model Description

The one-sided specification differs from the two-sided model in that only one agent needs to extend a link and bear the cost, in order for a pair of agents to benefit from each others effort level. This allows us to use Nash equilibrium. Note that under Pairwise Nash equilibrium pairs of agents can create only *one* link at a time and *both* agents may adjust their effort levels. Under Nash equilibrium we consider deviations where an agent may extend *multiple* links (and simultaneously delete any subset of existing ones), but *only the deviating agent* may adjust effort levels. In the following we describe the model for the one-sided case. The proofs are mostly similar to the ones in the two-sided specification and therefore relegated to the Appendix.

Let again $N = \{1, 2, ..., n\}$ be the set of players, with $n \geq 3$ . As before, each player $i$ chooses a personal effort level $x_i \in X$ and a set of links, which are represented as a row vector $\mathbf{g}_i = (g_{i,1},..., g_{ii-1,}g_{ii+1},..., g_{in})$, where $g_{ij} \in \{0, 1\}$ for each $j \in N\backslash\{i\}$. Again assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of strategies of $i$ is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ again specifies the individual effort level of each player, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and a set of links $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Agent $i$ is said to sustain or extend a link to $j$, if $g_{i,j} = 1$ and to receive a link from $j$, if $g_{j,i} = 1$. The network of relations $\mathbf{g}$ is a directed graph, i.e. it is possible that $g_{i,j} \neq g_{j,i}$. Let $N_i(\mathbf{g}) = \{j \in N : g_{i,j} = 1\}$ be the set of agents $i$ has extended a link to and define $\eta_i(\mathbf{g}) = |N_i(\mathbf{g})|$. Call the closure of $\mathbf{g}$ an undirected network, denoted by $\bar{\mathbf{g}} = cl(\mathbf{g})$, where $\bar{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each $i$ and $j$ in $N$. Denote with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ the set of players that are directly connected to $i$. The effort level of $i's$ direct neighbors can then be written as $y_i = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. We will drop the subscript of $y_i$ when it is clear from the context. Given a network $\mathbf{g}$, $\mathbf{g} + g_{i,j}$ and $\mathbf{g} - g_{i,j}$ have the following interpretation. When $g_{i,j} = 0$ in $\mathbf{g}$, $\mathbf{g} + g_{i,j}$ adds the link $g_{i,j} = 1$, while if $g_{i,j} = 1$ in $\mathbf{g}$, then $\mathbf{g} + g_{i,j} = \mathbf{g}$. Similarly, if $g_{i,j} = 1$ in $\mathbf{g}$, $\mathbf{g} - g_{i,j}$ deletes the link $g_{i,j}$, while if $g_{i,j} = 0$ in $\mathbf{g}$, then $\mathbf{g} - g_{i,j} = \mathbf{g}$. The network is said to be empty and denoted by $\bar{\mathbf{g}}^e$ if $\bar{g}_{i,j} = 0 \; \forall i, j \in N$ and complete and denoted by $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1 \; \forall i, j \in N$.

Payoffs are of player $i$ under strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi(x_i, y_i) - \eta_i(\mathbf{g})k,$$

where $k$ denotes the cost of extending a link. The assumptions on the payoff function are as in the one-sided specification. A Nash equilibrium is a strategy profile $\mathbf{s}^* = (\mathbf{x}^*, \mathbf{g}^*)$ such that

$$\Pi_i(\mathbf{s}_i^*, \mathbf{s}_{-i}^*) \geq \Pi_i(\mathbf{s}_i, \mathbf{s}_{-i}^*), \forall \mathbf{s}_i \in S_i, \forall i \in N,$$

Denote the directed equilibrium network by $\mathbf{g}^*$ and the undirected equilibrium network by $\bar{\mathbf{g}}^*$.

## 3.2 Analysis

Note that in Proposition 1, Lemma 1 and Lemma 2 we assume the network to be fixed and therefore these results carry over to the one-sided specification. We start by showing that, in any $NE$, there can be at most one link between any pair of players.

**Lemma 8:** *In any* $NE$ $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$*, there is at most one directed link between any pair of agents* $i, j \in N$.

***Proof.*** See the Appendix.

In Lemma 9 we show, again due to the convexity of the value function, that in any Nash equilibrium, if $i$ extends a link to $l$, then $i$ must also be connected to agent $k$, for any $k$ such that $x_k^* \geq x_l^*$. Note that we do not require that $i$ extends a link to $k$, but only that $i$ and $k$ are connected. That is, $k$ may be extending the link to agent $i$.

**Lemma 9:** *In any* $NE$ $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$*, if* $g_{i,l}^* = 1$ *then* $\bar{g}_{i,k}^* = 1$ $\forall k : x_k^* \geq x_l^*$.

***Proof.*** *See the Appendix.*

The following Lemma shows that if $i$ extends a link to $l$, then any agent $k$ with a higher or equal effort level than $i$ must also be connected to $l$. Again this follows from the convexity of the value function.

**Lemma 10:** *In any* $NE$ $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$*, if* $g_{i,l}^* = 1$ *then* $\bar{g}_{k,l}^* = 1$ $\forall k : x_k^* \geq x_i^*$.

***Proof.*** *See the Appendix.*

Similar to the two-sided specification, we again define two bounds, $k^1$and $k^2$. In Proposition 7 we show that for $k$ smaller than $k^1$, the unique Nash equilibrium is such that the network is complete, while for $k$ larger or equal than $k^1$, there exists a Nash equilibrium such that the network is empty. Proposition 8 shows that for linking cost larger than $k^2$, the unique Nash equilibrium is such that the network is empty, while for $k$ smaller or equal to $k^2$, there exists a Nash equilibrium such that the network is complete. Note that the thresholds $k^1$ and $k^2$ are defined differently from the two-sided specification. Due to the convexity of the value function, the most profitable deviation in the empty network is to extend a link to all remaining agents (where no agent other than the deviating agent adjust his effort level). The second threshold, $k^2$, is the maximal linking cost that can be sustained in the complete network. Note that the network is directed in the one-sided specification and, due to the convexity of the value function, the agent with the fewest incoming links has the greatest incentives to deviate. That is, the network that may sustain the maximum linking cost is the one where incoming and outgoing links are evenly distributed. With $n$ agents there are $\frac{n(n-1)}{2}$ pairs of agents. For $n$ odd this implies that when incoming and outgoing links are evenly distributed, each agent has $\frac{n-1}{2}$ incoming and $\frac{n-1}{2}$ outgoing links. For $n$ even, $\frac{n}{2}$ agents have $\frac{n}{2}$ incoming and $\frac{n-2}{2}$ outgoing links and $\frac{n}{2}$ agents have $\frac{n-2}{2}$ incoming and $\frac{n}{2}$ outgoing links. For simplicity we assume in the following that $n$ is odd. Analogous results are easily derived for $n$ even.

**Definition 2:** $k^1 = \frac{v((n-1)\bar{x}(0))-v(0)}{n-1}$ and $k^2 = \frac{1}{\frac{n-1}{2}}(v((n-1)x^*(\mathbf{g}^c)) - v(\frac{n-1}{2}x^*(\mathbf{g}^c))$.

**Proposition 7:** *If $k < k^1$, then the unique NE network is the complete network. If $k \geq k^1$, then there exists a NE such that the network is empty.*

***Proof.*** *See the Appendix.*

Before proceeding to Proposition 8, we show that in any Nash equilibrium network that is neither empty or complete, there exists and agent that extends at least one link and has less than $\frac{n-1}{2}$ incoming links. This result is useful when proving that the network that can be sustained at the highest linking cost is the complete network with evenly distributed incoming links.

**Lemma 11:** *In any NE network that is neither empty nor complete, there exists an agent with $\eta_i(\mathbf{g}) \geq 1$ and $\eta_i(\bar{\mathbf{g}}) - \eta_i(\mathbf{g}) < \frac{n-1}{2}$.*

***Proof.*** *See the Appendix.*

**Proposition 8:** *If $k > k^2$, then the unique NE is the empty network. If $k \leq k^2$, then there exists a NE such that the network is complete.*

***Proof.*** *See the Appendix.*

Lemma 12 shows that $k^1 < k^2$. We have therefore shown that, for linking cost smaller than $k^1$, the unique $NE$ is the complete network, while, for linking cost larger than $k^2$, the unique $NE$ network is the empty network. For linking cost $k \in [k^1, k^2]$ the complete and the empty network are Nash equilibria.

**Lemma 12:** $0 < k^1 < k^2$.

***Proof.*** *See the Appendix.*

The following Lemma shows that in any Nash equilibrium, if a pair of agents exert same effort levels, then they must share the same neighborhoods. The proof is a direct consequence of the convexity of the value function.

**Lemma 13:** *In any NE* $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$, $x_i^* = x_k^* \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.

***Proof.*** *See the Appendix.*

Lemma 14 shows that in any Nash equilibrium, if an agent $i$ exerts a weakly lower effort level than another agent $k$, then agent $i's$ neighborhood is contained in $k's$ neighborhood.

**Lemma 14:** *In any NE* $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$, $x_i^* \leq x_k^* \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.

***Proof.*** *See the Appendix.*

In Proposition 9 we show that in any Nash equilibrium, such that there exists a pair of agents with different effort levels, the network displays a core-periphery structure.

**Proposition 9:** *In any NE with a pair of agents i and j, such that* $x_i^* \neq x_j^*$, *the network is a core-periphery network.*

***Proof.*** *See the Appendix.*

Next, we define a *periphery-sponsored core-periphery network* as a core-periphery network where all agents in the periphery extend links to all agents in the core. A *core-sponsored core-periphery network* is a core-periphery network where all agents in the core extend links to all agents in the periphery.

**Definition 3:** A network $\mathbf{g}$ is a $periphery - sponsored\ core - periphery\ network$ if the set of agents $N$ can be partitioned into two sets $C(\mathbf{g})$ (the core) and $P(\mathbf{g})$ (the periphery), such that $\bar{g}_{i,j} = 1\ \forall i, j \in C(\mathbf{g})$, $\bar{g}_{i,j} = 0\ \forall i, j \in P(\mathbf{g})$ and $g_{i,j} = 1 \forall i \in C(\mathbf{g})$ and $\forall i \in P(\mathbf{g})$. A network $\mathbf{g}$ is a $core - sponsored\ core - periphery\ network$ if the set of agents $N$ can be partitioned into two sets $C(\mathbf{g})$ (the core) and $P(\mathbf{g})$ (the periphery), such that $\bar{g}_{i,j} = 1$ $\forall i, j \in C(\mathbf{g})$, $\bar{g}_{i,j} = 0\ \forall i, j \in P(\mathbf{g})$ and $g_{i,j} = 1 \forall i \in C(\mathbf{g})$ and $\forall i \in P(\mathbf{g})$.

**Lemma 15:** *There does not exist a NE such that the network is a core sponsored complete core-periphery network.*

***Proof.*** *See the Appendix.*

The results obtained so far were obtained for the general class of payoff functions defined in the model description. For Proposition 10 we assume special case of the linear-quadratic specification of Calvó-Armengol, Pattacchini and Zenou (2005 and 2009). In the following we obtain a necessary and sufficient condition for the existence of a $PNE$ with a star network. The conditions are then restated as sufficient conditions in Corollary 2: If $n \geq 6$ and either $\beta$, the own concavity parameter is sufficiently large, or $\lambda$, the parameter determining strategic complementarities, is sufficiently small, then there exists a linking cost $k$, such that star network is a $NE$. Note further that the convexity of the value function in the linear-quadratic case is given by $\frac{\lambda^2}{\beta}$, so that one can also state Corollary 1 in terms of the convexity of the value function.

**Proposition 10:** *If best response functions are linear, there exists a linking cost $k$, such that a $NE$ with a periphery-sponsored core-periphery network exists if and only if the number of agents in the core, $c$, is smaller than $\frac{n-1}{2}$ and $\lambda \leq \frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}}$.*

***Proof.*** *See the Appendix.*

**Corollary 3:** *If best response functions are linear and$\lambda$ sufficiently small, or $\beta$ sufficiently large, then for any periphery sponsored core-periphery network with $c < \frac{n-1}{2}$, there exists a linking cost $k$, such that $\mathbf{g}^*$ is a Nash equilibrium.*

***Proof.*** *See the Appendix.*

# 4 Conclusion

This paper provides a model of endogenous network formation with peer effects for a general class of payoff functions, where peer effects are assumed to induce positive local externalities and strategic complementarities in effort levels. These features are descriptive of a wide range of social and economic and phenomena, such as educational attainment, crime, labour market participation and R&D expenditures of firms. We solve the model for a two-sided specification, where both agents need to agree to form a link, and a one-sided specification, where links can be created unilaterally. In both cases the only Pairwise Nash equilibrium and Nash equilibrium network structures are of three types: the empty, complete, and core-periphery networks. For the case of linear-quadratic payoff functions, we provide necessary and sufficient conditions for the existence of a star (in the two-sided specification) and a periphery-sponsored core-periphery network (in the one sided-specification).

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# 6 APPENDIX A - The Two-Sided Model

**Proposition 6:** *If best response functions are linear, there exists a linking cost $k$, such that a PNE with a star network exists if and only if $\beta > (2+\sqrt{2})\lambda$ and one of the following conditions holds:*

- $1+\frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3} \leq n < 1+\frac{\beta^2}{\lambda^2}$ or
- $1+\frac{\beta^2}{\lambda^2} < n \leq 1+\frac{\beta^2}{\lambda^2}+\sqrt{\frac{\beta^2(\beta+\lambda)^2(\beta^2-4\beta\lambda+2\lambda^2)}{(\beta-\lambda)^2\lambda^4}}$.

***Proof.*** First, note that in a star network, all agents in the periphery access the same effort level, $x_c^*$, and therefore all agents in the periphery display the same effort level, $x_p^*$. The agent in the center, $c$, therefore maximizes $x_c \in argmax_{x_c \in X} x_c - \frac{\beta}{2}x_c^2 + \lambda x_c(n-1)x_p$, while for an agent in the periphery we have $x_p \in argmax_{x_p \in X} x_p - \frac{\beta}{2}x_p^2 + \lambda x_p x_c$. The reaction functions are given by $x_c(x_p) = \frac{1+\lambda x_p(n-1)}{\beta}$ and $x_p(x_c) = \frac{1+\lambda x_c}{\beta}$. Equilibrium effort levels are given by $x_c^* = \frac{\beta+\lambda(n-1)}{\beta^2-\lambda^2(n-1)}$ and $x_p^* = \frac{\beta+\lambda}{\beta^2-\lambda^2(n-1)}$. Plugging equilibrium effort levels back into the payoff function, yields equilibrium gross payoffs of $\pi_c^* = \frac{\beta(\beta+\lambda(n-1))^2}{2(\beta^2-\lambda^2(n-1))^2}$ and $\pi_p^* = \frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2(n-1))^2}$. Next, we calculate deviation payoffs of when two agents in the periphery create a link. Denote with $x_p^{1'}$ and $x_p^{2'}$ the effort levels of the two agents involved in the new link. A deviating agent maximizes $x_p^{1'} \in argmax_{x_p^{1'} \in X} x_p^{1'} - \frac{\beta}{2}(x_p^{1'}) + \lambda x_p^{1'}(x_c + x_p^{2'})$, which yields the following reaction function $x_p^{1'}(x_c^*, x_p^{2'}) = \frac{1+\lambda(x_c^*+x_p^{2'})}{\beta}$. Due to symmetry, deviation effort levels are given by $x_p^{1'}(x_c^*, x_p^{2'}) = x_p^{2'}(x_c^*, x_p^{1'}) = x_p' = \frac{\beta^2+\lambda\beta}{(\beta-\lambda)(\beta^2-\lambda^2(n-1))}$ and corresponding deviation gross payoffs by $\pi_p' = \frac{\beta^3(\beta+\lambda)^2}{2(\beta-\lambda)^2(\beta^2-\lambda^2(n-1))^2}$. For the existence of a star network we now need two conditions to hold. First, we want to find a linking cost $k$, such that an agent in the periphery finds it profitable to link to the center of the star, but, given the link with the center, does not find it profitable to link to another agent in the periphery. This condition can be written as $\frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2(n-1))^2} - \frac{1}{2\beta} \geq \frac{\beta^3(\beta+\lambda)^2}{2(\beta-\lambda)^2(\beta^2-\lambda^2(n-1))^2} - \frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2(n-1))^2}$, where $\frac{1}{2\beta}$ are the payoffs of a singleton, so that on the left hand side we have marginal payoffs of linking to the center and on the right hand side marginal payoffs of linking to another agent in the periphery, given the link with the center of the star. Second, we want to find a linking cost $k$, such that the center of the star finds it profitable to link to the periphery, but that again the periphery does not find it profitable to link to another agent in the periphery. This condition can be written as $(\frac{\beta(\beta+\lambda(n-1))^2}{2(\beta^2-\lambda^2(n-1))^2} - \frac{1}{2\beta})/(n-1) \geq \frac{\beta^3(\beta+\lambda)^2}{2(\beta-\lambda)^2(\beta^2-\lambda^2(n-1))^2} - \frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2(n-1))^2}$. Combining these two conditions one can show that they simultaneously hold if and only if $\beta > (2+\sqrt{2})\lambda$ and one of the following conditions holds: $1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3} \leq n < 1 + \frac{\beta^2}{\lambda^2}$ or $1 + \frac{\beta^2}{\lambda^2} < n \leq 1 + \frac{\beta^2}{\lambda^2} + \sqrt{\frac{\beta^2(\beta+\lambda)^2(\beta^2-4\beta\lambda+2\lambda^2)}{(\beta-\lambda)^2\lambda^4}}$.[9] *Q.E.D.*

**Corollary 1:** *If best response functions are linear and $\lambda$ sufficiently small, or $\beta$ sufficiently large, then there exists a linking cost $k$, such that the star network is a PNE for $n \geq 6$.*

***Proof.*** First the case where $\lambda$ is sufficiently small. Note that then $\beta > (2+\sqrt{2})\lambda$ holds. Furthermore, for given $\beta$ an $\lambda$ sufficiently small, $1 + \frac{\beta^2}{\lambda^2}$ is arbitrary large so that $n < 1 + \frac{\beta^2}{\lambda^2}$. For $1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3} \leq n$ to hold note that $lim_{\lambda\to 0}\ (1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3}) = 6$. Next, $\beta$ sufficiently large. Again, $\beta > (2+\sqrt{2})\lambda$ holds. Furthermore, for given $\lambda$ and $\beta$ sufficiently large, $n < 1 + \frac{\beta^2}{\lambda^2}$ holds. For $1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3} \leq n$ to hold note that $lim_{\beta\to\infty}\ (1 + \frac{\beta^2(5\beta-3\lambda)}{(\beta-\lambda)^3}) = 6$.[10] *Q.E.D.*

[9] These calculations were executed in Mathematica and the codes are available upon request.
[10] This calculations was executed in Mathematica.

# 7 APPENDIX B - The One-Sided Model

**Lemma 8:** *In any NE* $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$ *there is at most one directed link between any pair of agents* $i, j \in N$.

***Proof.*** Assume that $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$ is a Nash equilibrium and that $g_{i,j} = g_{j,i} = 1$. But then $i$ can profitably deviate by cutting the link to $j$, such that $g_{i,j} = 0$. Gross payoffs remain unchanged, while $i$'s linking total cost decrease by $k$. *Q.E.D.*

**Lemma 9:** *In any NE* $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$*, if* $g^*_{i,l} = 1$ *then* $\bar{g}^*_{i,k} = 1$ $\forall k : x^*_k \geq x^*_l$.

***Proof.*** For $g^*_{i,j} = 1$ to be part of a $NE$, it must be that $v(\sum_{j \in N_i(\bar{\mathbf{g}})} x^*_j) - v(\sum_{j \in N_i(\bar{\mathbf{g}})} x^*_j - x^*_l) \geq k$ . Assume, contrary to the above statement, that $\bar{g}^*_{i,k} = 0$ for some $k$ with $x^*_k \geq x^*_l$. This, however, can not be a $NE$, since $i$ finds it profitable to then extend a link to agent $k$. To see this, note that $v(\sum_{j \in N_i(\mathbf{g})} x^*_j + x_k) - v(\sum_{j \in N_i(\mathbf{g})} x^*_j) > v(\sum_{j \in N_i(\mathbf{g})} x^*_j) - v(\sum_{j \in N_i(\mathbf{g})} x^*_j - x^*_l) \geq k$, where the inequalities follow from the convexity of the value function. We have reached a contradiction and therefore $\bar{g}^*_{i,k} = 1$ for all agents $k$ with $x^*_k \geq x^*_l$. *Q.E.D.*

**Lemma 10:** *In any NE* $\mathbf{s}^*$=$(\mathbf{x}^*, \mathbf{g}^*)$*, if* $g^*_{i,l} = 1$ *then* $\bar{g}^*_{k,l} = 1$ $\forall k : x^*_k \geq x^*_i$.

***Proof.*** For $g^*_{i,j} = 1$ to be part of a $NE$, it must be that $v(\sum_{j \in N_i(\bar{\mathbf{g}})} x^*_j) - v(\sum_{j \in N_i(\bar{\mathbf{g}})} x^*_j - x^*_l) \geq k$ . Assume, contrary to the above statement, that $\bar{g}^*_{k,l} = 0$ for some $k$ with $x^*_k \geq x^*_i$. Note next that, for $x^*_k \geq x^*_i$to hold we must have $\sum_{j \in N_k(\bar{\mathbf{g}})} x^*_j \geq \sum_{j \in N_i(\bar{\mathbf{g}})} x^*_j$, which follows directly from strict strategic complementarities. Therefore, $v(\sum_{j \in N_k(\mathbf{g})} x^*_j + x^*_l) - v(\sum_{j \in N_k(\mathbf{g})} x^*_j) > v(\sum_{j \in N_i(\mathbf{g})} x^*_j) - v(\sum_{j \in N_i(\mathbf{g})} x^*_j - x_l) \geq k$, where the inequalities again follow from the convexity of the value function and we have reached a contradiction. *Q.E.D.*

**Definition 2:** $k^1 = \frac{v((n-1)\bar{x}(0)) - v(0)}{n-1}$ and $k^2 = \frac{1}{\frac{n-1}{2}}(v((n-1)x^*(\mathbf{g}^c)) - v(\frac{n-1}{2}x^*(\mathbf{g}^c))$.

**Proposition 7:** *If* $k < k^1$*, then the unique NE network is the complete network. If* $k \geq k^1$*, then there exists a NE such that the network is empty.*

***Proof.*** If $k < k^1$ then an agent finds it profitable to create a link to all remaining $n-1$ agents in an empty network, since average payoffs per link are given by $\frac{v((n-1)\bar{x}(0)) - v(0)}{n-1}$ with $\frac{v((n-1)\bar{x}(0)) - v(0)}{n-1} > k$. This is the most profitable deviation in an empty network, due to the convexity of the value function. Assume there exists a $\mathbf{g}^* \notin \{\mathbf{g}^e, \mathbf{g}^c\}$ with $k < k^1$. Consider the deviation of an agent $i$, with $\eta_i(\bar{\mathbf{g}}^*) < n-1$, who links to all agents he is not connected to in $\mathbf{g}^*$, i.e. $k \notin N_i(\bar{\mathbf{g}}^*)$. To simplify notation, we write $\eta_i$ for$\eta_i(\bar{\mathbf{g}}^*)$ in the following. Average marginal payoffs per link of proposed deviation are given by

$$\frac{v(\sum_{j \in N_i \setminus \{i\}} x^*_j) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x^*_j)}{n - \eta_i - 1}.$$

We can now write

$$\frac{v(\sum_{j\in N_i\setminus\{i\}} x_j^*)-v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x_j^*)}{n-1-\eta_i} \geq \frac{v((n-1)\bar{x}(0))-v(\eta_i\bar{x}(0))}{n-1-\eta_i}.$$

To see that the inequality holds, note first that $\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x_j^* \geq \eta_i\bar{x}(0)$, since $\eta_i(\bar{\mathbf{g}}^*) =\mid N_i(\bar{\mathbf{g}}^*) \mid$and $\bar{x}(0)$ are the lowest possible effort levels in any $NE$. Second, that $\sum_{j\in N_i\setminus\{i\}} x_j^* - \sum_{j\in N_i(\bar{\mathbf{g}}^*)} x_j^* \geq (n-1-\eta_i)\bar{x}(0)$. That is, when linking to the remaining $n-1-\eta_i$ agents in proposed deviation, the minimum additional effort level accessed is given by $(n-1-\eta_i)\bar{x}(0)$. The condition above then follows from the convexity of the value function. Note next that

$$\frac{v((n-1)\bar{x}(0))-v((\eta_i\bar{x}(0))}{n-1-\eta_i} > \frac{v((n-1)\bar{x}(0))-v(0)}{n-1}$$

also holds, again from the convexity of the value function, and we therefore have

$$\frac{v(\sum_{j\in N_i\setminus\{i\}} x_j^*)-v(\sum_{j\in N_i(\bar{\mathbf{g}}^*)} x_j^*)}{n-1-\eta_i} \geq \frac{v((n-1)\bar{x}(0))-v(0)}{n-1} > k.$$

Therefore, if $k < k^1$ a profitable deviation exists in any $\mathbf{g}^* \neq \mathbf{g}^c$. It is easy to see that there then exists a $NE$ with $\mathbf{g}^* = \mathbf{g}^c$ for $k < k^1$. If, on the other hand $k \geq k^1$, then no agent can profitably deviate in the empty network, and a $NE$ exists such that $\mathbf{g}^* = \mathbf{g}^e$. *Q.E.D.*

**Lemma 11:** *In any $NE$ network that is neither empty nor complete, there exists an agent with $\eta_i(\mathbf{g}) \geq 1$ and $\eta_i(\bar{\mathbf{g}}) - \eta_i(\mathbf{g}) < \frac{n-1}{2}$.*

***Proof.*** We discern two cases. First, everyone agent extends at least one link and receives at least $\frac{n-1}{2}$ links. That is, $\eta_i(\mathbf{g}) \geq 1$ and $\eta_i(\bar{\mathbf{g}}) - \eta_i(\mathbf{g}) \geq \frac{n-1}{2}$ $\forall i \in N$. But then there are at least $\frac{n(n-1)}{2}$ links in the network and the network is complete. Second, not everyone extends a link. Assume there are $k < n$ agents who extend a link. Since there are no incoming links from the remaining $n-k$ agents, the maximum number of incoming links among the $k$ agents extending a link is given by $\frac{k(k-1)}{2}$. That is, on average an agent has $\frac{k-1}{2}$ incoming links. The maximum of the minimum number of incoming links is given by $\frac{k-1}{2}$. Since $k < n$, there must be one agent with at most $\frac{k-1}{2} < \frac{n-1}{2}$ incoming links. *Q.E.D.*

**Proposition 8:** *If $k > k^2$, then the unique $NE$ is the empty network. If $k \leq k^2$, then there exists a $NE$ such that the network is complete.*

***Proof.*** We will first show that the highest cost that can be sustained under the complete network is given by $k^2$. Denote the $NE$ effort level in a complete network with $x^{c*}$. In the complete network the agent extending the highest number of links (and therefore receiving the fewest number of links) is the one with the highest incentives to delete his links. To see this, write

$$\frac{v((n-1)x^{c*})-v((n-1-h)x^{c*})}{n-1-h} > \frac{v((n-1)x^{c*})-v((n-1-h')x^{c*})}{n-1-h'},$$

where $h' > h > 0$. The inequality holds by the convexity of the value function. The network that minimizes the maximum number of links extended by agents in a network is such that each agent extends $\frac{n-1}{2}$ links (and receives $\frac{n-1}{2}$ links). Therefore, the highest linking cost that can be sustained in a complete network are given by $k = k^2$. It is easy to see that for $k < k^2$ there exists a $NE$ such that $\bar{\mathbf{g}}^* = \bar{\mathbf{g}}^c$. Assume next, and contrary to the above statement, that for $k > k^2$ there exists a $NE$ such that $\bar{\mathbf{g}}^* \neq \bar{\mathbf{g}}^e$. $k^2$ was derived as the maximal payoffs sustainable in a complete network, and therefore for $k > k^2$ there is does not exist a $NE$ such that $\bar{\mathbf{g}}^* = \bar{\mathbf{g}}^c$. Next, we show that for $k > k^2$ there also does not exist a $NE$ with $\bar{\mathbf{g}}^* \notin \{\bar{\mathbf{g}}^e, \bar{\mathbf{g}}^c\}$. Assume the contrary. Pick an agent with less than $\frac{n-1}{2}$ incoming links, $\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*) < \frac{n-1}{2}$, and at least one outgoing link, $\eta_i(\mathbf{g}^*) \geq 1$. We know from Lemma 8 that such an agent exists in $\bar{\mathbf{g}}^* \notin \{\bar{\mathbf{g}}^e, \bar{\mathbf{g}}^c\}$. We consider a deviation where this agent deletes all his links. To see that this is profitable, note that in the complete network, the average marginal payoff from extending links to all remaining agents is larger for an agent with $\frac{n-1}{2}$ incoming links, than for an agent with fewer incoming links, i.e. for an agent with$\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*) < \frac{n-1}{2}$. From $\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*) < \frac{n-1}{2}$ we have $n-1-(\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*)) > \frac{n-1}{2}$. The following inequality then holds again by the convexity of $v$.

$$\frac{v((n-1)x^{c*}) - v(\frac{n-1}{2}x^{c*})}{\frac{n-1}{2}} > \frac{v((n-1)x^{c*}) - v((\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))x^{c*})}{n-1-(\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))}.$$

Note that, given $\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*) < \frac{n-1}{2}$ incoming links, average marginal payoffs are highest when linking to all remaining agents with effort level $x^{c*}$. As$\eta_i(\bar{\mathbf{g}}^*)$ is at most $n-1$, we can write

$$\frac{v((n-1)x^{c*}) - v((\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))x^{c*})}{n-1-(\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))} \geq \frac{v(\eta_i(\bar{\mathbf{g}}^*)x^{c*}) - v((\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))x^{c*})}{\eta_i(\mathbf{g}^*)}.$$

Last, note that effort levels are maximal by Lemma 2 and by the convexity of $v$ we therefore have

$$\frac{v(\eta_i(\bar{\mathbf{g}}^*)x^{c*}) - v((\eta_i(\bar{\mathbf{g}}^*) - \eta_i(\mathbf{g}^*))x^{c*})}{\eta_i(\mathbf{g}^*)} > \frac{v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^*) - v(\sum_{j \in N_i(\mathbf{g}^*)} x_j^*)}{\eta_i(\mathbf{g}^*)}.$$

Average marginal payoffs are highest in the complete network where each agent extends $\frac{n-1}{2}$ links and therefore for $k > k^2$ the empty network is the unique $NE$.

*Q.E.D.*

**Lemma 12:** $0 < k^1 < k^2$.

***Proof.*** Recall the definitions of $k^1 = \frac{v((n-1)\bar{x}(0)) - v(0)}{n-1}$ and $k^2 = \frac{2(v((n-1)x^{c*}) - v((\frac{n-1}{2})x^{c*}))}{n-1}$. The inequalities then follow from $\bar{x}(0) > 0$, $x^{c*} > \bar{x}(0)$ and the convexity of the value function. *Q.E.D.*

**Lemma 13:** *In any $NE$ $\mathbf{s}^*=(\mathbf{x}^*, \mathbf{g}^*)$, $x_i^* = x_k^* \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.*

***Proof.*** First, $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\} \Rightarrow x_i^* = x_k^*$. If $\bar{g}_{i,k}^* = 0$, then $i$ and $k$ access the same effort level, i.e. $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* = y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*$ and therefore $x_i^* = x_k^*$. Assume next that $\bar{g}_{i,k}^* = 1$ and, without loss of generality that $x_i^* > x_k^*$. But then $k$ accesses a higher effort level than $i$, $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* < y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*$, and we have reached a contradiction. Second, $x_i^* = x_k^* \Rightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} = N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$. Assume to the contrary that $x_i^* = x_k^*$ and $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$. Note that for $x_i^* = x_k^*$, effort levels accessed must be equal by strict strategic complementarities, so that $y_i = y_k$. For $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \neq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$ to hold, there must exist an agent $l$, such that $l \in N_k(\bar{\mathbf{g}}^*)$ and $l \notin N_i(\bar{\mathbf{g}}^*)$. For the link $\bar{g}_{k,l}^* = 1$ to be in place in $\bar{\mathbf{g}}^*$ we must have that $v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^* - x_l^*) \geq k$. But from $y_i = y_k$ and the convexity of the value function $v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* + x_l) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*) > v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^* - x_l^*) \geq k$ holds and we reach a contradiction. *Q.E.D.*

**Lemma 14:** *In any NE* $\mathbf{s}^*{=}(\mathbf{x}^*, \mathbf{g}^*)$, $x_i^* \leq x_k^* \Leftrightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$.

***Proof.*** First, $N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\} \Rightarrow x_i^* \leq x_k^*$. If $\bar{g}_{i,k}^* = 0$, then $k$ accesses a weakly higher effort level, i.e. $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* \leq y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*$ and therefore $x_i^* \leq x_k^*$. Assume next that $\bar{g}_{i,k}^* = 1$ and, without loss of generality, that $x_i^* > x_k^*$. But then $k$ accesses a strictly higher effort level than $i$, $y_i = \sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* < y_k = \sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*$, and we have reached a contradiction. Second, $x_i^* \leq x_k^* \Rightarrow N_i(\bar{\mathbf{g}}^*) \setminus \{k\} \subseteq N_k(\bar{\mathbf{g}}^*) \setminus \{i\}$. Assume to the contrary that $x_i^* \leq x_k^*$ and there exists an agent $l$ such that $l \in N_i(\bar{\mathbf{g}}^*)$ and $l \notin N_k(\bar{\mathbf{g}}^*)$. For the link $\bar{g}_{i,l}^* = 1$ to be in place in $\bar{\mathbf{g}}^*$, either $g_{i,l}^* = 1$ or $g_{l,i}^* = 1$. If $g_{i,l}^* = 1$, then $v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^*) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* - x_l^*) \geq k$ must hold. But from $y_i \leq y_k$ and the convexity of the value function can write $v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^* + x_l) - v(\sum_{j \in N_k(\bar{\mathbf{g}}^*)} x_j^*) > v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^*) - v(\sum_{j \in N_i(\bar{\mathbf{g}}^*)} x_j^* - x_l^*) \geq k$ and we have reached a contradiction. We can apply an analogous argument for $g_{l,i}^* = 1$. *Q.E.D.*

**Proposition 9:** *In any NE with a pair of agents i and j, such that* $x_i^* \neq x_j^*$, *the network is a core-periphery network.*

***Proof.*** Rank agents by their effort levels in increasing order, such that $x_1^* \leq x_2^* \leq ... \leq x_{n-1}^* \leq x_n^*$. We know from Lemma 1 that the network is not complete, since there exists a pair of agents $i$ and $j$ such that $x_i^* \neq x_j^*$. The network is not empty, as agents have identical payoff functions and singleton agents display same effort levels, $\bar{x}(0)$. We start by showing that the two lowest ranked agents, agent 1 and agent 2, are not connected. Two cases are to be discerned. First, $g_{1,2}^* = 1$. From Lemma 9 we know that agent 1 must be connected to all agents remaining agents, since $x_j^* \geq x_2^*\ \forall j \geq 2$. Lemma 14 implies that the network is complete since $N_i(\mathbf{g}^*) \setminus \{j\} \subseteq N_j(\mathbf{g}^*) \setminus \{i\}$ holds for agents $j$ with $x_j^* \geq x_1^*\ \forall j \geq 1$. But then $x_i^* = x_j^*\ \forall i, j \in N$ by Lemma 1 and we have reached a contradiction. Second, $g_{2,1}^* = 1$. From Lemma 10 we know that agent 1 is connected to all agents, since $x_j^* \geq x_2^*\ \forall j \geq 2$ and the above argument applies. Since the network is neither empty, nor complete, at least one link exists. Pick the agent $i$ with the lowest subscript that is involved in a link and, if $i$ is involved in more than one link, consider the link to the agent with the lowest subscript $j$. We discern two cases, $g^*_{i,j} = 1$ and $g^*_{j,i} = 1$. First, $g^*_{i,j} = 1$. We can discern two subcases. First, agent $i$ and $j$ are adjacent. As $i$ is the agent with the lowest subscript involved in a link, all agents with lower subscripts have no links. All agents with a subscript higher or equal to $i$ are connected to each other. To see this, note that by Lemma 9, agent $i$ is connected to all agents with a subscript higher or equal than $j$. But then by Lemma 14, $\bar{g}^*_{l,m} = 1 \; \forall l, m \geq i$. The periphery, $P(\mathbf{g}^*)$, consists of agents with subscripts $k < i$, while the core, $C(\mathbf{g}^*)$, consists of agents with subscripts $k \geq i$. The argument for the case where $g^*_{j,i} = 1$ is analogous. Assume next that $i$ and $j$ are not adjacent. Note that since $g^*_{i,j} = 1$ and from $x^*_i \leq x^*_{j-1}$, we know by Lemma 10 that the link between $j-1$ and $j$, $\bar{g}^*_{j-1,j} = 1$, also exists. Next, check for the link $\bar{g}^*_{j-2,j-1}$. If $\bar{g}^*_{j-2,j-1} = 0$, then by Lemma 10 no agent with a subscript lower than $j-2$ is connected to $j-1$. Furthermore, no pair of agents with subscripts of lower or equal than $j-2$ is connected. Assume to the contrary that there exists a pair of nodes $l, m$ with $l \leq m < j-2$ and $\bar{g}^*_{l,m} = 1$. By Lemma 9 we must then have that $\bar{g}^*_{l,j-1} = 1$. This, however, contradicts Lemma 14, since $\bar{g}^*_{j-2,j-1} = 0$. The periphery, $P(\mathbf{g}^*)$, consists of agents with subscripts $k < j$, while the core, $C(\mathbf{g}^*)$, consists of agents with subscripts $k \geq j$. If $\bar{g}^*_{j-2,j-1} = 1$, check for the link $\bar{g}^*_{j-3,j-2}$. If $\bar{g}^*_{j-3,j-2} = 0$, then by above argument the periphery, $P(\mathbf{g}^*)$, consists of agents with subscripts $k < j-1$, while the core, $C(\mathbf{g}^*)$, consists of agents with subscripts $k \geq j-1$. If $\bar{g}^*_{j-3,j-2} = 1$, proceed in descending order until a pair of adjacent agents is found that is not connected and define the core and periphery accordingly. Note that such a pair of agents exists, since $i$ and $j$ were assumed to not be adjacent and therefore $\bar{g}^*_{i,i+1} = 0$. This concludes the proof. *Q.E.D.*

**Lemma 15:** *There does not exist a NE such that the network is a core sponsored complete core-periphery network.*

***Proof.*** Assume to the contrary that such a network is a Nash equilibrium. But then an agent in the periphery receives links from all agents in the core, while an agent in the core receives at most $c-1$ links. Therefore, by the convexity of the value function, if an agent finds it profitable to link to all agents in the periphery, then agents in the periphery find it profitable to link to all remaining agents in the periphery. *Q.E.D.*

**Proposition 10:** *If best response functions are linear, there exists a linking cost k, such that a NE with a periphery-sponsored core-periphery network exists if and only if the number of agents in the core, c, is smaller than* $\frac{n-1}{2}$ *and* $\lambda \leq \frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}}$.

***Proof.*** Denote with $c = \mid C(\mathbf{g}^*) \mid$ . First, note that in a complete core-periphery network, all agents in the periphery access the same effort level, $cx^*_c$, and therefore all agents in the periphery display the same effort level, $x^*_p$. Agents in the core display identical effort levels by an argument analogous to the one in Lemma 1. The agents in the core maximize $x_c \in argmax_{x_c \in X} x_c - \frac{\beta}{2}x_c^2 + \lambda x_c((n-1)x_p + (c-1)\hat{x}_c)$, where $x_p$ is the effort level of agents in the periphery and $\hat{x}_c$ and are effort levels of (other) agents in the core. For an agent in the periphery we have $x_p \in argmax_{x_p \in X} x_p - \frac{\beta}{2}x_p^2 + \lambda x_p(cx_c)$. The reaction functions are given by $x_c(x_p, \hat{x}_c) = \frac{1+\lambda x_c'(c-1)+\lambda x_p(n-c)}{\beta}$ and $x_p(x_c) = \frac{1+\lambda c x_c}{\beta}$, respectively. Equilibrium effort levels are given by $x_c^* = \frac{\beta+\lambda(n-c)}{\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1)}$ and $x_p^* = \frac{\beta+\lambda}{\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1)}$. Plugging equilibrium effort levels back into the payoff function, yields equilibrium gross payoffs of $\pi_c^* = \frac{\beta(\beta+\lambda(n-c))^2}{2(\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1))^2}$ and $\pi_p^* = \frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1))^2}$. Next, we calculate deviation payoffs of of an agent in the periphery linking to all remaining agents in the periphery. A deviating agent maximizes $x_p^d \in argmax_{x_p^d \in X} x_p^d - \frac{\beta}{2}(x_p^2)^2 + \lambda x_p^d(cx_c + (n-c-1)x_p)$, which yields the following reaction function $x_p^d(x_c^*, x_p^*) = \frac{1+\lambda c x_c^* + \lambda x_p^*(n-c-1)}{\beta}$. The deviation effort level is given by $x_p^d(x_c^*, x_p^*) = \frac{(\beta+\lambda)(\beta+\lambda(n-k-1))}{\beta(\beta^2-\lambda^2 c(n-c)-\lambda\beta(c-1))}$ and corresponding deviation gross payoffs by $\pi_p^d = \frac{(\beta+\lambda)^2(\beta+\lambda(n-k-1))^2}{2\beta(\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1))^2}$. For the existence of a periphery-sponsored core-periphery network we need two conditions to hold. First, we want to find a linking cost $k$, such that an agent in the core finds it profitable to link to agents in the core and, second, that agents in the periphery find it profitable to link to the core, but not to the periphery. Note that if agents in the periphery find it profitable to extend links to the core, agents in the core find it profitable to extend the if agents in the periphery. To see this, note that an agent in the core has $n-c-1$ incoming links and extends $c-1$ links to the core, while an agent in the periphery has *zero* incoming links and extends $c$ links to the core. The relevant condition then reads $(\frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1))^2} - \frac{1}{2\beta})/c \geq (\frac{(\beta+\lambda)(\beta+\lambda(n-k-1))}{\beta(\beta^2-\lambda^2 c(n-c)-\lambda\beta(c-1))} - \frac{\beta(\beta+\lambda)^2}{2(\beta^2-\lambda^2 c(n-c)-\beta\lambda(c-1))^2})/(n-c-1)$. One can then show that this condition holds if and only if the number of agents in the core, $c$, is smaller than $\frac{n-1}{2}$ and $\lambda \leq \frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}}$.[11] *Q.E.D.*

**Corollary 3:** *If best response functions are linear and$\lambda$ sufficiently small, or $\beta$ sufficiently large, then for any periphery sponsored core-periphery network with $c < \frac{n-1}{2}$, there exists a linking cost $k$, such that $\mathbf{g}^*$ is a Nash equilibrium.*

***Proof.*** Note first that $\lambda \leq \frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}}$ is increasing in $\beta$ and therefore, for $\beta$ sufficiently large, $\lambda \leq \frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}}$holds. Next, note that $\frac{c(n-c)\beta}{n-1+c((c-n)^2-1)} + \sqrt{\frac{(n-c-1)^2(c(n+1)-c^2+1)\beta^2}{(n-c-1+(n-c)^2)}} > 0$ for $c < \frac{n-1}{2}$ and $\beta > 0$. Therefore, for $\lambda$ sufficiently small, the inequality again holds. *Q.E.D.*

[11] This calculation was executed in Mathematica and the codes are available upon request.